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Study of Finsler manifolds with direction independence of the mean Landsberg tensor

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Abstract. Finsler manifolds some of whose characteristic tensors are direction independent provide stimulation for current research. In this paper, we show that the direction independence of the mean Landsberg tensor implies the vanishing of these tensor.

Keywords: Mean Landsberg curvature, Landsberg curvature, Berwald curvature, S-curvature.

1. **Introduction**

There are two version of curvatures in Finsler geometry, namely , Rieamannian and non-Riemannian curvatures. Among the non-Riemannian curvatures, the mean Landsberg curvature $\mathbf{J}$ has a important place. The mean Landsberg curvature $\mathbf{J}$ measure the rate of changes of the mean Cartan torsion $\mathbf{I}$ along the Finslerian geodesics. More precisely,

$$\mathbf{J} = D_0\mathbf{I},$$

where D_0 denotes the horizontal derivation along Finslerian geodesics. A Finsler metric F is said weakly Landsberg metric if it has vanishing mean

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Landsberg curvature $\mathbf{J} = 0$. This non-Riemannian curvature has been observed in many situations, including when working with the Gauss-Bonnet theorem in the Finslerian setting. Consider the Riemannian metric $\hat{\mathbf{g}}_x := g_{ij}(x,y)\delta y^i \otimes \delta y^j$ on T_xM_0, where $g_{ij} := 1/2[F^2]_{y^iy^j}$ and $\{\delta y^i := dy^i + N^i_j dx^j\}$ is the natural coframe on T_xM associated with the natural basis $\{\partial/\partial x^i|_x\}$ for T_xM. The constancy of the volume function of the unit tangent sphere $S_xM \subset (T_xM, \hat{\mathbf{g}}_x)$ is required to establish a Gauss-Bonnet theorem for Finsler manifolds. In [4], it is proved that the volume function is a constant for every weakly Landsberg metric.

Theorem 1.1. *Let (M, F) be a Finsler manifold. If the mean Landsberg curvature tensor of F depends only on the position, then F reduces to a weakly Landsberg metric.*

The class of two-dimensional Finsler spaces have many special features with respect to the higher dimensional Finsler spaces. For example, a well-known result explains that any two- dimensional Finsler space is of scalar flag curvature. In [6], Berwald made some important pioneering work for two-dimensional Finsler spaces. In this paper, we prove the following.

Corollary 1.2. *Every Finsler surface with the mean Landsberg curvature tensor depends only on the position is a Landsberg surface.*

A Randers metric $F = \alpha + \beta$ on a manifold M is just a Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ perturbated by a one form $\beta = b_i(x)y^i$ on M such that $\|\beta\|_\alpha < 1$. In the same time, another event was happened by a geometrician L. Berwald in connection with a two-dimensional Finsler space with rectilinear extremal and was investigated by V.K. Kropina [13]. Consequently, other match of Randers metric called Kropina metric $F = \alpha^2/\beta$ was born. By considering Kropina and Randers metrics, Matsumoto introduced the notion of (α, β)-metrics. An (α, β)-metric is a Finsler metric on M defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, α is a Riemannian metric and β is a 1-form on M [17]. A Finsler metric F on a manifold M is called a generalized Berwald metric if there exists a covariant derivative ∇ on M such that the parallel translations induced by ∇ preserve the Finsler function F. In this case, F is called a generalized Berwald metric on M and (M, F) is called a generalized Berwald manifold.

Corollary 1.3. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an non-Randers type generalized Berwald (α, β)-metric on a manifold M of dimension $n \geq 3$ such that $\phi'(0) \neq 0$. Suppose that the mean Landsberg curvature tensor of F depends only on the position. Then, F is a Berwald metric if and only if it $\mathbf{S} = 0$.*

For a Finsler manifold (M, F), the group of isometries $I(M, F)$ is a Lie transformation group of manifold M (see [10]). Homogeneous Finsler manifolds are those Finsler manifolds (M, F) that the orbit of the natural action of $I(M, F)$ on M at any point of M is the whole M. In this case, M is the quotient manifold $I(M, F)/H$, where H is the stablizer subgroup at a point in M. In this paper, we prove the following.

Corollary 1.4. *Every Homogeneous Finsler surface with the mean Landsberg curvature tensor depends only on the position is a Riemannian or a locally Minkowskian surface.*

2. **Preliminaries**

Let M be an n-dimensional C^∞ manifold, $TM = \bigcup_{x\in M} T_xM$ the tangent bundle and $TM_0 := TM - \{0\}$ the slit tangent bundle. A Finsler structure on M is a function $F : TM \to [0, \infty)$ with the following properties:
(i) F is C^∞ on TM_0;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, i.e., $F(x, \lambda y) = \lambda F(x, y)$, $\forall\lambda > 0$;
(iii) The following quadratic form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ is positively defined on TM_0

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(y + su + tv)\Big]_{s=t=0}, \quad u, v \in T_xM.$$

Then the pair (M, F) is called a Finsler manifold.

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, one can define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian.

For $y \in T_xM_0$, define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by

$$\mathbf{I}_y(u) := \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u, \partial_i, \partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in TM_0}$ is called the mean Cartan torsion. By definition, $\mathbf{I}_y(y) = 0$ and $\mathbf{I}_{\lambda y} = \lambda^{-1}\mathbf{I}_y$, $\lambda > 0$. Therefore, $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$.

For a vector $y \in T_xM_0$, define the Matsumoto torsion $\mathbf{M}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{M}_y(u, v, w) := \mathbf{C}_y(u, v, w) - \frac{1}{n+1}\Big\{\mathbf{I}_y(u)\mathbf{h}_y(v, w) + \mathbf{I}_y(v)\mathbf{h}_y(u, w) + \mathbf{I}_y(w)\mathbf{h}_y(u, v)\Big\}, \tag{2.1}$$

where $\mathbf{h}_y(u,v) := \mathbf{g}_y(u,v) - F^{-2}(y)\mathbf{g}_y(y,u)\mathbf{g}_y(y,v)$ is the angular metric. A Finsler metric F is said to be C-reducible if $\mathbf{M}_y = 0$. In [14], Matsumoto and Hōjō proved the following.

Lemma 2.1. *A Finsler metric F on a manifold of dimension $n \geq 3$ is a Randers metric or Kropina metric if and only if $\mathbf{M}_y = 0$ for all $y \in TM_0$.*

For a Finsler manifold (M,F), a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},$$

where $G^i = G^i(x,y)$ are local functions on TM given by

$$G^i := \frac{1}{4} g^{il} \Big\{ \frac{\partial^2[F^2]}{\partial x^k \partial y^l} y^k - \frac{\partial[F^2]}{\partial x^l} \Big\}, \quad y \in T_xM.$$

$\mathbf{G}$ is called the associated spray to (M,F).

Define $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{B}_y(u,v,w) := B^i{}_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i}|_x$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

$\mathbf{B}$ is called the Berwald curvature and F is called a Berwald metric if $\mathbf{B} = \mathbf{0}$.

For $y \in T_xM$, define the Landsberg curvature $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{L}_y(u,v,w) := -\frac{1}{2} \mathbf{g}_y\big(\mathbf{B}_y(u,v,w), y\big).$$

In local coordinates, $\mathbf{L}_y(u,v,w) := L_{\ ijk}(y) u^i v^j w^k$, where

$$L_{ijk} := -\frac{1}{2} y_l B^l{}_{ijk}.$$

$\mathbf{L}_y(u,v,w)$ is symmetric in u, v and w and $\mathbf{L}_y(y,v,w) = 0$. $\mathbf{L}$ is called the Landsberg curvature. A Finsler metric F is called a Landsberg metric if $\mathbf{L} = 0$. Also, the Landsberg curvature of F can be defined by following

$$\mathbf{L}_y(u,v,w) := \frac{d}{dt}\Big[\mathbf{C}_{\dot\sigma(t)}\big(U(t), V(t), W(t)\big)\Big]_{t=0},$$

where $y \in T_xM$, $\sigma = \sigma(t)$ is the geodesic with $\sigma(0) = x$, $\dot\sigma(0) = y$ and $U(t), V(t), W(t)$ are linearly parallel vector fields along σ with $U(0) = u, V(0) = v, W(0) = w$. Then the Landsberg curvature $\mathbf{L}_y$ is the rate of change of $\mathbf{C}_y$ along geodesics

For $y \in T_xM$, define $\mathbf{J}_y : T_xM \to \mathbb{R}$ by $\mathbf{J}_y(u) := J_i(y) u^i$, where

$$J_i := g^{jk} L_{ijk}.$$

By definition, $\mathbf{J}_y(y) = 0$. $\mathbf{J}$ is called the mean Landsberg curvature or J-curvature. A Finsler metric F is called a weakly Landsberg metric if $\mathbf{J}_y = 0$. By definition, every Landsberg metric is a weakly Landsberg metric. Mean Landsberg curvature can be defined as following

$$J_i := y^m \frac{\partial I_i}{\partial x^m} - I_m \frac{\partial G^m}{\partial y^i} - 2G^m \frac{\partial I_i}{\partial y^m}.$$

By definition, we get

$$\mathbf{J}_y(u) := \frac{d}{dt}\Big[\mathbf{I}_{\dot\sigma(t)}\big(U(t)\big)\Big]_{t=0},$$

where $y \in T_xM$, $\sigma = \sigma(t)$ is the geodesic with $\sigma(0) = x$, $\dot\sigma(0) = y$ and $U(t), V(t), W(t)$ are linearly parallel vector fields along σ with $U(0) = u, V(0) = v, W(0) = w$. Then the mean Landsberg curvature $\mathbf{J}_y$ is the rate of change of $\mathbf{I}_y$ along geodesics for any $y \in T_xM_0$. It has been shown that on a weakly Landsberg manifold, the volume function $Vol(x)$ is a constant [5].

There is an induced Riemannian metric of Sasaki type on TM_0. Aikou proved that if $\mathbf{L} = 0$, then all the slit tangent spaces T_xM_0 are totally geodesic in TM_0 [1]. In [16], Shen showed that if $\mathbf{J} = 0$, then all the slit tangent spaces T_xM_0 are minimal in TM_0.

Let $F = F(x, y)$ be a Finsler metric on an n-dimensional manifold M. The distortion $\tau = \tau(x, y)$ on TM associated with the Busemann-Hausdorff volume form $dV_{BH} = \sigma(x)dx$ is defined by

$$\tau(x, y) = \ln \frac{\sqrt{\det\big(g_{ij}(x, y)\big)}}{\sigma(x)}.$$

By definition, the distortion τ is homogeneous of degree 1 with respect to y, i.e., the following holds

$$\tau(\lambda y) = \lambda\tau(y), \qquad \lambda > 0,\ y \in T_xM_0.$$

A natural volume form $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ of a Finsler metric F on an n-dimensional manifold M is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n)}{\mathrm{Vol}\Big\{(y^i) \in \mathbb{R}^n \ \Big|\ F\big(y^i \frac{\partial}{\partial x^i}|_x\big) < 1\Big\}},$$

where

$$\mathbb{B}^n = \{y \in \mathbb{R}^n |\ |y| < 1\}.$$

The S-curvature is defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i}\Big[\ln \sigma_F(x)\Big],$$

where $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$.

3. Proof of Main Results

In this section, we are going to prove Theorem 1.1 and Corollaries 1.2 and 1.3. First, we prove Theorem 1.1 and then by using it, we prove the other results.

Proof of Theorem 1.1: With our condition

$$J_{i\cdot m} = 0 \tag{3.1}$$

We have

$$J_k = I_{k|l} y^l \tag{3.2}$$

So

$$0 = J_{k\cdot m} = I_{k|l\cdot m} y^k + I_{k|m} \tag{3.3}$$

Using the Ricci identity, we get

$$I_{k|l\cdot m} = I_{k\cdot m|l} - I_r B^r_{kml}. \tag{3.4}$$

Contracting this with y^l leads to

$$I_{k|l\cdot m} y^l = I_{k\cdot m|l} y^l \tag{3.5}$$

Then

$$I_{k\cdot m|l} y^l + I_{k|m} = 0 \tag{3.6}$$

With interchanging indices m and k, we obtain

$$I_{m\cdot k|l} y^l + I_{m|k} = 0 \tag{3.7}$$

The following holds

$$\begin{aligned}\frac{\partial \tau}{\partial y^i} =& \frac{\partial}{\partial y^i}\left[\ln \frac{\sqrt{\det\left(g_{ij}(x,y)\right)}}{\sigma(x)}\right]\\ =& \frac{1}{2} g^{ij} \frac{\partial g_{ij}}{\partial y^k}\\ =& g^{ij} C_{ijk}\\ =& I_k.\end{aligned}$$

Thus

$$I_{k\cdot m} = I_{m\cdot k}. \tag{3.8}$$

If we subtract (0.7) from (0.6) then

$$I_{k|m} - I_{m|k} = 0 \tag{3.9}$$

By contracting this with y^k we find

$$I_{m|k} y^k = 0. \tag{3.10}$$

This means F is a weakly Landsberg metric. □

Proof of Corollary 1.2: Every Finsler surface is C-reducible

$$C_{ijk} = \frac{1}{3}\Big\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\Big\}. \tag{3.11}$$

Taking a horizontal derivation of (3.11) along Finslerian geodesics implies that

$$L_{ijk} = \frac{1}{3}\Big\{h_{ij}J_k + h_{jk}J_i + h_{ki}J_j\Big\}. \tag{3.12}$$

By Theorem 1.1, we have $\mathbf{J} = 0$. Then by (3.12), it follows that F is a Landsberg metric. □

For an (α,β)-metric, let us put

$$Q := \frac{\phi'}{\phi - s\phi'}, \quad \Delta := 1 + sQ + (b^2 - s^2)Q'.$$

Now, let $\phi = \phi(s)$ be a positive C^∞ function on $(-b_0, b_0)$. For a number $b \in [0, b_0)$, let

$$\Phi := -(Q - sQ')(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q''.$$

In [7], Cheng-Shen characterized (α,β)-metrics with isotropic S-curvature on a manifold M of dimension $n \geq 3$. Soon, they found that their result holds for the class of (α,β)-metrics with constant length one-forms, only [8]. In [20], Vincze showed that an (α,β)-metric satisfying $\phi'(0) \neq 0$ is a generalized Berwald manifold if and only if β has constant length with respect to α. Then, we conclude the following.

Lemma 3.1. *Let $F = \alpha\phi(\beta/\alpha)$ be an non-Randers type generalized Berwald (α,β)-metric on a manifold M of dimension $n \geq 3$ such that $\phi'(0) \neq 0$. Then, F is of isotropic S-curvature $\mathbf{S} = (n+1)cF$, if and only if one of the following holds:*

(i) *β satisfies*

$$r_{ij} = \epsilon\Big\{b^2 a_{ij} - b_i b_j\Big\}, \qquad s_j = 0, \tag{3.13}$$

where $\epsilon = \epsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2}, \tag{3.14}$$

where k is a constant. In this case, $\mathbf{S} = (n+1)k\epsilon F$.

(ii) *β satisfies*

$$r_{ij} = 0, \qquad s_j = 0. \tag{3.15}$$

In this case, $\mathbf{S} = 0$.

Proof of Corollary 1.3: By assumption, F has vanishing S-curvature, $\mathbf{S} = 0$. By Lemma 3.1, we have

$$r_{ij} = 0, \qquad s_j = 0. \tag{3.16}$$

On the other hand, by Theorem 1.1, F has vanishing mean Landsberg curvature. By Corollary 3.3 in [9], β is a closed 1-form, namely

$$s_{ij} = 0. \tag{3.17}$$

By (3.16) and (3.17), we find that β is a parallel 1-form with respect to the Riemannian metric α. In this case, F is a Berwald metric. The converse is trivial according to [19]. □

Proof of Corollary 1.4: By Corollary 1.2, F is a Landsberg metric. In [18], Tayebi-Najafi considered the homogeneous Landsberg surfaces and prove that any homogeneous Landsberg surface is Riemannian or locally Minkowskian. This completes the proof. □

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