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Some rigidity results on homogeneous Finsler spaces equipped with Killing frames

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Abstract. Utilizing Killing frames on homogeneous Finsler manifolds, we express the Berwald and mean Berwald curvatures in terms of Killing frames and get some rigidity results among them we prove that homogeneous isotropic weakly Berwald metrics reduce to weakly Berwald metric.

Keywords: Homogeneous Finsler metrics,, Berwald metric, Weakly Berwald metric.

1. **Introduction**

In Riemannian geometry, a Killing vector field is a vector field on a Riemannian manifold $(M, \mathbf{g})$ that preserves the metric $\mathbf{g}$. Killing fields are the infinitesimal generators of isometries; that is, flows generated by Killing fields are continuous isometries of the manifold. More simply, the flow generates a symmetry, in the sense that moving each point on an object the same distance

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in the direction of the Killing vector will not distort distances on the object. Thus a vector field X is a Killing field if the Lie derivative with respect to X of the metric $\mathbf{g}$ vanishes, $\mathcal{L}_X\mathbf{g} = 0$. A typical use of the Killing field is to express a symmetry in general relativity in which the geometry of spacetime as distorted by gravitational fields is viewed as a 4-dimensional pseudo-Riemannian manifold. In a static configuration, in which nothing changes with time, the time vector will be a Killing vector, and thus the Killing field will point in the direction of forward motion in time.

In Finsler geometry, the Riemannian and non-Riemannian curvatures are mostly defined with standard local coordinates

$$\mathbf{C} := \Big\{x = (x^i) \in M,\ y = y^i\partial/\partial x^i \in T_xM\Big\}.$$

But in the homogeneous context, generally speaking, the local coordinates are not compatible with the homogeneous structure, and invariant frames or Killing frames seem more suitable. A Killing frame for a Finsler manifold (M, F) is a set of local vector fields $\{X_i\}_{i=1}^n$, $n := dim(M)$, defined on an open subset U around a given point, such that: (1) The values $X_i(x)$, $\forall i$, give bases for each tangent space $T_x(M)$, $x \in U$; (2) In U, eachX_i satisfies $\tilde{X}_i(F) = 0$.

Though Killing frames are rare in the general study of Finsler geometry, they can be easily found for a homogeneous Finsler space at any given point. Let the homogeneous Finsler space (M, F) be presented as $M = G/H$, where H is the isotropy subgroup for the given x. The tangent space TM_x can be identified as the quotient $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H, respectively. Take any basis $\{v_i\}_{i=1}^n$ of $\mathfrak{m}$, with the pre-images $\{\hat{v}_1, \ldots, \hat{v}_n\}$ in $\mathfrak{g}$. The Killing vector fields $\{X_i\}_{i=1}^n$ on M corresponding to $\hat{v}_i$s defines a Killing frame around x. The choice of $\hat{v}_i$s or X_is identifies the quotient space $\mathfrak{m}$ with a subspace of $\mathfrak{g}$, and then we can write the decomposition of linear space

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}. \tag{1.1}$$

In homogeneous Finsler geometry, curvatures can be reduced to some tensors on $\mathfrak{m}$, and differential equations are reduced to linear equations. So we may avoid some extremely complicated calculations, and find the intrinsic nature of curvatures from the Lie algebra structures. The most successful examples for this approach include the following. We can use Killing frames (i.e., local frames provided by Killing vector fields) to present the S-curvature, and use the Finslerian submersion technique to deduce the flag curvature formula, for a homogeneous Finsler manifold [1][2][4][5][7][15][16]. Notice in Finsler geometry, curvatures are mostly defined with standard local coordinates $\mathbf{C}$. But in the homogeneous context, generally speaking, the local coordinates are not compatible with the homogeneous structure, and invariant frames or Killing frames seem more suitable. In [3], by applying invariant frames, Huang provides explicit formulas for all curvatures in homogeneous Finsler geometry.

In [4], by using the Killing fields, Hu-Deng give formula for the Riemann curvature of a homogeneous Finsler manifolds which is a generalization of the formula for homogeneous Riemannian manifolds. Then Xu-Deng proved that a homogeneous Finsler space has isotropic S-curvature if and only if it has vanishing S-curvature [14].

2. **Killing Frame**

For the Killing frame $\{X_1, \ldots, X_n\}$ around $x \in M$, a set of y-coordinates $y = (y^i)$ can be defined by $y = y^i X_i$. Accordingly, we have the fundamental tensor $g_{ij} = \frac{1}{2}[F^2]_{y^i y^j}$, and the inverse matrix of (g_{ij}) is denoted as (g^{ij}). When both the Killing frame and the local coordinates $\{x = (\bar{x}^{\bar{i}}), y = \bar{y}^{\bar{j}} \partial_{\bar{x}^{\bar{j}}}\}$ are used, the terms and indices for the local coordinates are marked with bars, and the indices with bars are moved up and down by the fundamental tensors $\bar{g}^{\bar{i}\bar{j}}$ or $\bar{g}_{\bar{i}\bar{j}}$ for the local coordinates. Let $f^i_{\bar{i}}$ and $f^{\bar{i}}_i$, $\forall i$ and $\bar{i}$, be the transition functions such that around x,

$$\partial_{\bar{x}^{\bar{i}}} = f^i_{\bar{i}} X_i, \quad \text{and} \quad X_i = f^{\bar{i}}_i \partial_{\bar{x}^{\bar{i}}}. \tag{2.1}$$

We summarize some easy and useful identities which show how the transition functions exchange the indices with and without bars:

$$\bar{y}^{\bar{i}} = f^{\bar{i}}_i y^i \quad \text{and} \quad y^i = f^i_{\bar{i}} \bar{y}^{\bar{i}} \tag{2.2}$$

$$\partial_{\bar{y}^{\bar{i}}} = f^i_{\bar{i}} \partial_{y^i} \quad \text{and} \quad \partial_{y^i} = f^{\bar{i}}_i \partial_{\bar{y}^{\bar{i}}}, \tag{2.3}$$

$$\bar{g}_{\bar{i}\bar{j}} = f^i_{\bar{i}} g_{ij} f^j_{\bar{j}} \quad \text{and} \quad g_{ij} = f^{\bar{i}}_i \bar{g}_{\bar{i}\bar{j}} f^{\bar{j}}_j, \tag{2.4}$$

$$\bar{g}^{\bar{i}\bar{j}} = f^{\bar{i}}_i g^{ij} f^{\bar{j}}_j \quad \text{and} \quad g^{ij} = f^i_{\bar{i}} \bar{g}^{\bar{i}\bar{j}} f^j_{\bar{j}}. \tag{2.5}$$

The S-curvature is a important non-Riemannian quantity. At the first time S-curvature introduced by Z.Shen in [6]. it is for measure the rate of change of the distortion along geodesics. There is a notion of distortion $\tau = \tau(x, y)$ on TM associated with a volume form $dV = \sigma(x)dx$, which is defined by

$$\tau(x, y) = \ln \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)}. \tag{2.6}$$

Then the S-curvature is defined by

$$\mathbf{S}(x, y) = \frac{d}{dt}\Big[\tau\Big(c(t), \dot{c}(t)\Big)\Big]|_{t=0}, \tag{2.7}$$

where $c(t)$ is the geodesic with $\dot{c}(0) = y$.

3. Results

Lemma 3.1. *Let (M,F) be an n-dimensional homogeneous Finsler manifold. Suppose that $\{X_i\}_{i=1}^n$ is a Killing frame around $x \in M$ for the Finsler metric F. Then for $y = \tilde{y}^i X_i(x) \in TM_x$, the geodesic spray coefficients can be presented as*

$$G^{\bar{i}} = -\frac{1}{2}\Big\{ y^i \bar{y}^{\bar{j}} \partial_{\bar{x}^{\bar{i}}} f_i^{\bar{i}} + \frac{1}{2} g^{il} c^k{}_{lj} [F^2]_{y^k} y^j f_i^{\bar{i}} \Big\}. \tag{3.1}$$

where $c^k{}_{lj}$ are defined by $[X_l, X_j](x) = c^k{}_{lj} X_k(x)$.

Proof. Let $\{X_1,\ldots,X_n\}$ be a Killing frame around $x \in M$ for the Finsler metric F. In [14], Xu-Deng have studied Killing frames of homogeneous Finsler spaces. For $y = \tilde{y}^i X_i(x) \in TM_x$, they obtain the geodesic spray $G(x,y)$ as follows

$$\mathbf{G}(x,y) = y^i \tilde{X}_i + \frac{1}{2} g^{il} c^k_{lj} [F^2]_{y^k} y^j \partial_{y^i}. \tag{3.2}$$

The spray coefficients are given by

$$G^{\bar{i}} = \frac{1}{4} \bar{g}^{\bar{i}\bar{l}} \Big[(F^2)_{\bar{x}^{\bar{j}} \bar{y}^{\bar{l}}} \bar{y}^{\bar{j}} - (F^2)_{\bar{x}^{\bar{l}}} \Big]. \tag{3.3}$$

By using (3.2) and (3.3), we get

$$\begin{aligned} \mathbf{G}(x,y) &= y^i (f_i^{\bar{i}} \partial_{x^i} + \bar{y}^{\bar{i}} \partial_{x^i} f_i^{\bar{j}} \partial_{\bar{y}^{\bar{j}}}) + \frac{1}{2} g^{il} c^k_{lj} [F^2]_{y^k} y^j f_i^{\bar{i}} \partial_{\bar{y}^{\bar{i}}} \\ &= \bar{y}^{\bar{i}} \partial_{x^i} + \Big\{ y^i \bar{y}^{\bar{j}} \partial_{x^j} f_i^{\bar{i}} + \frac{1}{2} g^{il} c^k_{lj} [F^2]_{y^k} y^j f_i^{\bar{i}} \Big\} \partial_{\bar{y}^{\bar{i}}}. \end{aligned} \tag{3.4}$$

Comparing (3.2) and (3.4), imply (3.1). □

Let (M,F) be an n-dimensional homogeneous Finsler manifold. Suppose that $\{X_i\}_{i=1}^n$ is a Killing frame around $x \in M$. Then for any $0 \neq y \in TM_x$, the S-curvature at (x,y) can be presented with the notations for the Killing frame as

$$\mathbf{S}(x,y) = \frac{1}{2} g^{il} c^k{}_{lj} \Big[F^2\Big]_{y^k} y^j I_i. \tag{3.5}$$

Now consider the case that $M = G/H$ is a homogeneous Finsler space, where H is the isotropy group of $x \in M$. Let the Killing vector fields X_i's be defined by $\hat{v}_i \in \mathfrak{g}$, $\forall i$. Then the tangent space TM_x can be identified with the n-dimensional subspace $\mathfrak{m}$ spanned by the values of all the $\hat{v}_i$'s at x. With respect to the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, there is a projection map $\mathrm{pr} : \mathfrak{g} \to \mathfrak{m}$. Note that for the bracket operation $[\cdot,\cdot]$ on $\mathfrak{g}$, we have $[\cdot,\cdot]_{\mathfrak{m}} = \mathrm{pr}[\cdot,\cdot]$. Then c^k_{ij}s can be determined by

$$[\hat{v}_i, \hat{v}_j]_{\mathfrak{m}} = -c^k_{ij} \hat{v}_k. \tag{3.6}$$

In [14], Xu-Deng proved the existence of Killing frames around x. Each Killing frame $\{X_i\}_{i=1}^n$ determines a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where X_i is determined by $\hat{v}_i$ in $\mathfrak{m}$. Let the operation $[\cdot,\cdot]_{\mathfrak{m}}$ be defined as before. The gradient field of

$\ln\sqrt{\det(g_{pq})}$ with respect to the fundamental tensor on $TM_x\backslash 0$ is the $\mathfrak{m}$-valued function

$$g^{il}I_i\hat{v}_l = g^{il}\Big[\ln\sqrt{\det(g_{pq})}\Big]_{y^i}\hat{v}_l. \tag{3.7}$$

We will denote it as $\nabla^{g_{ij}}\ln\sqrt{\det(g_{pq})}(y)$ for $y\in\mathfrak{m}$. Let $\langle\cdot,\cdot\rangle_y$ be the inner product defined by the fundamental tensor g_{ij} at y. Then by (3.6) we can rewrite (3.5) as

$$\mathbf{S}(x,y) = g^{il}c^k_{lj}g_{kh}y^hy^jI_i = \Big\langle [y,\nabla^{g_{ij}}\ln\sqrt{\det(g_{pq})}(y)]_{\mathfrak{m}}, y\Big\rangle_y, \tag{3.8}$$

which gives a formula for the S-curvature of a homogenous Finsler space.

Proposition 3.2. *Let (M,F) be an n-dimensional homogeneous Finsler manifold. Suppose that $\{X_i\}_{i=1}^n$ is a Killing frame around $x\in M$ for the Finsler metric F. Then for $y=\tilde{y}^iX_i(x)\in TM_x$, the Berwald curvature of F is given by*

$$\begin{aligned}
B^{\bar{i}}{}_{\bar{p}\bar{q}\bar{m}} = -\frac{1}{2}f^p_{\bar{p}}f^q_{\bar{q}}f^m_{\bar{m}}f^{\bar{i}}_i\Bigg\{ & \Big[\partial_{y^p}\partial_{y^q}\partial_{y^m}g^{il}(c^k_{lj}[F^2]_{y^k}y^j)\Big]\\
& +\Big[\partial_{y^q}\partial_{y^m}g^{il}(c^k_{lj}y^jg_{kp}+\frac{1}{2}c^k_{lp}F^2]_{y^k})\Big]\\
& +\Big[\partial_{y^p}\partial_{y^m}g^{il}(c^k_{lj}y^jg_{kq}+\frac{1}{2}c^k_{lq}F^2]_{y^k})\Big]\\
& +\Big[\partial_{y^p}\partial_{y^q}g^{il}(c^k_{lj}y^jg_{km}+\frac{1}{2}c^k_{lm}F^2]_{y^k})\Big]\\
& +\Big[\partial_{y^p}\partial_{y^q}g_{km}(c^k_{lj}y^jg^{il})\Big]+\Big[(\partial_{y^m}g^{il}\partial_{y^p}g_{kq}+\partial_{y^q}g^{il}\partial_{y^p}g_{km}\\
& +\partial_{y^p}g^{il}\partial_{y^q}g_{km})(c^k_{lj}y^j)\Big]+\Big[(\partial_{y^m}g^{il})(c^k_{lp}g_{kq}+c^k_{lq}g_{kp})\Big]\\
& +\Big[(\partial_{y^q}g^{il})(c^k_{lp}g_{km}+c^k_{lm}g_{kp})\Big]+\Big[(\partial_{y^p}g^{il})(c^k_{lq}g_{km}+c^k_{lm}g_{kq})\Big]\\
& +\Big[(\partial_{y^q}g_{km}c^k_{lp}+\partial_{y^p}g_{km}c^k_{lq}+\partial_{y^p}g_{kq}c^k_{lm})g^{il}\Big]\Bigg\}.
\end{aligned} \tag{3.9}$$

The mean Berwald curvature of F can be express as follows

$$\begin{aligned}
E_{\bar{s}\bar{t}} = \frac{1}{4}\Bigg\{ & \Big[f^s_{\bar{s}}f^t_{\bar{t}}(\partial_{y^s}\partial_{y^t}g^{il}c^k_{lj}[F^2]_{y^k}y^jI_i)\Big]+\Big[f^t_{\bar{t}}\partial_{y^t}g^{il}c^k_{lj}(2f^s_{\bar{s}}g_{ks}y^jI_i\\
& +f^s_{\bar{s}}[F^2]_{y^k}y^j\partial_{y^s}I_i+f^j_{\bar{s}}[F^2]_{y^k}I_i)\Big]+\Big[f^s_{\bar{s}}\partial_{y^s}g^{il}c^k_{lj}(2f^t_{\bar{t}}g_{kt}y^jI_i\\
& +f^t_{\bar{t}}[F^2]_{y^k}y^j\partial_{y^t}I_i+f^j_{\bar{t}}[F^2]_{y^k}I_i)\Big]+\Big[g^{il}c^k_{lj}\big(2f^j_{\bar{s}}f^t_{\bar{t}}g_{kt}I_i\\
& +2f^s_{\bar{s}}f^t_{\bar{t}}g_{kt}y^j\partial_{y^s}I_i+2f^s_{\bar{s}}f^j_{\bar{t}}g_{ks}I_i+f^s_{\bar{s}}f^j_{\bar{t}}[F^2]_{y^k}\partial_{y^s}I_i\\
& +2f^s_{\bar{s}}f^t_{\bar{t}}g_{ks}y^j\partial_{y^t}I_i+f^j_{\bar{s}}f^t_{\bar{t}}[F^2]_{y^k}\partial_{y^t}I_i+f^s_{\bar{s}}f^t_{\bar{t}}[F^2]_{y^k}\partial_{y^s}\partial_{y^t}I_i)\Big]\Bigg\}.
\end{aligned} \tag{3.10}$$

For a two-dimensional plane $P \subset T_xM$ and $y \in T_xM_0$, the flag mean Berwald curvature $\mathbf{E}(P, y)$ is defined by

$$\mathbf{E}(P, y) := \frac{F^3(x, y)\mathbf{E}_y(u, u)}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - [\mathbf{g}_y(y, u)]^2}, \tag{3.11}$$

where $P := span\{y, u\}$. F is called of isotropic mean Berwald curvature if for any flag (P, y), the following holds

$$\mathbf{E}(P, y) = \frac{n+1}{2}c \iff E_{ij} = \frac{n+1}{2}cF_{y^iy^j} \iff E_{ij} = \frac{n+1}{2}cF^{-1}h_{ij}, \tag{3.12}$$

where $c = c(x)$ is a scalar function on M. We have the following.

Theorem 3.3. *Let (M, F) be an n-dimensional homogeneous Finsler manifold. Then F has isotropic E-curvature if and only if it has vanishing E-curvature.*

Proof. It is proved that, F has isotropic E-curvature $\mathbf{E} = (n + 1)/2cF^{-1}\mathbf{h}$ if and only if it has almost isotropic S-curvature $\mathbf{S} = (n + 1)cF + \eta$, where $\eta = \eta_i(x)y^i$ is a 1-form on M. Let us consider the S-curvature at a fixed point x. The function $\ln\sqrt{\det(g_{pq})}$ is homogeneous of degree 0, so it must reach its maximum or minimum at some nonzero y, where the gradient field vanishes. Then by (3.8), $\mathbf{S}(x, y) = 0$. If the S-curvature is almost isotropic, i.e., if $\mathbf{S} = (n + 1)c(x)F + \eta$, then $c(x) = 0$. In this case, $\mathbf{S} = \eta$. Taking two vertical derivation of it implies that $\mathbf{E} = 0$. □

A Finsler metric F has isotropic Berwald metric if its Berwald curvature is given by following

$$\begin{aligned}\mathbf{B}_y(u, v, w) = cF^{-1}\Big\{&\mathbf{h}(u, v)\Big(w - \mathbf{g}_y(w, \ell)\ell\Big) + \mathbf{h}(v, w)\Big(u - \mathbf{g}_y(u, \ell)\ell\Big)\\ &+\mathbf{h}(w, u)\Big(v - \mathbf{g}_y(v, \ell)\ell\Big) + 2F\mathbf{C}_y(u, v, w)\ell\Big\},\end{aligned} \tag{3.13}$$

where $c \in C^\infty(M)$. In this case, F is called an isotropic Berwald metric. It is obvious that Berwald metrics is a isotropic Berwald metric condition with $c = 0$. Geometrically, Berwald metrics are those Finsler metrics which are affinely equivalent to Riemannian ones.

Corollary 3.4. *Let (M, F) be an n-dimensional homogeneous Finsler manifold. Then F has isotropic Berwald curvature if and only if it has vanishing Berwald curvature.*

Proof. In [9], it is proved that every isotropic Berwald metric (3.13) has isotropic S-curvature $\mathbf{S} = (n+1)cF$. By Theorem 3.3, we get $c = 0$. Putting it in (3.13) implies that $\mathbf{B} = 0$. □

By definition, $\mathbf{L}/\mathbf{C}$ is regarded as the relative rate of change of $\mathbf{C}$ along Finslerian geodesics. Let (M, F) be a Finsler manifold. Then, F is called relatively isotropic Landsberg metric if it satisfies $\mathbf{L}+cF\mathbf{C}=0$, where $c=c(x)$ is a scalar function on M. We get the following.

Corollary 3.5. *Every homogeneous relatively isotropic Landsberg metric of isotropic mean Berwald curvature is a Berwald metric.*

Proof. Every n-dimensional ($n\geq 3$) relatively isotropic Landsberg metric of isotropic mean Berwald curvature is a isotropic Berwald metric. In [8], it is proved that every relatively isotropic Landsberg surface ($n=2$) of isotropic mean Berwald curvature is a isotropic Berwald surface. By Corollary 3.4, it follows that F reduces to a Berwald metric. □

Let (M,F) be a Finsler manifold. Then, F is called a weakly Douglas metric if it satisfies

$$D^i_{jkl}=T_{jkl}y^i, \tag{3.14}$$

where T_{jkl} is a Finslerian tensor on M.

Corollary 3.6. *Let (M,F) be an n-dimensional ($n\geq 3$) weakly Douglas manifold. Suppose that F has isotropic mean Berwald curvature. Then F is a Berwald metric.*

Proof. Taking vertical derivative from (3.14) with respect to y^s, we get

$$\begin{aligned}\frac{\partial D^i{}_{jkl}}{\partial y^s}&=\frac{\partial B^i{}_{jkl}}{\partial y^s}-\frac{2}{n+1}\Big\{\frac{\partial E_{jk}}{\partial y^s}\delta^i_l+\frac{\partial E_{kl}}{\partial y^s}\delta^i_j+\frac{\partial E_{lj}}{\partial y^s}\delta^i_k+\frac{\partial^2 E_{jk}}{\partial y^s\partial y^l}y^i+\frac{\partial E_{jk}}{\partial y^l}\delta^i_s\Big\}\\&=\frac{\partial T_{jkl}}{\partial y^s}y^i+T_{jkl}\delta^i_s.\end{aligned} \tag{3.15}$$

Contracting i and s in (3.15) and using the relations

$$\frac{1}{2}\frac{\partial B^s{}_{jkl}}{\partial y^s}=\frac{\partial E_{jk}}{\partial y^l}=\frac{\partial E_{kl}}{\partial y^j}=\frac{\partial E_{lj}}{\partial y^k} \tag{3.16}$$

we get

$$0=\frac{\partial D^s{}_{jkl}}{\partial y^s}=(n-2)T_{jkl}. \tag{3.17}$$

Therefore, for $n>2$, we get $T_{jkl}=0$ and F is a Douglas metric. By Theorem 3.3, F reduces to a Berwald metric. □

Let (M,F) be a Finsler manifold. Then, F is called a generalized Douglas-Weyl metric if its Douglas tensor satisfies

$$D^i{}_{jkl|m}y^m=T_{jkl}y^i \tag{3.18}$$

that is hold for some tensor T_{jkl}, where $D^i{}_{jkl|m}$ denotes the horizontal covariant derivatives of Douglas curvature $D^i{}_{jkl}$ with respect to the Berwald connection of F. For a manifold M, let $\mathcal{G}DW(M)$ denotes the class of all Finsler metrics satisfying in above relation for some tensor T_{jkl}. It is proved that $\mathcal{G}DW(M)$ is closed under projective changes.

Corollary 3.7. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a homogeneous non-Randers type (α, β)-metric on n-dimensional manifold M $(n \geq 3)$. Then F is a generalized Douglas-Weyl metric with isotropic S-curvature if and only if it is a Berwald metric.*

Proof. In [11], it is proved that a non-Randers type (α, β)-metric is a generalized Douglas-Weyl metric with vanishing S-curvature if and only if it is a Berwald metric. By using Theorem 3.3, we get the proof. □

A Finsler metric F has almost vanishing H-curvature if its non-Riemannian quantity H satisfies following

$$\mathbf{H} = \frac{n+1}{2F}\,\theta\mathbf{h},$$

where $c = c(x)$ is a scalar function and $\theta = \theta_i(x)dx^i$ is a 1-form on M ([10][12]).

Corollary 3.8. *Let $F := \alpha + \beta$ be a homogeneous Randers metric an n-dimensional manifold M. Suppose that F has almost vanishing H-curvature. Then $\mathbf{S} = 0$.*

Proof. In [13], Xia proved that a Randers metric has almost vanishing H-curvature if and only if it has isotropic S-curvature $\mathbf{S} = (n+1)cF$. By Theorem 3.3, we get $\mathbf{S} = 0$. The converse is trivial. □

Corollary 3.9. *Let $F := \alpha + \beta$ be a homogeneous Randers metric of weakly isotropic flag curvature on a manifold M of dimension $n \geq 3$. Let F has constant flag curvature.*

Proof. Let F be of weakly isotropic flag curvature

$$\mathbf{K} = \frac{3\theta}{F} + \sigma, \tag{3.19}$$

where $\sigma = \sigma(x)$ is a scalar function and $\theta = \theta_i(x)y^i$ is a 1-form on M. By Najafi-Shen-Tayebi theorem, F satisfies following

$$\mathbf{H} = \frac{(n^2-1)\theta}{2F} \tag{3.20}$$

where $\theta = \theta_i(x)dx^i$ is a 1-form on M. By corollary 3.8, $\theta = 0$. By the Schur lemma in Finsler geometry, if $\mathbf{K} = \sigma(x)$ is a scalar function on M, then it must be constant in dimension $n \geq 3$. □

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