Journal of Finsler Geometry and its Applications
Vol. 5, No. 2 (2024), pp 127-142
https://doi.org/10.22098/jfga.2024.15799.1136

Special CR maximal dimensional submanifolds in the Kenmotsu space forms

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Abstract. The $(n+1)$-dimensional almost metric contact submanifolds wuth maximal $CR-$ submanifolds of $(n-1)$ in the Kenmotsu space forms classified such that $n > 5$ and $h(FX, Y) - h(X, FY) = g(FX, Y)\zeta$ for vector fields X, Y tangent to M, where h and F denote the second fundamental form and a skew-symmetric endomorphism acting on the tangent space of M, respectively, and ζ a non zero normal vector field to M.

Keywords: CR maximal dimensional submanifolds, Kenmotsu manifolds, Kenmotsu Space Form.

1. Introduction

Let M be a connected $(n+1)-$dimensional submanifold of codimension $q+1$ of a Kenmotsu space form $(\overline{M}, \phi, \xi, \eta, g)$, where $n > 5$. Then it is known

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AMS 2020 Mathematics Subject Classification: 53C25, 53C40

that if the maximal ϕ-invariant subspace of each tangent space is $(n-1)$-dimensional, M admits a naturally induced metric structure [4], [5]. For the hypersurface case, the maximal ϕ-invariant subspace is necessarily $(n-1)$-dimensional and when the ambient space $\overline{M}$ is a Kenmotsu space form, it is the maximal holomorphic subspace. On the other hand, for arbitrary codimension $q+1$, less detailed results are known, but more may be expected.

Kim et al. studied in [10] the maximal dimensional contact CR-submanifolds in unit sphere which satisfy the condition

$$h(FX, Y) + h(X, FY) = 0.$$

They determined such submanifolds under the additional condition, where F denotes a skew-symmetric endomorphism induced from ϕ acting on the tangent bundle TM and h the second fundamental form on M. Also, Okumura et al. studied in [4] the maximal dimensional contact CR-submanifolds in complex space form with the same condition. Recently, in [9] Kim et al. and the author in [6] introduced the same submanifolds in Sasakian space form and Kenmotsu space form, respectively.

Afterward Kim et al. studied in [11] the maximal dimensional contact CR-submanifold in unit sphere which satisfy the condition

$$h(FX, Y) - h(X, FY) = g(FX, Y)\zeta$$

for a normal non-zero vector field ζ to M. Also Okumura et al. in [5] and the author et al. in [7] studied the maximal dimensional contact CR-submanifold in complex space forms and Sasakian space forms with the same condition, respectively.

In this paper, we study $(n+1)-$dimensional contact CR-submanifolds of $(n-1)$ contact CR-dimension in a Kenmotsu space form and determine such submanifolds in a complete simply connected Kenmotsu space form of constant ϕ-holomorphic sectional curvature c, under the assumption

$$h(FX, Y) - h(X, FY) = g(FX, Y)\zeta$$

for a normal non-zero vector field ζ to M. As our main results, we obtain:

Theorem. Let M be a $CR-$submanifold of $(n-1)$ contact CR-dimension in the Kenmotsu space form $\overline{M}^{2n+1}(c)$. If, for any vector fields X, Y tangent to M, the above condition holds on M, then

- **•:** for $c \neq -1$, $\overline{M}^{2n+1}(c)$ does not admit any $CR-$submanifolds of $(n-1)$ contact $CR-$ dimension.
- **•:** for $c = -1$, either M is a totally geodesic submanifold either or M is locally isometric to a product of $C \times M_\lambda$, which C is a geodesic curve and M_λ is submanifold of M or M is locally isometric to a product $M_1 \times M_2$, where M_1 and M_2 are F-anti-invariant submanifolds in M.

All manifolds, submanifolds and geometric objects will be assumed to be connected, differentiable and of class C^∞, and all maps also be of class C^∞ if not stated otherwise.

2. **Preliminaries**

A differentiable manifold $\overline{M}^{2n+1}$ is said to have an almost contact structure if it admits a (non-vanishing) vector field ξ, a one-form η and a $(1,1)-$tensor field ϕ satisfying

$$\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi,$$

where I denotes the field of identity transformations of the tangent spaces at all points. These conditions imply that $\phi\xi = 0$ and $\eta \circ \phi = 0$, and that the endomorphism ϕ has rank $2n$ at every point in $\overline{M}^{2n+1}$. A manifold $\overline{M}^{2n+1}$, equipped with an almost contact structure (ϕ, ξ, η), is called an almost contact manifold.

Suppose that $\overline{M}^{2n+1}$ is a manifold carrying an almost contact structure. A Riemannian metric $\overline{g}$ on $\overline{M}^{2n+1}$ satisfying

$$\overline{g}(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields X and Y, is called compatible with the almost contact structure. It is known that an almost contact manifold always admits at least one compatible metric. Note that

$$\eta(X) = \overline{g}(X, \xi),$$

for all vector fields X tangent to $\overline{M}^{2n+1}$, which means that η is the metric dual of the characteristic vector field ξ.

A manifold $\overline{M}^{2n+1}$ is said to be a contact manifold if it carries a global one-form η such that

$$\eta \wedge (d\eta)^n \neq 0,$$

everywhere on M. The one-form η is called the contact form.

A submanifold M of a Riemannian contact manifold $\overline{M}^{2n+1}$ tangent to ξ is called an invariant (resp. anti-invariant) submanifold if $\phi(T_pM) \subset T_pM$, for each $p \in M$ (resp. $\phi(T_pM) \subset T_p^\perp M$, for each $p \in M$).

A submanifold M tangent to ξ of a contact manifold $\overline{M}^{2n+1}$ is called a contact CR-submanifold if there exists a pair of orthogonal differentiable distributions D and $D^\perp$ on M such that:

(1) $TM = D \oplus D^\perp \oplus \mathbb{R}\xi$, where $\mathbb{R}\xi$ is the $1-$dimensional distribution spanned by ξ;
(2) D is invariant by ϕ, i.e., $\phi(D_p) \subset D_p$, for each $p \in M$;
(3) $D^\perp$ is anti-invariant by ϕ, i.e., $\phi(D_p^\perp) \subset T_p^\perp M$, for each $p \in M$.

Let $(\overline{M},\phi,\xi,\eta,\overline{g})$ be a $(2n+1)$-dimensional contact manifold such that

$$\overline{\nabla}_X\xi = X-\eta(X)\xi, \qquad (\overline{\nabla}_X\phi)Y = \overline{g}(X,\phi Y)\xi-\eta(Y)\phi X,$$

where $\overline{\nabla}$ is the Levi-Chivita connection of $\overline{M}$, then $\overline{M}$ is called a Kenmotsu manifold. The plane section π of $T\overline{M}$ is called a $\phi-$section if $\phi\pi_x \subseteq \pi_x$, for each $x\in\overline{M}$. Also $\overline{M}$ is called of constant $\phi-$sectional curvature if the sectional curvature of $\phi-$sections is constant. A Kenmotsu space form is a Kenmotsu manifold of constant $\phi-$sectional curvature. In this case the Riemannian curvature tensor field $\overline{R}$ is given by

$$\begin{aligned}\overline{R}(X,Y)Z &= \frac{c+3}{4}\{\overline{g}(Y,Z)X-\overline{g}(X,Z)Y\}\\ &-\frac{c-1}{4}\{\eta(Z)[\eta(Y)X-\eta(X)Y]+[\overline{g}(Y,Z)\eta(X)-\overline{g}(X,Z)\eta(Y)]\xi\\ &\qquad -\overline{g}(\phi Y,Z)\phi X+\overline{g}(\phi X,Z)\phi Y+2\overline{g}(\phi X,Y)\phi Z\},\end{aligned}$$

for each $X,Y,Z\in\chi(\overline{M})$.

Let M be an $(n+1)-$dimensional submanifold tangent to the structure vector field ξ of $\overline{M}$. If the ϕ-invariant subspace D_x has constant dimension for any $x\in M$, then M is called a contact $CR-$submanifold and the constant is called contact $CR-$dimension of M (cf. [1, 4, 5, 6, 7, 9, 10, 11]).

3. CR maximal dimensional submanifold structure

Let $(\overline{M}(c),\overline{g})$ be an $(n+p)-$dimensional Kenmotsu space form with contact structure (ϕ,ξ,η) and let M be an $n-$dimensional submanifold tangent to the structure vector field ξ of $\overline{M}(c)$ with the immersion ι of M into $\overline{M}(c)$. Then the tangent bundle TM is identified with a subbundle of $T\overline{M}$ and a Riemannian metric g of M is induced from the Riemannian metric $\overline{g}$ in such a way that $g(X,Y)=\overline{g}(\iota X,\iota Y)$, where X,Y in TM, while we denote the differential of the immersion also by ι. The normal bundle $T^{\perp}M$ is the subbundle of $T\overline{M}$ consisting of all X of $T\overline{M}$ which are orthogonal to TM with respect to Riemannian metric $\overline{g}$.

Now, let M be a CR submanifold of maximal CR dimension, that is, at each point x of M, if we denote by D_x the ϕ-invariant subspace of the tangent space T_xM, then ξ cannot be contained in D_x at any point $x\in M$, thus the assumption $\dim D_x^{\perp}=2$ being constant and equal to 2 at each point $x\in M$ yields that M can be dealt with as a contact CR-submanifold, where $D_x^{\perp}$ denotes the complementary orthogonal subspace to D_x in T_xM. Further, the tangent space T_xM satisfies $\dim(T_xM\cap\phi T_xM)=n-2$.

Moreover, then it follows that M is even-dimensional and that there exists a unit vector field N normal to M such that

$$\phi TM\subset TM\oplus span\{N\}.$$

In [6], the author showed the following equalises

$$g(U,X)=u(X), \tag{3.1}$$
$$F^2X=-X+\eta(X)\xi+u(X)U, \tag{3.2}$$
$$u(FX)=\eta(FX)=0,\quad FU=F\xi=0,\quad PN=0, \tag{3.3}$$
$$u(\xi)=\eta(U)=0,\quad U_i=0 \qquad i=1,\dots,p-1. \tag{3.4}$$

Further, let us denote by $\overline{\nabla}$ and ∇ the Levi-Civita connection on $\overline{M}(c)$ and M, respectively, and by $\nabla^{\perp}$ the normal connection induced from $\overline{\nabla}$ in the normal bundle $TM^{\perp}$ of M. Then Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y=\nabla_X Y+h(X,Y), \tag{3.5}$$
$$\overline{\nabla}_X N=-AX+\nabla^{\perp}_X N=-AX+\sum_{a=1}^{q}\{s_a(X)N_a+s_{a^*}(X)N_{a^*}\}, \tag{3.6}$$
$$\overline{\nabla}_X N_a=-A_aX-s_a(X)N+\sum_{b=1}^{q}\{s_{ab}(X)N_b+s_{ab^*}(X)N_{b^*}\}, \tag{3.7}$$
$$\overline{\nabla}_X N_{a^*}=-Aa^*X-s_{a^*}(X)N+\sum_{b=1}^{q}\{s_{a^*b}(X)N_b+s_{a^*b^*}(X)N_{b^*}\}, \tag{3.8}$$
$$h(X,Y)=g(AX,Y)N+\sum_{a=1}^{q}\{g(A_aX,Y)N_a+g(A_{a^*}X,Y)N_{a^*}\}. \tag{3.9}$$

for any tangent vector fields X, Y to M. Also we have

$$A_aX=-FA_{a^*}X+s_{a^*}(X)U,\quad trA_{a^*}=-s_a(U), \tag{3.10}$$
$$A_{a^*}X=FA_aX-s_a(X)U,\quad trA_a=-s_{a^*}(U), \tag{3.11}$$
$$s_a(X)=-u(A_{a^*}X),\quad s_{a^*b^*}(X)=s_{ab}(X), \tag{3.12}$$
$$s_{a^*}(X)=u(A_aX),\quad s_{a^*b}(X)=-s_{ab^*}(X), \tag{3.13}$$
$$g((FA_a+A_aF)X,Y)=s_a(X)u(Y)-s_a(Y)u(X), \tag{3.14}$$
$$g((FA_{a^*}+A_{a^*}F)X,Y)=s_{a^*}(X)u(Y)-s_{a^*}(Y)u(X). \tag{3.15}$$
$$(\nabla_Y F)X=g(FY,X)\xi-\eta(X)FY-g(AY,X)U+u(X)AY, \tag{3.16}$$
$$g(FX,FY)=g(X,Y)-\eta(X)\eta(Y)-u(X)u(Y), \tag{3.17}$$
$$\nabla_X U=FAX-u(X)\xi, \tag{3.18}$$
$$\nabla_X \xi=X-\eta(X)\xi, \tag{3.19}$$
$$A\xi=0,\quad A_a\xi=0,\quad A_{a^*}\xi=0, \qquad a=1,\dots,q. \tag{3.20}$$

If the ambient manifold $\overline{M}$ is a Kenmotsu space form $\overline{M}(c)$, i.e., a Kenmotsu space form of constant ϕ-holomorphic sectional curvature c, then the curvature tensor $\overline{R}$ of $\overline{M}(c)$ has a special form and the Gauss equation becomes

$$\begin{aligned} R(X,Y)Z &= \frac{c-3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ &+\frac{c+1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &-g(Y,Z)\eta(X)\xi + g(FY,Z)FX - g(FX,Z)FY - 2g(FX,Y)FZ\} \\ &+g(AY,Z)AX + g(AX,Z)AY \\ &+\sum_{a=1}^{q}\{g(A_aY,Z)A_aX - g(A_aX,Z)A_aY\} \\ &+\sum_{a=1}^{q}\{g(A_{a^*}Y,Z)A_{a^*}X - g(A_{a^*}X,Z)A_{a^*}Y\}, \end{aligned} \tag{3.21}$$

for any vector fields X, Y, Z tangent to M, where R denotes the Riemannian curvature tensor of M. In this case, we can see that the equations of Codazzi and Ricci-Kühne imply

$$\begin{aligned} (\nabla_X A)Y - (\nabla_Y A)X = \frac{c+1}{4}\{u(X)FY - u(Y)FX - 2g(FX,Y)U\} \\ +\sum_{a=1}^{q}\{s_a(X)A_aY - s_a(Y)A_aX + s_{a^*}(X)A_{a^*}Y - s_{a^*}(Y)A_{a^*}X\}, \end{aligned} \tag{3.22}$$

$$\begin{aligned} (\nabla_X A_a)Y - (\nabla_Y A_a)X = s_a(Y)AX - s_a(X)AY \\ +\sum_{b=1}^{q}\{s_{ab}(X)A_bY - s_{ab}(Y)A_bX + s_{ab^*}(X)A_{b^*}Y - s_{ab^*}(Y)A_{b^*}X\}, \end{aligned} \tag{3.23}$$

$$\begin{aligned} (\nabla_X A_{a^*})Y - (\nabla_Y A_{a^*})X = s_{a^*}(Y)AX - s_{a^*}(X)AY \\ +\sum_{b=1}^{q}\{s_{a^*b}(X)A_bY - s_{a^*b}(Y)A_bX + s_{a^*b^*}(X)A_{b^*}Y - s_{a^*b^*}(Y)A_{b^*}X\}, \end{aligned} \tag{3.24}$$

$$\begin{aligned} \overline{g}(\overline{R}(X,Y)\xi_a,\xi) = g((AA_a - A_aA)X,Y) + (\nabla_X s_a)(Y) - (\nabla_Y s_a)(X) \\ +\sum_{b=1}^{q}\{s_{ab}(Y)s_b(X) - s_{ab}(X)s_b(Y) + s_{ab^*}(Y)s_{b^*}(X) - s_{ab^*}(X)s_{b^*}(Y)\} \end{aligned} \tag{3.25}$$

for any vector fields X, Y tangent to M.

4. **Proof of the Main Theorem**

In this section we let M be an $(n+1)$-dimensional contact CR-submanifold of $(n-1)$ contact CR-dimension immersed in a Kenmotsu space form $\overline{M}(c)$ and let us use the same notation as stated in the previous section.

We assume that the equality

$$h(FX,Y) - h(X,FY) = g(FX,Y)\zeta \tag{4.1}$$

holds on M for a normal vector field ζ to M. We also use the orthonormal basis (3.4) of normal vectors to M and set

$$\zeta = \rho N + \sum_{a=1}^{q}(\rho_a N_a + \rho_{a^*} N_{a^*}).$$

Then by means of (3.9) the condition (4.1) is equivalent to

$$(AF + FA)X = \rho FX, \tag{4.2}$$

$$(A_a F + FA_a)X = \rho_a FX, \quad (A_{a^*} F + FA_{a^*})X = \rho_{a^*} FX \tag{4.3}$$

for all $a = 1, \ldots, q$. Moreover, the last two equations combined with (3.14) and (3.15) yield

$$s_a(X)u(Y) - s_a(Y)u(X) = \rho_a g(FX, Y), \tag{4.4}$$

$$s_{a^*}(X)u(Y) - s_{a^*}(Y)u(X) = \rho_{a^*} g(FX, Y), \tag{4.5}$$

from which, putting $Y = U$ and $Y = \xi$ into (4.4) and (4.5), respectively, and using (3.1), we obtain

$$s_a(X) = s_a(U)u(X), \quad s_{a^*}(X) = s_{a^*}(U)u(X), \tag{4.6}$$
$$s_a(\xi) = 0, \quad s_{a^*}(\xi) = 0, \quad a = 1, \ldots, q.$$

Substituting (4.6) into (4.5), we have

$$\rho_a = 0, \quad \rho_{a^*} = 0, \quad a = 1, \ldots, q$$

and consequently with the aid of (4.3) we obtain

$$FA_a + A_a F = 0, \quad FA_{a^*} + A_{a^*} F = 0, \quad a = 1, \ldots, q. \tag{4.7}$$

As a direct consequence of (4.2) and (4.7), it follows from (3.1), (3.2), (3.12), (3.20) and (3.21) that

$$AU = \lambda U, \quad \lambda := u(AU) \tag{4.8}$$

and, for $a = 1, \ldots, q$,

$$A_a U = u(A_a U)U = s_{a^*}(U)U, \quad A_{a^*} U = u(A_{a^*} U)U = -s_a(U)U. \tag{4.9}$$

Inserting FX into (4.2) instead of X and using (3.2), (3.20) and (4.8), we have

$$AX = \{(\lambda - \rho)u(X) + \eta(X)\}U + \{u(X) - \rho\eta(X)\}\xi + FAFX + \rho X. \tag{4.10}$$

On the other hand, $FD_x = D_x$ at each point $x \in M$, and thus there exists a local orthonormal basis $\{E_i, E_{i^*}, U, \xi\}_{i=1,\ldots,l}$ of tangent vectors to M such that

$$E_{i^*} = FE_i, \quad i = 1, \ldots, l := \frac{n-1}{2}. \tag{4.11}$$

Lemma 4.1. *Let M be an $(n+1)-$dimensional contact CR-submanifold of $(n-1)$ contact CR-dimension immersed in a Kenmotsu space form $\overline{M}(c)$. If the condition (4.1) is satisfied on M for a non-zero normal vector field ρ to M, then U is an eigenvector of the shape operator A with respect to distinguished normal vector field ξ, at any point of M.*

Using Gauss equation (3.21) and Ricci-Kuhne formula (3.25), we obtain

$$\begin{aligned} 0=\overline{g}(\overline{R}(X,Y)\xi_a,\xi) &= g(AA_aX,Y)-g(A_aAX,Y) \\ &+(\nabla X s_a)(Y)-(\nabla_Y s_a)(X) \\ &+\sum_{b=1}^{q}\{s_b(Y)s_{ba}(X)+s_b(Y)s_{b*a}(X) \\ &-s_b(X)s_{ba}(Y)-s_{b^*}(X)s_{b*a}(Y)\}. \end{aligned} \tag{4.12}$$

Lemma 4.2. *Let M be an $(n+1)-$dimensional contact CR-submanifold of $(n-1)$ contact CR-dimension immersed in a Kenmotsu space form $M(c)$. If, for any vector fields X,Y tangent to M, the equality (4.1) holds on M for a non-zero normal vector field ρ to M, then*

$$s_a=0,\quad s_{a^*}=0,\quad a=1,\ldots,q,$$

namely, the distinguished normal vector field N is parallel with respect to the normal connection. Moreover,

$$A_a=0,\quad A_{a^*}=0,\quad a=1,\ldots,q.$$

Proof. First, differentiating the relation (3.11) and using (3.16), (3.18), (4.8) and (4.9), we obtain

$$g((\nabla_X A_{a^*})Y,U)=-g(A_aAX,Y)+\lambda s_{a^*}(U)u(X)u(Y)-(\nabla_X s_a)(Y). \tag{4.13}$$

Reversing X and Y and subtracting thus yields

$$\begin{aligned} g((\nabla_X A_{a^*})Y-(\nabla_Y A_{a^*})X,,U) &= g((AA_a-A_aA)X,Y) \\ &-(\nabla_X s_a)(Y)+(\nabla_Y s_a)(X). \end{aligned} \tag{4.14}$$

Substituting (3.25) into (4.14) and using (4.8), we have

$$\begin{aligned} g((AA_a-A_aA)\,X,Y) \quad - \quad & (\nabla X s_a)(Y)+(\nabla_Y s_a)(X)= \\ & \sum_{b=1}^{q}\{s_{a^*b}(X)g(A_bY,U)-s_{a^*b}(Y)g(A_bX,U)\} \\ & +\sum_{b=1}^{q}\{s_{a^*b^*}(X)g(A_{b^*}Y,U)-s_{a^*b^*}(Y)g(A_{b^*}X,U)\} \end{aligned} \tag{4.15}$$

Now, using (3.11), (3.12), (3.13), relations (4.12) and (4.15) yield

$$g((AA_a-A_aA)X,Y)=0, \tag{4.16}$$

for all $X, Y \in T(M)$. On the other hand, differentiating (4.9) and using (3.18) and (4.2), we obtain

$$\begin{aligned} g((\nabla_X A_{a^*})Y &- (\nabla_Y A_{a^*})X, U) + g((A_{a^*}FA + AFA_{a^*})X, Y) \\ &= Y(s_a(U))u(X) - X(s_a(U))u(Y) - \rho s_a(U)g(FX, Y) \\ &\quad + s_a(U)u(X)\eta(Y) - s_a(U)u(Y)\eta(U). \end{aligned} \tag{4.17}$$

From (3.17) and using (3.11), (3.12), (3.20), (4.6) and (4.8), we compute

$$g((A_{a^*}FA + AFA_{a^*})X, Y) = g((A_a A - AA_a)X, Y). \tag{4.18}$$

From (3.12), (3.13), Codazzi equation (3.24) and (4.8), yields

$$\begin{aligned} g((\nabla_X A_{a^*})Y - (\nabla_Y A_{a^*})X, U) &= \lambda s_{a^*}(Y)u(X) - \lambda s_{a^*}(X)u(Y) \\ &\quad + \sum_{b=1}^{q}\{s_{a^*b}(X)s_{b^*}(Y) - s_{a^*b}(Y)s_{b^*}(X)\} \\ &\quad + \sum_{b=1}^{q}\{s_{a^*b^*}(Y)s_b(X) - s_{a^*b^*}(X)s_b(Y)\}. \end{aligned} \tag{4.19}$$

Therefore, using (4.17), (4.18) and (4.19), we get

$$\begin{aligned} Y(s_a(U))u(X) &- X(s_a(U))u(Y) - \rho s_a(U)g(FX, Y) \\ &\quad + s_a(U)u(X)\eta(Y) - s_a(U)u(Y)\eta(U) \\ &= g((A_a A - AA_a)X, Y) + \lambda s_{a^*}(Y)u(X) - \lambda s_{a^*}(X)u(Y) \\ &\quad + \sum_{b=1}^{q}\{s_{a^*b}(X)s_{b^*}(Y) - s_{a^*b}(Y)s_{b^*}(X)\} \\ &\quad + \sum_{b=1}^{q}\{s_{a^*b^*}(Y)s_b(X) - s_{a^*b^*}(X)s_b(Y)\}. \end{aligned} \tag{4.20}$$

Putting $Y = U$ into(4.20) and taking account of (4.6), it follows that

$$\begin{aligned} X(s_a(U)) &= U(s_a(U))u(X) - s_a(U)\eta(X) \\ &\quad - \sum_{b=1}^{q}\{s_{a^*b}(X)s_{b^*}(U) - s_{a^*b^*}(X)s_b(U) \\ &\qquad - s_{a^*b}(U)s_{b^*}(U)u(X) + s_{a^*b^*}(U)s_b(U)u(X)\}. \end{aligned} \tag{4.21}$$

Also, with using (4.6) and (4.8), we conclude $g((A_a A - AA_a)X, U) = 0$. Therefore, relation (4.20) with (4.21) and using (4.6), we have

$$g((AA_a - A_a A)X, Y) = \rho s_a(U)g(FX, Y). \tag{4.22}$$

Thus (4.16) and (4.22) imply $s_a(U) = 0$ and consequently, from (4.6) we conclude $s_a(X) = 0$. In entirely the same way, we obtain $s_{a^*} = 0$, which completes the proof. □

Now from lemma 4.2, we would have

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c+1}{4}\{u(X)FY - u(Y)FX - 2g(FX,Y)U\}. \quad (4.23)$$

Since A is self adjoint, (3.20) and (4.8) show that D is an invariant subspaces under A. Hence there exists a locally orthonormal frame

$$X_1, \ldots, X_{2n-2},$$

for D, where

$$AX_i = \alpha_i X_i, \qquad i = 1, \ldots, 2n-2.$$

Proposition 4.3. *Let M be an $(n+1)$-dimensional contact $CR-$submanifold of $(n-1)$ contact $CR-$dimension in the Kenmotsu space form $\overline{M}(c)$. If the equality (4.1) holds on M for a non-zero normal vector field ρ to M, then for eigenvalues of the shape operator A of M we have*

$$X(\lambda) = X(\alpha_i) = 0, \text{ for all } X \perp \xi,$$
$$\xi(\lambda) = -\lambda, \quad \xi(\alpha_i) = -\alpha_i.$$

Proof. Differentiating (4.8) covariantly and using (3.18), (3.20), (4.2) and (4.8), we have

$$\begin{aligned} g((\nabla_X A)Y - (\nabla_Y A)X, U) &= -2g(AFAX,Y) + X(\lambda)u(Y) - Y(\lambda)u(X) \\ &\quad + \lambda\rho g(FX,Y) - \lambda u(X)\eta(Y) + \lambda u(Y)\eta(X). \end{aligned}$$

Moreover, using (3.3) and (4.23), we have

$$\begin{aligned} -\frac{c+1}{2}g(FX,Y) &= -2g(AFAX,Y) + X(\lambda)u(Y) - Y(\lambda)u(X) \\ &\quad + \lambda\rho g(FX,Y) - \lambda u(X)\eta(Y) + \lambda u(Y)\eta(X). \quad (4.24) \end{aligned}$$

Putting $Y = U$ into the the last equation and using (3.3), we obtain

$$X(\lambda) = U(\lambda)u(X) - \lambda\eta(X). \quad (4.25)$$

Choosing $X \in D$ in (4.25) we get

$$X(\lambda) = 0, \quad (4.26)$$

and as well choosing $X = \xi$ in (4.25) we have

$$\xi(\lambda) = -\lambda. \quad (4.27)$$

Substituting (4.25) into (4.24), we obtain

$$-\frac{c+1}{2}g(FX,Y) = -2g(AFAX,Y) + \lambda\rho g(FX,Y)$$

Putting $X = X_i$ into the the last equation and using (4.2), we have

$$\alpha_i^2 - \rho\alpha_i + \frac{\lambda\rho}{2} + \frac{c+1}{4} = 0. \quad (4.28)$$

Differentiating (4.8) covariantly respect to U and using (3.18), (3.19), (4.8) and (4.23), we have

$$U(\lambda) = 0.$$

Putting $X = X_i$ and $Y = \xi$ into the (4.23) and using (3.3), (3.19), (3.20) and (4.26), we have

$$\xi(\alpha_i) = -\alpha_i.$$

Taking $Y = U$ and $X = X_i$ into the (4.23) and using (3.18) and (4.8), we obtain

$$U(\alpha_i) = 0.$$

Putting $X = X_i$ and $Y = X_j$ into the (4.23), we obtain

$$X(\alpha_i) = 0,$$

which completes the proof. □

Proposition 4.4. *Let M be an $(n+1)$-dimensional contact $CR-$submanifold of $(n-1)$ contact $CR-$dimension in Kenmotsu space form $\overline{M}(c)$. If the equality (4.1) holds on M for a non-zero normal vector field ρ to M, then $c = -1$.*

Proof. Putting $X = FX_i$ and $Y = \xi$ in (4.23) and using proposition 4.3, (3.3), (3.19), (4.2), it follows that

$$\xi(\rho) = -\rho. \tag{4.29}$$

With differentiating of the equation (4.28) and relations proposition 4.3, (4.27) and (4.29) we have

$$\alpha_i^2 + \alpha_i\lambda + \frac{\lambda\rho}{2} = 0, \tag{4.30}$$

therefore $c = -1$. □

Hence, we can state the following:

Theorem 4.5. *A Kenmotsu space form with $c \neq -1$ does not admit any CR-submanifold of $(n-1)$ contact $CR-$dimension for which equality (4.1) holds for a non-zero normal vector field ρ to M.*

Proposition 4.6. *Let M be an $(n+1)$-dimensional contact $CR-$submanifold of $(n-1)$ contact $CR-$dimension in the Kenmotsu space form $\overline{M}(-1)$. If the equality (4.1) holds on M for a non-zero normal vector field ρ to M, then for eigenvalues of the shape operator A of M we have*

$$X(\rho) = 0, \text{ for all } X \perp \xi, \quad \xi(\rho) = -\rho.$$

Proof. Taking $X = FX_i$ and $Y = U$ in (4.23) and using proposition 4.3, (3.3), (3.19), (3.20), (4.2) and (4.8), follows that

$$U(\rho) = 0. \tag{4.31}$$

Differentiating (4.2) covariantly and using (3.16), (3.20), (4.2) and (4.8), we have

$$\begin{aligned} X(\rho)FY &= (\nabla_X A)FY + F(\nabla_X A)Y + u(Y)A^2X + (\lambda-\rho)u(Y)AX \\ &+\eta(Y)FAX - \{(\lambda-\rho)g(AX,Y) - g(AX,AY)\}U \\ &+\{g(FX,AY) - \rho g(FX,Y)\}\xi \end{aligned}$$

from which, using (3.3) and the orthonormal basis given by (4.11),

$$\begin{aligned} &\sum_{i=1}^{n+1} g((\nabla_{E_i}A)FY, E_i) - \sum_{i=1}^{l} g((\nabla_{E_i}A)FE_i - (\nabla_{FE_i}A)E_i, Y) \\ &\quad +(trA^2 + (\lambda-\rho)trA - \lambda(\lambda-\rho) - \lambda^2)u(Y) = (FY)(\rho). \end{aligned} \tag{4.32}$$

On the other hand, using (3.3) and (3.19), we have

$$\sum_{i=1}^{n+1} g((\nabla_{E_i}A)FY, E_i) = \sum_{i=1}^{n+1} g((\nabla_{FY}A)E_i, E_i) = 0, \tag{4.33}$$

and

$$\sum_{i=1}^{l} g((\nabla_{E_i}A)FE_i - (\nabla_{FE_i}A)E_i, Y) = 0. \tag{4.34}$$

Substituting (4.25) into (4.24) and use (4.2) implies

$$(\frac{\lambda\rho}{2} + \frac{c+1}{4})FX + \rho FAX - FA^2X = 0.$$

Applying F to this equation and using (3.2), (3.3), (3.20), (4.2) and (4.8), we can easily obtain

$$\begin{aligned} A^2X &= (\lambda^2 - \lambda + \frac{\lambda\rho}{2} + \frac{c+1}{4})u(X)U + (\frac{\lambda\rho}{2} + \frac{c+1}{4})\eta(X)\xi \\ &-(\frac{\lambda\rho}{2} + \frac{c+1}{4})X - \rho AX. \end{aligned} \tag{4.35}$$

Moreover, taking the trace of (4.35) with respect to the orthonormal bais (4.11) and using (3.20), (4.8) and (4.10), we can find

$$trA = \lambda + \frac{\rho(n-1)}{2}, \tag{4.36}$$

$$trA^2 = \frac{(n-1)\rho(\lambda-\rho)}{2} - \lambda^2 + \frac{(n-1)(c+1)}{4}, \tag{4.37}$$

Substituting (4.33),(4.34) and (4.36) into (4.32) and taking account of (3.20), (4.8), (4.10), (4.35) and since $c = -1$, we can see that

$$(FX)(\rho) = 0.$$

Thus we have for all $X \in D$

$$X(\rho) = 0. \tag{4.38}$$

Hence (4.38) with (4.29) and (4.31) completes the proof. □

Lemma 4.7. *Let M be an $(n+1)$-dimensional contact $CR-$submanifold of $(n-1)$ contact $CR-$dimension in the Kenmotsu space form $\overline{M}(c)$. If the equality (4.1) holds on M for a non-zero normal vector field ρ to M, then the shape operator A has one eigenvalues $\lambda = 0$ of multiplicities $n+1$ or 2 eigenvalues $0, \lambda$ of multiplicities 1 and n, or 4 eigenvalues*

$$0, \quad \lambda, \quad \frac{\rho - \sqrt{\rho^2 - 2\lambda\rho}}{2}, \quad \frac{\rho + \sqrt{\rho^2 - 2\lambda\rho}}{2}$$

of multiplicities 1, 1, $\frac{n-1}{2}$ and $\frac{n-1}{2}$, respectively. Moreover, if A has exactly 2 eigenvalues $0, \lambda$, then the eigenvalue α corresponding to an eigenvector of A, orthogonal to U and ξ, satisfies $\alpha = \lambda = \rho/2$ and vice-versa.

Proof. If $\lambda = 0$, the relation (4.30) implies that $\alpha_i = 0$. Otherwise, since $\lambda \neq 0$ from (4.30) the shape operator A has 2 eigenvalues $0, \lambda$ of multiplicities 1 and n, or 4 constant eigenvalues

$$0, \quad \lambda, \quad \frac{\rho - \sqrt{\rho^2 - 2\lambda\rho}}{2}, \quad \frac{\rho + \sqrt{\rho^2 - 2\lambda\rho}}{2}$$

whose multiplicities are 1, 1, $\frac{n-1}{2}$ and $\frac{n-1}{2}$, respectively, with the help of (3.20) and (4.8). Moreover, if A has exactly 2 eigenvalues 0 and λ, then $\alpha = \lambda = \rho/2$. □

Therefore we have one of the main result.

Theorem 4.8. *Let M be a $CR-$submanifold of $(n-1)$ contact CR-dimension in the Kenmotsu space form $\overline{M}^{2n+1}(-1)$. If, for any vector fields X, Y tangent to M, the equality (4.1) holds on M for a non-zero normal vector field ρ to M and the shape operator A of M has exactly one eigenvalue, then M is totally geodesic submanifold.*

Let's assume now that A has exactly two distinct the eigenvalues. From lemma 4.7, we put

$$T_\lambda = \{X \in TM | AX = \lambda X\} = D \oplus \mathbb{R}U.$$

Then, we get the distributions T_λ.

Lemma 4.9. *The distributions T_λ is involutive.*

Proof. Let us choose $X, Y \in T_\lambda$ and using (3.19), we have

$$g(\nabla_X Y, \xi) = -g(X, Y),$$

therefore

$$g([X,Y],\xi)=0. \tag{4.39}$$

Now, for $X,U\in T_\lambda$ and using (3.18), we have

$$g(\nabla_X U,\xi)=-u(X).$$

Also, from the Codazzi equation, proposition 4.3, proposition4.6 and (4.23), we get

$$g(\nabla_U X,\xi)=-u(X),$$

therefore

$$g([X,U],\xi)=0. \tag{4.40}$$

With selection $X,Y\in D$ and using (4.23), (4.26) and the Codazzi equation, it follows that

$$0=(\nabla_X A)Y-(\nabla_Y A)X=\lambda\nabla_X Y-A\nabla_X Y-\lambda\nabla_Y X+A\nabla_Y X,$$

so

$$g([X,Y],\xi)=\frac{1}{\lambda}g(A\nabla_X Y-A\nabla_Y X,\xi)=0. \tag{4.41}$$

Relations (4.39), (4.40) and (4.41) imply that, for all $X,Y\in T_\lambda$, we have

$$g([X,Y],\xi)=0.$$

Hence, $[X,Y]\in T_\lambda$. This shows that the distribution T_λ is involutive. □

Now we consider the integral submanifolds M_λ for the distributions T_λ in M and we consider the integral curve of the vector field ξ and show it $C(t)$. In other words $C'(t)=\xi$. Hence the following theorem holds:

Theorem 4.10. *Let M be a $CR-$submanifold of $(n-1)$ contact CR-dimension in the Kenmotsu space form $\overline{M}^{2n+1}(-1)$. If, for any vector fields X,Y tangent to M, the equality (4.1) holds on M for a non-zero normal vector field ρ to M and the shape operator A of M has exactly two eigenvalues, then M is locally isometric to a product of $C\times M_\lambda$, which C is a geodesic curve and M_λ is submanifold of M.*

Let's assume now that A has exactly four distinct the eigenvalues

$$0,\quad \lambda,\quad \alpha=\frac{\rho-\sqrt{\rho^2-2\lambda\rho}}{2},\quad \beta=\frac{\rho+\sqrt{\rho^2-2\lambda\rho}}{2}.$$

For eigenvalues of A, we put

$$T_1=D_1\oplus\mathbb{R}\xi=\{X\in D|AX=\alpha X\}\oplus\mathbb{R}\xi,$$
$$T_2=D_2\oplus\mathbb{R}U=\{X\in D|AX=\beta X\}\oplus\mathbb{R}U.$$

Then, we get two distributions T_1 and T_2.

Also, from lemma 4.7 we have $\alpha + \beta = \rho$ and for the vector field X on M, if we have $AX = \alpha X$, from (4.2) we have $AFX = \beta FX$. So that D_1 and D_2 is F-anti-invariant subspace.

Lemma 4.11. *The distributions T_1 and T_2 are both involutive.*

Proof. By choosing $X, Y \in T_1$ and $U \in T_2$. Then, using (3.18), we have

$$g(\nabla_X Y, U) = 0,$$

therefore

$$g([X, Y], U) = 0. \tag{4.42}$$

Now, for $X, \xi \in T_1$ and $Z \in T_2$. Then, using (3.19), we have

$$g(\nabla_X \xi, Z) = 0.$$

Also, from the Codazzi equation, proposition 4.3, proposition 4.6 and (4.23), we get

$$g(\nabla_\xi X, Z) = 0,$$

therefore

$$g([X, \xi], Z) = 0. \tag{4.43}$$

With selection $X, Y \in D_1$ and $Z \in D_2$ and using (4.23), (4.26) and the Codazzi equation, it follows that

$$0 = (\nabla_X A)Y - (\nabla_Y A)X = \alpha\nabla_X Y - A\nabla_X Y - \alpha\nabla_Y X + A\nabla_Y X,$$

so

$$g([X, Y], Z) = \frac{1}{\alpha} g(A\nabla_X Y - A\nabla_Y X, Z) = 0. \tag{4.44}$$

Relations (4.42), (4.43) and (4.44) imply that, for all $X, Y \in T_1$ and $Z \in T_2$, we have

$$g([X, Y], Z) = 0.$$

Hence, $[X, Y] \in T_1$. This shows that the distribution T_1 is involutive. In entirely the same way, we prove that T_2 is involutive. □

Now we consider the integral submanifolds M_1 and M_2 respectively for the distributions T_1 and T_2 in M. So that M_1 and M_2 is F-anti-invariant submanifolds.

Thus we have one of the main result.

Theorem 4.12. *Let M be a $CR-$submanifold of $(n-1)$ contact CR-dimension in the Kenmotsu space form $\overline{M}^{2n+1}(-1)$. If, for any vector fields X, Y tangent to M, the equality (4.1) holds on M for a non-zero normal vector field ρ to M and the shape operator A of M has exactly four eigenvalues, then M is locally isometric to a product $M_1 \times M_2$, where M_1 and M_2 are $F-$anti-invariant submanifolds in M.*

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Received: 08.09.2024
Accepted: 07.12.2024