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On $C3$-like Finsler spaces of relatively isotropic mean Landsberg curvature

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Abstract. In this paper, we study the class of $C3$-like Finsler metrics with relatively isotropic mean Landsberg. We find some conditions under which these metrics reduce to relatively isotropic Landsberg metrics.

Keywords: Relatively isotropic mean Landsberg metric, relatively isotropic Landsberg metric.

1. Introduction

There are some interesting special forms of Cartan torsion and Landsberg tensor which have been obtained by some Finslerians [2][4][13][15]. The Finsler spaces having such special forms have been called C-reducible, semi-C-reducible, C2-Like, L-reducible (or P-reducible), general relatively isotropic Landsberg, and etc [5][6]. Let us remark the notion of Cartan torsion and Landsberg tensor. For a Finsler manifold (M, F), the second derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ is an inner product $\mathbf{g}_y$ on T_xM. The third order derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ is a symmetric trilinear forms $\mathbf{C}_y$ on T_xM. We call $\mathbf{g}_y$ and $\mathbf{C}_y$ the fundamental form and the Cartan torsion, respectively. In [4], Matsumoto introduced the notion of C-reducible Finsler metrics and proved that any Randers metric is

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C-reducible. Later on, Matsumoto-Hōjō proves that the converse is true too [1]. A Randers metric $F = \alpha + \beta$ is just a Riemannian metric α perturbated by a one form β, which has important applications both in mathematics and physics [14]. The rate of change of $\mathbf{C}_y$ along geodesics is the Landsberg curvature $\mathbf{L}_y$ on T_xM for any $y \in T_xM_0$. F is said to be Landsbergian if $\mathbf{L} = 0$.

In [10], Prasad-Singh by considering the special form of Cartan torsion of 3-dimensional Finsler spaces introduced a new class of Finsler spaces named by $C3$-like spaces which contains the class of semi-C-reducible spaces, as special case (see [7], [8], [9]). A Finsler metric F on a manifold M of dimension $n \geq 3$ is called $C3$-like if its Cartan tensor is given by

$$C_{ijk} = \Big\{a_i h_{jk} + a_j h_{ki} + a_k h_{ij}\Big\} + \Big\{b_i I_j I_k + I_i b_j I_k + I_i I_j b_k\Big\}, \tag{1.1}$$

where $a_i = a_i(x,y)$ and $b_i = b_i(x,y)$ are homogeneous scalar functions on TM of degree -1 and 1, respectively. We have some special cases as follows:

(1) If $a_i = 0$, then we have
$$C_{ijk} = \Big\{b_i I_j I_k + I_i b_j I_k + I_i I_j b_k\Big\}.$$
Contracting it with g^{ij} implies that
$$b_i = \frac{1}{3\|\mathbf{I}\|^2} I_i.$$
Then F is a $C2$-like metric;

(2) If $b_i = 0$, then we have
$$C_{ijk} = \Big\{a_i h_{jk} + a_j h_{ki} + a_k h_{ij}\Big\}.$$
Contracting it with g^{ij} implies that
$$a_i = \frac{1}{n+1} I_i.$$
Then F is a C-reducible metric;

(3) Let us put
$$a_i = \frac{p}{n+1} I_i, \quad b_i = \frac{q}{3\|\mathbf{I}\|^2} I_i,$$
where $p = p(x,y)$ and $q = q(x,y)$ are scalar functions on TM. In this case, F reduces to a semi-C-reducible metric.

It is remarkable that, in [2] Matsumoto-Shibata introduced the notion of semi-C-reducibility and proved that every non-Riemannian (α,β)-metric on a manifold M of dimension $n \geq 3$ is semi-C-reducible. Therefore the study of the class of $C3$-like Finsler spaces will enhance our understanding of the geometric meaning of (α,β)-metrics.

Theorem 1.1. *Let (M,F) be an n-dimensional $C3$-like Finsler manifold $n \geq 3$ such that $b_i = b_i(x,y)$ is constant along Finslerian geodesics. Suppose that one of the following holds:*

: *(i)* $\Im = -1/2$;

: *(ii)* $a'_i = 2ca_i$;

where $\Im := b_m I^m$ *and* $a'_i = a_{i|j}y^j$. *Then* F *is isotropic mean Landsberg metric* $\mathbf{J} = cF\mathbf{I}$ *if and only if it is isotropic Landsberg metric* $\mathbf{L} = cF\mathbf{C}$.

2. **Preliminaries**

Let M be a n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M.

A Finsler metric on M is a function $F : TM \to [0,\infty)$ which has the following properties:
(i) F is C^∞ on $TM_0 := TM \setminus \{0\}$;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM,
(iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u,v) := \frac{1}{2}\left[F^2(y+su+tv)\right]|_{s,t=0}, \quad u,v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\left[\mathbf{g}_{y+tw}(u,v)\right]|_{t=0}, \quad u,v,w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that **C=0** if and only if F is Riemannian.

For $y \in T_xM_0$, define mean Cartan torsion $\mathbf{I}_y$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where

$$I_i := g^{jk}C_{ijk}.$$

Here, $u = u^i\partial/\partial x^i|_x$. By Diecke Theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$.

For $y \in T_xM_0$, define the Matsumoto torsion $\mathbf{M}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{M}_y(u,v,w) := M_{ijk}(y)u^iv^jw^k$ where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\{I_ih_{jk} + I_jh_{ik} + I_kh_{ij}\},$$

and $h_{ij} := FF_{y^iy^j} = g_{ij} - \frac{1}{F^2}g_{ip}y^pg_{jq}y^q$ is the angular metric. A Finsler metric F is said to be C-reducible if $\mathbf{M}_y = 0$. This quantity is introduced by Matsumoto [4]. Matsumoto proves that every Randers metric satisfies that $\mathbf{M}_y = 0$. A Randers metric $F = \alpha + \beta$ on a manifold M is just a Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ perturbated by a one form $\beta = b_i(x)y^i$ on M such that $\|\beta\|_\alpha < 1$. Later on, Matsumoto-Hōjō proves that the converse is true too.

Lemma 2.1. ([1]) A Finsler metric F on a manifold of dimension $n \geq 3$ is a Randers metric if and only if $\mathbf{M}_y = 0$, $\forall y \in TM_0$.

A Finsler metric is called semi-C-reducible if its Cartan tensor is given by

$$C_{ijk} = \frac{p}{1+n}\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\} + \frac{q}{C^2}I_iI_jI_k,$$

where $p = p(x,y)$ and $q = q(x,y)$ are scalar function on TM and $C^2 = I^iI_i$. Multiplying the definition of semi-C-reducibility with g^{jk} shows that p and q must satisfy $p + q = 1$. If $p = 0$, then F is called $C2$-like metric. In [2], Matsumoto and Shibata proved that every (α,β)-metric is semi-C-reducible. Let us remark that an (α,β)-metric is a Finsler metric on M defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, α is a Riemannian metric and β is a 1-form on M [3].

Theorem 2.2. ([2][3]) Let $F = \phi(\frac{\beta}{\alpha})\alpha$ be a non-Riemannian (α,β)-metric on a manifold M of dimension $n \geq 3$. Then F is semi-C-reducible.

The horizontal covariant derivatives of $\mathbf{C}$ along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ defined by

$$\mathbf{L}_y(u,v,w) := L_{ijk}(y)u^iv^jw^k,$$

where $L_{ijk} := C_{ijk|s}y^s$, $u = u^i\frac{\partial}{\partial x^i}|_x$, $v = v^i\frac{\partial}{\partial x^i}|_x$ and $w = w^i\frac{\partial}{\partial x^i}|_x$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y\in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$.

There are many connections in Finsler geometry [11][12]. In this paper, we use the Berwald connection and the h- and v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively.

3. Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. For this aim, we need the following.

Lemma 3.1. *Let (M,F) be an n-dimensional $C3$-like Finsler manifold $n \geq 3$. Suppose that F is not Riemannian. Then the following hold:*

$$a_i(x,y)y^i = 0, \qquad b_i(x,y)y^i = 0. \tag{3.1}$$

Proof. F is $C3$-like metric

$$C_{ijk} = \Big\{a_ih_{jk} + a_jh_{ki} + a_kh_{ij}\Big\} + \Big\{b_iI_jI_k + I_ib_jI_k + I_iI_jb_k\Big\}, \tag{3.2}$$

where $a_i = a_i(x,y)$ and $b_i = b_i(x,y)$ are scalar functions on TM. Multiplying (3.2) with g^{ij} implies that

$$I_i = a_ih_{jk} + b_iI_jI_k. \tag{3.3}$$

Contracting (3.2) with y^i yields

$$a_iy^ih_{jk} + b_iy^iI_jI_k = 0. \tag{3.4}$$

Multiplying (3.4) with g^{jk} gives us

$$(n-1)a_iy^i + \|\mathbf{I}\|^2 b_iy^i = 0, \tag{3.5}$$

which by considering the assumption $\|\mathbf{I}\| \neq 0$ is equal to

$$b_iy^i = -\frac{1}{\|\mathbf{I}\|^2}(n-1)a_iy^i. \tag{3.6}$$

Putting (3.6) in (3.4) implies

$$\Big[h_{jk} - \frac{1}{\|\mathbf{I}\|^2}(n-1)I_jI_k\Big]a_iy^i = 0. \tag{3.7}$$

By contracting (3.7) with I^j and using

$$h_{jk}I^j = I_k$$

we get

$$(n-2)a_iy^iI_k = 0. \tag{3.8}$$

Since F is not Riemannian and $n \geq 3$, then (3.8) gives us

$$a_iy^i = 0. \tag{3.9}$$

Putting (3.9) in (3.6) yields

$$b_iy^i = 0. \tag{3.10}$$

This completes the proof. □

Lemma 3.2. *Let (M, F) be a $C3$-like Finsler manifold. Suppose that $b_i = b_i(x, y)$ is constant along Finslerian geodesics and $I^mb_m = -1/2$. Then F is isotropic mean Landsberg metric if and only if it is isotropic Landsberg metric.*

Proof. F is $C3$-like metric

$$C_{ijk} = \Big\{a_ih_{jk} + a_jh_{ki} + a_kh_{ij}\Big\} + \Big\{b_iI_jI_k + I_ib_jI_k + I_iI_jb_k\Big\}, \tag{3.11}$$

where $a_i = a_i(x, y)$ and $b_i = b_i(x, y)$ are scalar functions on TM. Multiplying (3.11) with g^{ij} implies that

$$a_i = \frac{1}{n+1}\Big\{(1-2\mathfrak{I})I_i - \|\mathbf{I}\|^2 b_i\Big\}, \tag{3.12}$$

where $\mathfrak{I} := b_mI^m$ and $\|\mathbf{I}\|^2 := I_mI^m$. By plugging (3.12) in (3.11), we get

$$\begin{aligned} C_{ijk} = & \frac{1}{n+1}\Big\{I_ih_{jk} + I_jh_{ki} + I_kh_{ij}\Big\} - \frac{2\mathfrak{I}}{n+1}\Big\{I_ih_{jk} + I_jh_{ki} + I_kh_{ij}\Big\} \\ & - \frac{\|\mathbf{I}\|^2}{n+1}\Big\{b_ih_{jk} + b_jh_{ki} + b_kh_{ij}\Big\} + \Big\{b_iI_jI_k + I_ib_jI_k + I_iI_jb_k\Big\}, \end{aligned} \tag{3.13}$$

or equivalently

$$\begin{aligned} M_{ijk} = -\frac{2\mathfrak{I}}{n+1}\Big\{I_ih_{jk} + I_jh_{ki} + I_kh_{ij}\Big\} - \frac{\|\mathbf{I}\|^2}{n+1}\Big\{b_ih_{jk} + b_jh_{ki} + b_kh_{ij}\Big\} \\ + \Big\{b_iI_jI_k + I_ib_jI_k + I_iI_jb_k\Big\}. \end{aligned} \tag{3.14}$$

By taking a horizontal derivation of (3.14), we have

$$\begin{aligned}
\widetilde{M}_{ijk} = & - \frac{2}{n+1}(J^m b_m + I^m b'_m)\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} \\
& - \frac{2\mathfrak{I}}{n+1}\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\} - \frac{\|\mathbf{I}\|^2}{n+1}\Big\{b'_i h_{jk} + b'_j h_{ki} + b'_k h_{ij}\Big\} \\
& - \frac{1}{n+1}(J^m I_m + I^m J_m)\Big\{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\Big\} \\
& + \Big\{b_i J_j I_k + b_i I_j J_k + b_j J_i I_k + b_j I_i J_k + b_k J_i I_j + b_k I_i J_j\Big\} \\
& + \Big\{b'_i I_j I_k + b'_j I_i I_k + b'_k I_i I_j\Big\},
\end{aligned} \tag{3.15}$$

where $b'_i = b_{i|s} y^s$ and

$$\widetilde{M}_{ijk} = L_{ijk} - \frac{1}{n+1}\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\}.$$

Let $b'_i = 0$. Then (3.15) reduces to following

$$\begin{aligned}
L_{ijk} = & \frac{1}{n+1}\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\} - \frac{2\mathfrak{I}}{n+1}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} \\
& - \frac{2}{n+1}\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\} b_m I^m \\
& - \frac{1}{n+1}(J^m I_m + I^m J_m)\Big\{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\Big\} \\
& + \Big\{b_i J_j I_k + b_i I_j J_k + b_j J_i I_k + b_j I_i J_k + b_k J_i I_j + b_k I_i J_j\Big\}
\end{aligned} \tag{3.16}$$

Let F is isotropic mean Landsberg metric

$$\mathbf{J} = cF\mathbf{I},$$

where $c = c(x)$ is a scalar function on M. Then (3.16) became as follows

$$\begin{aligned}
L_{ijk} = & \frac{cF}{n+1}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} - \frac{4cF\mathfrak{I}}{n+1}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} \\
& - \frac{2cF\|\mathbf{I}\|^2}{n+1}\Big\{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\Big\} \\
& + 2cF\Big\{b_i I_j I_k + I_i b_j I_k + I_i I_j b_k\Big\}.
\end{aligned} \tag{3.17}$$

By (3.13) we have

$$\begin{aligned}
\Big\{b_i I_j I_k + I_i b_j I_k + I_i I_j b_k\Big\} = C_{ijk} & - \frac{1}{n+1}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} \\
& + \frac{\mathfrak{I}}{n+1}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} \\
& + \frac{\|\mathbf{I}\|^2}{n+1}\Big\{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\Big\}
\end{aligned} \tag{3.18}$$

Putting (3.18) in (3.17) yields

$$L_{ijk}=2cFC_{ijk}-\frac{cF(1+2\mathfrak{I})}{n+1}\Big\{I_ih_{jk}+I_jh_{ki}+I_kh_{ij}\Big\}. \tag{3.19}$$

Since $\mathfrak{I}=-1/2$, then (3.19) reduces to $L_{ijk}=2cFC_{ijk}$. □

Lemma 3.3. *Let (M,F) be a $C3$-like Finsler manifold, such that $b_i=b_i(x,y)$ is constant along Finslerian geodesics and $a'_i=2ca_i$. Then F is isotropic mean Landsberg metric $\mathbf{J}=cF\mathbf{I}$ if and only if it is isotropic Landsberg metric $\mathbf{L}=cF\mathbf{C}$.*

Proof. Let F be a $C3$-like metric

$$C_{ijk}=\Big\{a_ih_{jk}+a_jh_{ki}+a_kh_{ij}\Big\}+\Big\{b_iI_jI_k+I_ib_jI_k+I_iI_jb_k\Big\}, \tag{3.20}$$

By taking a horizontal derivation of (3.20), we get

$$\begin{aligned}L_{ijk}= &\Big\{a'_ih_{jk}+a'_jh_{ki}+a'_kh_{ij}\Big\}+\Big\{b'_iI_jI_k+b'_jI_iI_k+b'_kI_iI_j\Big\}\\ &+\Big\{b_iJ_jI_k+b_iI_jJ_k+b_jJ_iI_k+b_jI_iJ_k+b_kJ_iI_j+b_kI_iJ_j\Big\}.\end{aligned} \tag{3.21}$$

Let F is isotropic mean Landsberg metric $J=cFI$. Then (3.21) became as follows

$$\begin{aligned}L_{ijk}=\Big\{a'_ih_{jk}+a'_jh_{ki}+a'_kh_{ij}\} + \ &\{b'_iI_jI_k+b'_jI_iI_k+b'_kI_iI_j\Big\}\\ &+\ 2cF\Big\{b_iI_jI_k+I_ib_jI_k+I_iI_jb_k\Big\}.\end{aligned} \tag{3.22}$$

By (3.20) we have

$$\Big\{b_iI_jI_k+I_ib_jI_k+I_iI_jb_k\Big\}=C_{ijk}-\Big\{a_ih_{jk}+a_jh_{ki}+a_kh_{ij}\Big\}. \tag{3.23}$$

Putting (3.23) in (3.22) yields

$$\begin{aligned}L_{ijk}=2cFC_{ijk}\ +\ &\Big\{(a'_i-2ca_i)h_{jk}+(a'_j-2ca_j)h_{ki}+(a'_k-2ca_k)h_{ij}\Big\}\\ +\ &\Big\{b'_iI_jI_k+b'_jI_iI_k+b'_kI_iI_j\Big\}.\end{aligned} \tag{3.24}$$

Since $b'_i=0$ and $a'_i=2ca_i$, then (3.24) reduces to

$$L_{ijk}=2cFC_{ijk}. \tag{3.25}$$

This completes the proof. □

Proof of Theorem 1.1: By Lemmas 3.2 and 3.3, we get the proof. □

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