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Analysis of generalized quasilinear hyperbolic and Boussinesq equations from the point of view of potential symmetry

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Abstract. Using the Lie classical method, the potential symmetry of the generalized hyperbolic quasilinear and Boussinesq equations is investigated. To find these symmetries in specific cases, we study various scientific examples that admit these symmetries. In addition, using this method, the potential symmetries of the conservative forms of the Boussinesq equation is determined.

Keywords: Pseudo-Riemannian space, Boussinesq equation, Potential symmetry, Generalized quasilinear hyperbolic equation, Potential equations, Conservative form.

1. **Introduction**

In various applied sciences, including engineering sciences, mathematical physics, quantum and particle physics, physical chemistry, etc., conversation laws are examined for a vast domain of nonlinear partial differential equations (PDEs) [15, 13]. Generally, the principal laws in physics that express that

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discrete quantities of an isolated system stay stable over time are called conservation laws.

The Lie symmetry method, known as the classical Lie method, is a basic method in this field. By this method, we can reduce the order of ordinary differential equations and also reduce PDEs and convert them to ODEs in certain cases [7, 21, 22]. In recent decades, due to the widespread applications of science, Lie classical groups, especially potential symmetries, have been considered by researchers [8, 23]. Furthermore, relation between a conservation law and symmetries perform a significant work in the analysis of qualitative attributes of the solutions [6, 19, 24, 25, 26]. Undoubtedly, the history of the potential symmetries go back to Bluman et al., in 1988 [3]. The potential procedure is used to earn an extensive range of symmetries of PDE, that is rewritten as a conservative form. Scientists studied the Lie symmetries of the potential system, which is created by adding potential variables as extra unfamiliar functions to the equation. By calculation the Lie point groups of transformation, that operate on the various spaces of the dependent and independent variables and their derivatives of the system, an other set of symmetries, so called potential symmetries, are obtained. These symmetries are different from point and Lie-Bäcklund symmetries [11]. Approximate symmetries are another type of symmetries that are used to obtain solutions for equations that have a small parameter [9, 10]. Invariant solutions for potential symmetries lead to finding more solutions for the PDE under study. Any Lie group results in potential symmetries, provided that at least one of the generators clearly depends on the potential variables. In searching for new solutions with the reduction method, these symmetries will be useful [4, 18, 16, 20, 24].

This study is dedicated to the potential symmetry analysis of generalized quasilinear hyperbolic equations, which is another type of second-order wave equations,

$$r(x)u_{tt} = [s(x,u)u_t + k(x,u)]_x. \tag{1.1}$$

Here $r \in \mathcal{C}^1(\mathbb{R})$ and $s, k \in \mathcal{C}^1(\mathbb{R}^2)$ are nonzero functions as $s_x = k_x = 0$. In [20], an analysis of potential symmetries has been obtained for a type of this equation.

In continuation, the Boussinesq equation is studied. This equation is applied to various physical phenomena, including diffusion long waves in shallow water [5]. The main form of this equation is

$$u_{tt} - u_{2x} + 3uu_{2x} + 3u_x^2 + \alpha u_{4x} = 0 \tag{1.2}$$

where α is real parameter and $u(t,x)$ is an arbitrary functional and t, x are time and space variables respectively. Considerable interest in the Boussinesq equation in the last few decades has led to the construction and study of many solutions and developments of this equation. For instance, using symbolic computation method, rogue wave solutions, three types of breather solutions and analytical N-soliton solutions have been obtained by Ma for generalized nonlinear Boussinesq equation [1, 14]. In [17], a complete study has been done on different types of this equation. In the second part of this study the potential symmetries of the conservation laws for this equation, will be obtained by the multiplier method.

The outline of this study is as follows. Section 2, is dedicated to recalling the principal definitions about potential symmetry. In Section 3, these symmetries of equation (1.1) for three scientific items are investigated. Finally, the potential symmetries of the conservative forms of the Boussinesq equation are analyzed in section 4.

2. Preliminaries

Assume $\mathbf{G}$ is a n-th order system of a PDE with independent variables (t, x) such that $x = (x^1, \cdots, x^p)$ and dependent variables $u = (u^1, \cdots, u^q)$ as

$$\mathbf{G}(t, x, u, u^{(1)}, \cdots, u^{(n)}) = 0. \tag{2.1}$$

Here we denote the n-th order partial derivative of the function u by $u^{(n)}$. To calculate the symmetries of equation (2.1), it is sufficient to rewrite it in the conservative form:

$$D_t\rho[u] + D_x\psi[u] = 0, \tag{2.2}$$

where

$$\rho[u] = \rho(t, x, u, u^{(1)}, \cdots, u^{(n-1)})$$

and

$$\psi[u] = \psi(t, x, u, u^{(1)}, \cdots, u^{(n-1)}).$$

Assuming the system $\mathbf{G}\{t, x; u\}$ can be explicitly rewritten in the conservative form (2.2), a potential variable $v(t, x)$ can be introduced as an additional unfamiliar function. The pair of potential equations $\mathbf{p}$ that derived from (2.2) are defined as follows

$$v_x = \rho[u],\ v_t = -\psi[u]. \tag{2.3}$$

With community of system $\mathbf{G}\{t, x; u\}$ and potential equations $\mathbf{p}$, the potential system $\mathbf{H}\{t, x; u, v\}$ are constructed. Indeed, the solution set of potential system $\mathbf{H}\{t, x; u, v\}$ and $\mathbf{G}\{t, x; u\}$ are equal. The infinitesimal symmetries are obtained from the following equations,

$$\mathbf{X}^n(v_x - \rho[u])|_{\mathbf{G}} = 0,\ \mathbf{X}^n(v_t + \psi[u])|_{\mathbf{G}} = 0. \tag{2.4}$$

By replacing v_x with $\rho[u]$, and v_t by $-\psi[u]$ in equations (2.4), the determining equations are created. The potential symmetries of (2.1) are resulted by solving this equations. With the generators ξ, τ, η and φ, the Lie symmetry for (2.3) is clearly defined. If the relation $\xi_v^2 + \tau_v^2 + \eta_v^2 = 0$ holds, point symmetries are obtained and if $\xi_v^2 + \tau_v^2 + \eta_v^2 > 0$ then potential symmetries are obtained for the equation (2.1). This symmetries give new solutions of (2.3). Finally, another solutions of (2.1) are induced.

3. Potential symmetry of the Hyperbolic equation

After rewriting equation (1.1) in conservative form, we get

$$D_t(r(x)u_t) - D_x(s(x,u)u_t + k(x,u)) = 0. \tag{3.1}$$

Substituting a potential variable $v(t,x)$ in (3.1), we have

$$v_x = r(x)u_t, \;\; v_t = s(x,u)u_t + k(x,u). \tag{3.2}$$

After solving characterize system,

$$\mathbf{X}^1(v_x - ru_t)|_{(7)} = 0, \;\; \mathbf{X}^1(v_t - su_t - k)|_{(7)} = 0, \tag{3.3}$$

the generators are obtained. $\mathbf{X}^1$ is defined as

$$\mathbf{X}^1 = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \varphi\frac{\partial}{\partial u} + \eta\frac{\partial}{\partial v} + \varphi_1^{(1)}\frac{\partial}{\partial u_x} + \varphi_2^{(1)}\frac{\partial}{\partial u_t} + \eta_1^{(1)}\frac{\partial}{\partial v_x} + \eta_2^{(1)}\frac{\partial}{\partial v_t}. \tag{3.4}$$

Where

$$\begin{cases} \varphi_1^{(1)} = \varphi_x + (\varphi_u - \xi_x)u_x - \tau_x u_t - \tau_u u_x u_t - \xi_u u_x^2 + \varphi_v v_x - \tau_v u_t v_x - \xi_v u_x v_x, \\ \varphi_2^{(1)} = \varphi_t + (\varphi_u - \tau_t)u_t - \tau_u u_t^2 - \xi_t u_x - \xi_u u_t u_x + \varphi_v v_t - \tau_v u_t v_t - \xi_v u_x v_t, \\ \eta_1^{(1)} = \eta_x + (\eta_v - \xi_x)v_x - \xi_v v_x^2 + \eta_u u_x - \tau_x v_t - \tau_u u_x v_t - \tau_v v_x v_t - \xi_u u_x v_x, \\ \eta_2^{(1)} = \eta_t + (\eta_v - \tau_t)v_t - \tau_v v_t^2 + \eta_u u_t - \tau_u u_t v_t - \xi_t v_x - \xi_u u_t v_x - \xi_v v_x v_t. \end{cases}$$

Therefore (3.3) becomes

$$\begin{cases} -\Big[(\varphi_t + \varphi_u u_t + \varphi_v)v_t - (\xi_t + \xi_u u_t + \xi_v v_t)u_x - (\tau_t + \tau_u u_t + \tau_v v_t)u_t\Big]f + \\ \quad \eta_x + \eta_u u_x + \eta_v v_x - \Big[\xi_x + \xi_u u_x + \xi_v v_x\Big]v_x - \Big[\tau_x + \tau_u u_x + \tau_v v_x\Big]v_t = 0, \\ -\Big[(\varphi_t + \varphi_u u_t + \varphi_v)v_t - (\xi_t + \xi_u u_t + \xi_v v_t)u_x - (\tau_t + \tau_u u_t + \tau_v v_t)u_t\Big]g + \\ \quad \eta_t + \eta_u u_t + \eta_v v_t - \Big[\xi_t + \xi_u u_t + \xi_v v_t\Big]v_x - \Big[\tau_t + \tau_u u_t + \tau_v v_t\Big]v_t = 0. \end{cases} \tag{3.5}$$

By replacing v_x with $r(x)u_t$, and v_t with $s(x,u)u_t + k(x,u)$ in (3.5), determinant equations are obtained.

In the following, we going to verify physical cases of $r(x)$, $s(x,u)$ and $k(x,u)$, which admits potential symmetries by calculating the point symmetries of them. These cases are very important in physics and mathematics. They are especially useful in investigating space-time metrics on pseudo-Riemannian spaces [2, 27, 28].

Case 1: $s(x,u) = u, r(x) = b$ and $k(x,u) = e^{\lambda u}$, here b and λ are changeless. The infinitesimal group are computed as:

$$\begin{cases} \xi = c_1 x + c_2, \\ \tau = c_3 t + c_4 v, \\ \varphi = c_5 x + c_6, \\ \eta = (c_1 - c_3)v. \end{cases} \tag{3.6}$$

Then, symmetries are obtained by the following generators:

$$\begin{aligned} \mathbf{X}_1 &= \frac{\partial}{\partial x}, \\ \mathbf{X}_2 &= v\frac{\partial}{\partial t}, \\ \mathbf{X}_3 &= x\frac{\partial}{\partial u}, \\ \mathbf{X}_4 &= \frac{\partial}{\partial u}, \\ \mathbf{X}_5 &= t\frac{\partial}{\partial t} - v\frac{\partial}{\partial v}, \\ \mathbf{X}_6 &= x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v}. \end{aligned} \tag{3.7}$$

Undoubtedly, $\mathbf{X}_2$ becomes the only potential symmetry of equation (1.1), because this symmetry satisfies the condition : $\xi_v^2 + \tau_v^2 + \varphi_v^2 = 1 > 0$.
Case 2: $s(x,u) = u, r(x) = x$ and $k(x,u) = u^2$. Hence, infinitesimal symmetries are determined as,

$$\begin{cases} \xi = c_1 x + c_2 v, \\ \tau = c_2 x u + c_3 v, \\ \varphi = \dfrac{-c_2 u}{x} v + c_1 u + c_4, \\ \eta = c_1 v + c_5. \end{cases} \tag{3.8}$$

Therefore, point symmetries are provided with these vector fields

$$\begin{aligned} \mathbf{X}_1 &= \frac{\partial}{\partial v}, \\ \mathbf{X}_2 &= v\frac{\partial}{\partial t}, \\ \mathbf{X}_3 &= \frac{\partial}{\partial u}, \\ \mathbf{X}_4 &= x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + u\frac{\partial}{\partial u}, \\ \mathbf{X}_5 &= v\frac{\partial}{\partial x} + xu\frac{\partial}{\partial t} - \frac{uv}{x}\frac{\partial}{\partial u}. \end{aligned} \tag{3.9}$$

Certainly, $\mathbf{X}_2$ and $\mathbf{X}_5$ are potential symmetries for equation (1.1). Because, respectively, we have

$$\xi_v^2 + \tau_v^2 + \varphi_v^2 = 1 > 0, \quad \xi_v^2 + \tau_v^2 + \varphi_v^2 = 1 + \frac{u^2}{x^2} > 0,$$

Case 3: $s(x,u)=u, r(x)=x$ and $k(x,u)=u$. The infinitesimal symmetries are defined as,

$$\begin{cases} \xi = c_4x + c_5v, \\ \tau = c_5xu + c_6, \\ \varphi = \dfrac{-c_5uv}{x} + \Big[(2t+u)c_1 - c_2\Big]v - 2c_1v + \Big[-c_1u^2 + (-2c_1t + c_2)u \\ \qquad +2c_1v\Big]t + x^2c_1u - \dfrac{1}{3}c_1u^3 + \dfrac{1}{2}c_2u^2 + c_3u, \\ \eta = \dfrac{c_1}{2}v^2 + \Big[(x^2-t^2)c_1 + c_2t + c_3\Big]v. \end{cases}$$

with following generators

$$\begin{aligned} &\mathbf{X}_1 = x\frac{\partial}{\partial x}, \\ &\mathbf{X}_2 = \frac{\partial}{\partial t} \\ &\mathbf{X}_3 = xu\frac{\partial}{\partial t} - \frac{uv}{x}\frac{\partial}{\partial u} + v\frac{\partial}{\partial x}, \\ &\mathbf{X}_4 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \\ &\mathbf{X}_5 = \Big(-v + ut + \frac{1}{2}u^2\Big)\frac{\partial}{\partial u} + tv\frac{\partial}{\partial v}, \\ &\mathbf{X}_6 = \Big[(2t+u)v - 2v + (-u^2 - 2tu + 2v)t + x^2u - \frac{1}{3}u^3\Big]\frac{\partial}{\partial u} + \Big(\frac{1}{2}v^2 + (x^2-t^2)v\Big)\frac{\partial}{\partial v}. \end{aligned}$$

In this case, $\mathbf{X}_3$, $\mathbf{X}_5$ and $\mathbf{X}_6$ are the potential symmetries for equation (1.1). Since, we have:

$$\begin{cases} \xi_v^2 + \tau_v^2 + \varphi_v^2 = 1 + \dfrac{u^2}{x^2} > 0, \\ \xi_v^2 + \tau_v^2 + \varphi_v^2 = 1 > 0, \\ \xi_v^2 + \tau_v^2 + \varphi_v^2 = (4t+u-2)^2 > 0. \end{cases} \tag{3.10}$$

4. potential Symmetry Analysis of the Boussinesq Equation

A set of conservation laws for the Boussinesq equation is obtained with the following forms [12],

$$\begin{aligned} &D_t(u_t) + D_x(3uu_x - u_x + \alpha u_{3x}) = 0, \\ &D_t(xu_t) + D_x\Big(3xuu_x + u - \frac{3}{2}u^2 - xu_x - \alpha u_{2x} + \alpha xu_{3x}\Big) = 0, \\ &D_t(tu_t - u) + D_x(3tuu_x - tu_x + \alpha tu_{3x}) = 0, \\ &D_t(-xu + xtu_t) + D_x\Big(3xtuu_x + tu - \frac{3}{2}tu^2 - xtu_x - 3t\alpha u_{2x} + 3\alpha txu_{3x}\Big) = 0. \end{aligned} \tag{4.1}$$

In the following, we going to verify the potential symmetries of the Boussinesq equation, which is rewritten in a form (4.1), by calculating the point symmetries of them.

Case 1: By putting the equation (1.2) in the conservative form, it's potential symmetries are calculated

$$D_t(u_t) + D_x(3uu_x - u_x + \alpha u_{3x}) = 0. \tag{4.2}$$

From the equation (4.2), the infinitesimal symmetries are resulted as,

$$\begin{aligned}
\xi &= \frac{1}{5}(c_1 x + 5c_5)t + c_4 x + c_6, \\
\tau &= \frac{1}{2}c_1 t^2 + c_2 t + c_3, \\
\eta &= -\frac{2}{5}(c_1 t + 5c_4)(u - \frac{1}{3}), \\
\varphi &= F_1(x) + \frac{1}{5}(c_1 x + 5c_5)u + \frac{1}{5}(5c_2 - 25c_4)v.
\end{aligned}$$

Then, symmetries are obtained by the following generators:

$$\begin{aligned}
&X_1 = \partial_t, \\
&X_2 = \partial_x, \\
&X_3 = x\partial_x - 2u\partial_u - 5v\partial_v, \\
&X_4 = t\partial_x + u\partial_v, \\
&X_5 = t\partial_t + v\partial_v.
\end{aligned} \tag{4.3}$$

Clearly, nothing of point symmetries are a potential symmetry for (1.2), because we have

$$\xi_v^2 + \tau_v^2 + \eta_v^2 = 0.$$

Case 2: Consider another conservative form as follows,

$$D_t(xu_t) + D_x\Big(3xuu_x + u - \frac{3}{2}u^2 - xu_x - \alpha u_{2x} + \alpha x u_{3x}\Big) = 0. \tag{4.4}$$

The following infinitesimals are derived from the equation (4.4),

$$\begin{aligned}
\xi &= \frac{1}{4}(c_1 t + 4c_4)x, \\
\tau &= \frac{1}{2}c_1 t^2 + c_2 t + c_3, \\
\eta &= -\frac{1}{2}(c_1 t + 4c_4)(u - \frac{1}{3}), \\
\varphi &= F_1(x) + \frac{1}{12}(12c_2 - 48c_4)v + \frac{1}{12}(3x^2 u - t^2)c_1 - \frac{2}{3}c_4 t.
\end{aligned}$$

Thus, point symmetries are provided as

$$\begin{aligned}
&X_1 = \partial_t \\
&X_2 = \partial_u, \\
&X_3 = x\partial_x - 2u\partial_u - 4v\partial_v - \frac{2}{3}t\partial_v, \\
&X_4 = t\partial_t + v\partial_v.
\end{aligned} \tag{4.5}$$

Also, none of them are potential symmetries. Since we have

$$\xi_v^2 + \tau_v^2 + \eta_v^2 = 0.$$

Case 3: Suppose the conservative form is as follows

$$D_t(-u + tu_t) + D_x(3tuu_x - tu_x + \alpha tu_{3x}) = 0. \tag{4.6}$$

Equation (4.6) admits the following infinitesimals,

$$\begin{aligned}
\xi &= (\frac{1}{5}\frac{c_1}{t} + c_4)x + c_5 + \frac{c_6}{t},\\
\tau &= \frac{c_2}{t} + c_1 + c_3 t,\\
\eta &= -\frac{2}{15}\frac{(5c_4 t + c_1)(3u-1)}{t},\\
\varphi &= F_1(x) - \frac{1}{5}\frac{(c_1 x + 5c_6)(u)}{t}) + (2c_3 -, 5c_4)v,
\end{aligned}$$

with following generators,

$$\begin{aligned}
X_1 &= \partial_x,\\
X_2 &= x\partial_x + 2(1-u)\partial_u - 5v\partial_v,\\
X_3 &= t\partial_t + 2v\partial_v.
\end{aligned}$$

Clearly, no potential symmetry is obtained, because $\xi_v^2 + \tau_v^2 + \eta_v^2 = 0$.

Case 4: We turn to the last conservative form of (4.1)

$$D_t(-xu + xtu_t) + D_x(3xtuu_x + tu - \frac{3}{2}tu^2 - xtu_x - 3t\alpha u_{2x} + 3\alpha txu_{3x}) = 0. \tag{4.7}$$

Then, the following infinitesimals are concluded,

$$\begin{aligned}
\xi &= \frac{1}{4}\frac{xc_1}{t} + c_4 x,\\
\tau &= \frac{c_2}{t} + c_1 + c_3 t,\\
\eta &= -\frac{1}{6}\frac{(4c_4 t + c_1)(3u-1)}{t}),\\
\varphi &= F_1(x) - \frac{1}{4}\frac{x^2 c_1 u}{t} - \frac{1}{3}t^2 c_4 - \frac{1}{6}c_1 t + (2c_3 - 2c_4)v.
\end{aligned}$$

As a result, point symmetry is obtained

$$\begin{aligned}
X_1 &= 3x\partial_x - 6(3u-1)\partial_u - (6v + t^2)\partial_v,\\
X_2 &= t\partial_t + 2v\partial_v.
\end{aligned}$$

Since all point symmetries satisfy condition $\xi_v^2 + \tau_v^2 + \eta_v^2 = 0$, no potential symmetry is achieved. It should be noted that the obtained conservation laws, besides being new, are non-equivalent and non-trivial. The reason for this is that the effect of the Euler operator on them is non-zero and also the effect of the Euler operator on the difference of their two primitives is also non-zero.

5. Conclusions

The present study is devoted to investigating the potential symmetry of generalized quasilinear hyperbolic and Boussinesq equations. The infinitesimals and potential symmetries for real scientific items $r(x)$, $s(x,u)$ and $k(x,u)$ of generalized quasilinear hyperbolic equation are achieved. Then, by calculating the point symmetries of conservative forms of the Boussinesq equation, we conclude that the equation under study has no potential symmetry.

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