Journal of Finsler Geometry and its Applications
Vol. 4, No. 1 (2023), pp 88-101
DOI: 10.22098/jfga.2023.12908.1089

Diverse Forms of Generalized Birecurrent Finsler Space

Alaa A. Abdallah[a], Ahmed A. Hamoud[b], A. Navlekar[c], Kirtiwant Ghadle[d], Basel Hardan[e], Homan Emadifar[f]*

[a]Department of Mathematics, Abyan University, Abyan, Yemen.
E-mail: maths.aab@bamu.ac.in
[b]Department of Mathematics, Taiz University, Taiz P.O. Box 6803, Yemen
E-mail: ahmed.hamoud@taiz.edu.ye
[c]Department of Mathematics, Pratishthan Mahavidyalaya, Paithan, India.
E-mail: dr.navlekar@gmail.com
[d]Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, India.
E-mail: ghadle.maths@bamu.ac.in
[e]Department of Mathematics, Abyan University, Abyan, Yemen.
E-mail: bassil2003@gmail.com
[f]Department of Mathematics, Hamedan Branch, Islamic Azad University, Hamedan, Iran.
E-mail: homan_emadi@yahoo.com

Abstract. The generalized birecurrent Finsler space have been introduced by the Finslerian geometers. The purpose of the present paper is to study three special forms of P^i_{jkh} in generalized $\mathfrak{B}P-$birecurrent space. We use the properties of $P2-$like space, P^*-space and $P-$reducible space in the main space to get new spaces that will be called a $P2-$like generalized $\mathfrak{B}P-$birecurrent space, P^*-generalized $\mathfrak{B}P-$birecurrent space and $P-$reducible generalized $\mathfrak{B}P-$birecurrent space, respectively. In addition, we prove that the Cartan's first curvature tensor S^i_{jkh} satisfies the birecurrence property. Certain identities belong to these spaces have been obtained. Further, we end up this paper with some demonstrative examples.

*Corresponding Author
AMS 2020 Mathematics Subject Classification: 53C42, 53C60.

Keywords: Cartan's first curvature tensor S^i_{jkh}, $P2-$like space, P^*-space, $P-$reducible space.

1. Introduction

Various special forms of $h(hv)-$curvature tensor P^i_{jkh} and $v(hv)-$torsion tensor P^i_{jk} which are called $P2-$like space, P^*-space and $P-$reducible space have been studied by scientists of Finsler geometry. A review of literature for some special Finsler spaces introduced by Dubey [9]. Tripathi and Pandey [23] discussed a special form of $h(hv)-$torsion tensor P_{ijk} in different Finsler spaces. Wosoughi [24] introduced a new special form in Finsler space and obtained the condition for Finsler space to be a Landsberg space. Furthermore, Narasimhamurthy et al. [2, 16] studied hypersurfaces of special Finsler spaces.

The properties of $P2-$like space, P^*-space and $P-$reducible space in the generalized $\mathfrak{B}P-$recurrent space have been discussed by [2, 4]. Also, Alaa et al. [3] introduced $P2-$like$-\mathfrak{B}C - RF_n$, $P^* - \mathfrak{B}C - RF_n$ and $P-$reducible $-\mathfrak{B}C - RF_n$.

Qasem and Hadi [19] and Assallal [7] studied the properties of $P2-$like space and P^*-space in generalized $\mathfrak{B}R-$birecurrent space and generalized P^h- birecurrent space, respectively. Otman [18] introduced the $P2-$like $-P^h-$birecurrent space and $P^* - P^h-$birecurrent space.

Dwivedi [10] obtained every $C-$reducible Finsler space is $P-$reducible and converse is not necessarily true. Zamanzadeh et al. [25] introduced a generalized $P-$reducible Finsler manifolds. In this paper, we merge the generalized $\mathfrak{B}P-$birecurrent space with special spaces in Finser space to get new spaces contain the same properties of the main space.

2. Preliminaries

In this section, some preliminary concepts which are necessary for the discussion of the following sections. An $n-$dimensional space X_n equipped with a function $F(x, y)$ which denoted by $F_n = (X_n, F(x, y))$ called a Finsler space if the function $F(x, y)$ satisfying the request conditions [1, 2, 6, 8, 17, 22].

The covariant vector y_i is defined by

$$y_i = g_{ij}(x, y)y^j \tag{2.1}$$

where the metric tensor $g_{ij}(x, y)$ is positively homogeneous of degree zero in y^i and symmetric in its indices which is defined by

$$g_{ij}\left(x, y\right) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2\left(x, y\right).$$

The metric tensor g_{ij} and its associative g^{ij} are related by

$$g_{ij}g^{ik} = \delta^k_j = \begin{cases} 1 \text{ if } j = k, \\ 0 \text{ if } j \neq k. \end{cases} \tag{2.2}$$

In view of (2.1) and (2.2), we have

$$a)\ \delta^i_j g_{ir} = g_{jr}, \quad b)\ \delta^i_j y_i = y_j \quad \text{and} \quad c)\ \delta^i_j y^j = y^i. \tag{2.3}$$

Matsumoto [14] introduced the $(h)hv-$torsion tensor C_{ijk} that is positively homogeneous of degree -1 in y^i and defined by

$$C_{ijk} = \frac{1}{2}\dot{\partial}_i\ g_{jk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k\ F^2.$$

This tensor satisfies the following

$$\begin{gathered} a)\ C^i_{jk} y_i = 0, \quad b)\ C^h_{ik} = g^{hj} C_{ijk}, \quad c)\ C^i_{ri} = C_r, \quad d)\ C_{ijk} = g_{hj} C^h_{ik}, \\ e)\ \delta^i_j C_{ikl} = C_{jkl}, \quad f)\ \delta^i_j C^j_{kh} = C^i_{kh} \quad \text{and} \quad g)\ C_{ijk} y^i = C_{kij} y^i = C_{jki} y^i = 0, \end{gathered} \tag{2.4}$$

where C^i_{jk} is called associate tensor of the $(h)hv-$torsion tensor C_{ijk}.

The unit vector l^i and associate vector l_i with the direction of y^i are given by

$$a)\ l^i = \frac{y^i}{F} \text{ and } b)\ l_i = \frac{y_i}{F}. \tag{2.5}$$

Cartan $h-$covariant differentiation with respect to x^k is given by [20]

$$X^i_{|k} = \partial_k X^i - (\dot{\partial}_r x^i) G^r_k + X^r \Gamma^{*i}_{rk}.$$

The $h-$covariant derivative of the vector y^i and associate metric tensor g^{ij} are vanish identically i.e.

$$a)\ y^i_{|k} = 0, \quad \text{and} \quad b)\ g^{ij}_{|k} = 0. \tag{2.6}$$

Berwald covariant derivative $\mathfrak{B}_k T^i_j$ of an arbitrary tensor field T^i_j with respect to x^k is given by [20]

$$\mathfrak{B}_k T^i_j = \partial_k T^i_j - (\dot{\partial}_r T^i_j) G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}.$$

Berwald covariant derivative of the vector y^i vanish identically i.e.

$$\mathfrak{B}_k y^i = 0. \tag{2.7}$$

The tensor P^i_{jkh} is called $hv-$curvature tensor (Cartan's second curvature tensor) which is positively homogeneous of degree -1 in y^i and defined by

$$P^i_{jkh} = \dot{\partial}_h \Gamma^{*i}_{jk} + C^i_{jr} P^r_{kh} - C^i_{jh|k}$$

and satisfies the relation

$$P^i_{jkh} y^j = \Gamma^{*i}_{jkh} y^j = P^i_{kh} = C^i_{kh|r} y^r, \tag{2.8}$$

where P^i_{kh} is called the $(v)hv-$torsion tensor. This tensor and its associative tensor P_{rkh} are related by

$$P^i_{kh} = g^{ir} P_{rkh}. \tag{2.9}$$

The associate tensor P_{ijkh} is given by

$$P^r_{jkh} = g^{ir} P_{ijkh}. \tag{2.10}$$

The $P-$Ricci tensor P_{jk}, curvature vector P_k and curvature scalar P are given by

$$a)\ P_{jk} = P^i_{jki},\ b)\ P_k = P^i_{ki} \text{ and } c)\ P = P_k y^k \tag{2.11}$$

respectively. Cartans second curvature tensor P^i_{jkh} satisfies the identity

$$P^i_{jkh} - P^i_{jhk} = -S^i_{jkh|r} y^r,$$

where S^i_{jkh} is called $v-$curvature tensor (Cartan's first curvature tensor) which is defined by [20]

$$S^i_{jkh} = C^i_{rk} C^r_{jh} - C^i_{rh} C^r_{jk}. \tag{2.12}$$

The associate curvature tensor S_{pjkh} of $v-$curvature tensor S^i_{jkh} is given by

$$S_{pjkh} = g_{ip} S^i_{jkh}. \tag{2.13}$$

In contracting the indices i and h in (2.12), we get

$$S^i_{jki} = S_{jk} = C^s_{rk} C^r_{js} - C_r C^r_{jk}. \tag{2.14}$$

Definition 2.1. *A Finsler space F_n is called a $P2-$like space if the Cartan's secend curvature tensor P^i_{jkh} is characterized by the condition* [15]

$$P^i_{jkh} = \varphi_j C^i_{kh} - \varphi^i C_{jkh}, \tag{2.15}$$

where φ_j and φ^i are non - zero covariant and contravariant vectors field, respectively.

Definition 2.2. *A Finsler space F_n is called a P^*-Finsler space if the $(v)hv$-torsion tensor P^i_{kh} is characterized by the condition* [13]

$$P^i_{kh} = \varphi C^i_{kh},\ \ \varphi \neq 0, \tag{2.16}$$

where $P^i_{jkh} y^j = P^i_{kh} = C^i_{kh|s} y^s$.

Definition 2.3. *A Finsler space F_n is called a $P-$reducible space if the associate tensor P_{jkh} of $(v)hv-$torsion tensor P^i_{kh} is characterized by one of the following conditions* [10, 21]

$$P_{jkh} = \lambda C_{jkh} + \varphi\Big(h_{jk} C_h + h_{kh} C_j + h_{hj} C_k\Big), \tag{2.17}$$

where λ and φ are scalar vectors positively homogeneous of degree one in y^j and h_{jk} is the angular metric tensor.

$$P_{jkh} = \frac{1}{(n+1)}\Big(h_{jk} P_h + h_{kh} P_j + h_{hj} P_k\Big), \tag{2.18}$$

where $P_{jkh} = C_{jkh|m} y^m$, $P^i_{ik} = P_k$ and $h_{ij} = g_{ij} - l_i l_j$.

Definition 2.4. *Let the current coordinates in the tangent space at the point x_0 be x^i, then the indicatrix I_{n-1} is a hypersurface defined by $F(x_0, x^i) = 1$ or by the parametric form defined by $x^i = x^i\left(u^a\right), a = 1, 2,\ \ldots,\ n-1$.*

The projection of any tensor T^i_j on indicatrix I_{n-1} is given by [11]

$$p.T^i_j = T^a_b h^i_a h^b_j, \tag{2.19}$$

where

$$h^i_c = \delta^i_c - l^i l_c. \tag{2.20}$$

Then, the projection of the vector y^i, unit vector l^i and metric tensor g_{ij} on the indicatrix are given by $p.y^i = 0$, $p.l^i = 0$ and $p.g_{ij} = h_{ij}$, where $h_{ij} = g_{ij} - l_i l_j$.

Alaa et al. [5] introduced the generalized $\mathfrak{B}P-$birecurrent space which Cartan's second curvature tensor P^i_{jkh} satisfies the condition

$$\mathfrak{B}_l\mathfrak{B}_m P^i_{jkh} = a_{lm}P^i_{jkh} + b_{lm}(\delta^i_j g_{kh} - \delta^i_k g_{jh}) - 2y^t\mu_m\mathfrak{B}_t(\delta^i_j C_{khl} - \delta^i_k C_{jhl}). \tag{2.21}$$

This space is denoted by $G(\mathfrak{B}P) - BRF_n$.

Let us consider a $G(\mathfrak{B}P) - BRF_n$.
Transvecting the condition (2.21) by y^j, using (2.1), (2.3), (2.4), (2.7) and (2.8), we get

$$\mathfrak{B}_l\mathfrak{B}_m P^i_{kh} = a_{lm}P^i_{kh} + b_{lm}(y^i g_{kh} - \delta^i_k y_h) - 2y^t\mu_m\mathfrak{B}_t(y^i C_{khl}). \tag{2.22}$$

Contracting the indices i and h in the condition (2.21), using (2.3), (2.4) and (2.11), we get

$$\mathfrak{B}_l\mathfrak{B}_m P_{jk} = a_{lm}P_{jk}. \tag{2.23}$$

Contracting the indices i and h in (2.22) and using (2.1), (2.3), (2.4) and (2.11), we get

$$\mathfrak{B}_l\mathfrak{B}_m P_k = a_{lm}P_k. \tag{2.24}$$

Transvecting (2.24) by y^k, using (2.7), (2.11) and put $(y_k y^k = 1)$, we get

$$\mathfrak{B}_l\mathfrak{B}_m P = a_{lm}P. \tag{2.25}$$

Berwald's covariant derivative of first and second order for the $(h)hv-$torsion tensor C_{ijk} and its associative C^i_{jk} satisfy [3, 12]

$$\begin{cases} a)\ \mathfrak{B}_m C^i_{kh} = \lambda_m C^i_{kh} + \mu_m(\delta^i_k y_h - \delta^i_h y_k) \\ b)\ \mathfrak{B}_m C_{jkh} = \lambda_m C_{jkh} + \mu_m(g_{jk}y_h - g_{jh}y_k) \\ c)\ \mathfrak{B}_l\mathfrak{B}_m C^i_{kh} = a_{lm}C^i_{kh} + b_{lm}(\delta^i_k y_h - \delta^i_h y_k) \\ d)\ \mathfrak{B}_l\mathfrak{B}_m C_{jkh} = a_{lm}C_{jkh} + b_{lm}(g_{jk}y_h - g_{jh}y_k). \end{cases} \tag{2.26}$$

3. A $P2-$Like$-$Generalized $\mathfrak{B}P-$Birecurrent Space

Definition 3.1. *The generalized $\mathfrak{B}P-$birecurrent space which is $P2-$like space i.e. satisfies the condition* (2.15)*, will be called a $P2-$like generalized $\mathfrak{B}P-$birecurrent space and will be denoted briefly by $P2-like-G(\mathfrak{B}P)-BRF_n$.*

Remark 3.2. *It will be sufficient to call the tensor which satisfies the condition of $P2-$like$-G(\mathfrak{B}P)-BRF_n$ as a generalized $\mathfrak{B}-$birecurrent.*

Let us consider a $P2-\text{like}-G(\mathfrak{B}P)-BRF_n$.
In next theorem we obtain the tensor $(\varphi_j C^i_{kh}-\varphi^i C_{jkh})$ satisfies the generalized birecurrence property.

Theorem 3.3. *The tensor $(\varphi_j C^i_{kh}-\varphi^i C_{jkh})$ is generalized $\mathfrak{B}-$birecurrent in $P2-like-G(\mathfrak{B}P)-BRF_n$.*

Proof. Taking $\mathfrak{B}-$covariant derivative for the condition (2.15) twice with respect to x^m and x^l, respectively, using the condition (2.21), we get

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m(\varphi_j C^i_{kh}-\varphi^i C_{jkh}) &= a_{lm}P^i_{jkh}+b_{lm}(\delta^i_j g_{kh}-\delta^i_k g_{jh})\\ &\quad -2y^t\mu_m\mathfrak{B}_t(\delta^i_j C_{khl}-\delta^i_k C_{jhl}).\end{aligned}$$

Using the condition (2.15) in above equation, we get

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m(\varphi_j C^i_{kh}-\varphi^i C_{jkh}) &= a_{lm}(\varphi_j C^i_{kh}-\varphi^i C_{jkh})+b_{lm}(\delta^i_j g_{kh}-\delta^i_k g_{jh})\\ &\quad -2y^t\mu_m\mathfrak{B}_t(\delta^i_j C_{khl}-\delta^i_k C_{jhl}).\end{aligned} \tag{3.1}$$

Hence, we have proved this theorem. □

Now, we infer a corollary related to the previous theorem.
Contracting the indices i and h in the condition (2.15), using (2.4) and (2.11), we get

$$P_{jk}=\varphi_j C_k-\varphi^i C_{jki}. \tag{3.2}$$

Taking $\mathfrak{B}-$covariant derivative for (3.2) twice with respect to x^m and x^l, respectively, using (2.23), we get

$$\mathfrak{B}_l\mathfrak{B}_m(\varphi_j C_k-\varphi^i C_{jki})=a_{lm}P_{jk}$$

Using (3.2) in above equation, we get

$$\mathfrak{B}_l\mathfrak{B}_m(\varphi_j C_k-\varphi^i C_{jki})=a_{lm}(\varphi_j C_k-\varphi^i C_{jki}) \tag{3.3}$$

Thus, we conclude the following corollary:

Corollary 3.4. *In $P2-like-G(\mathfrak{B}P)-BRF_n$, the behavior of the tensor $(\varphi_j C_k-\varphi^i C_{jki})$ as birecurrent .*

4. A P^*-Generalized $\mathfrak{B}P-$Birecurrent Space

Definition 4.1. [17] *The generalized $\mathfrak{B}P-$birecurrent space which is P^*-space i.e. satisfies the condition* (2.16)*, will be called a P^*- generalized $\mathfrak{B}P-$birecurrent space and will be denoted briefly by $P^* - G(\mathfrak{B}P) - BRF_n$.*

Remark 4.2. *All results in $P2-$like$-G(\mathfrak{B}P) - BRF_n$ which obtained in the previous section are satisfied in $P^* - G(\mathfrak{B}P) - BRF_n$.*

Let us consider a $P^* - G(\mathfrak{B}P) - BRF_n$.
In next theorem we obtain the Berwalds covariant derivative of second order for some tensors are non - vanishing.

Theorem 4.3. *Berwalds covariant derivative of second order for the tensors (φC^i_{kh}), (φC_k) and (φC) are non-vanishing in $P^* - G(\mathfrak{B}P) - BRF_n$.*

Proof. Taking $\mathfrak{B}-$covariant derivative for the condition (2.16) twice with respect to x^m and x^l, respectively, using (2.22), we get

$$\mathfrak{B}_l\mathfrak{B}_m(\varphi C^i_{kh}) = a_{lm}P^i_{kh} + b_{lm}(y^i g_{kh} - \delta^i_k y_h) - 2y^t\mu_m\mathfrak{B}_t(y^i C_{khl}).$$

Using the condtion (2.16) in above equation, we get

$$\mathfrak{B}_l\mathfrak{B}_m(\varphi C^i_{kh}) = a_{lm}(\varphi C^i_{kh}) + b_{lm}(y^i g_{kh} - \delta^i_k y_h) - 2y^t\mu_m\mathfrak{B}_t(y^i C_{khl}). \quad (4.1)$$

Contracting the indices i and h in the condition (2.16), using (2.4) and (2.11), we get

$$P_k = \varphi C_k. \quad (4.2)$$

Taking $\mathfrak{B}-$covariant derivative for (4.2) twice with respect to x^m and x^l, respectively, using (2.24), we get

$$\mathfrak{B}_l\mathfrak{B}_m(\varphi C_k) = a_{lm}P_k.$$

Using (4.2) in above equation, we get

$$\mathfrak{B}_l\mathfrak{B}_m(\varphi C_k) = a_{lm}(\varphi C_k). \quad (4.3)$$

Transvecting (4.2) by y^k, using (2.11) and put $(C_k y^k = C)$, we get

$$P = \varphi C. \quad (4.4)$$

Taking $\mathfrak{B}-$covariant derivative for (4.4) twice with respect to x^m and x^l, respectively, using (2.25), we get

$$\mathfrak{B}_l\mathfrak{B}_m(\varphi C) = a_{lm}P.$$

Using (4.4) in above equation, we get

$$\mathfrak{B}_l\mathfrak{B}_m(\varphi C) = a_{lm}(\varphi C). \quad (4.5)$$

The equations (4.1), (4.3) and (4.5) prove that the tensors (φC^i_{kh}), (φC_k) and (φC) are non-vanishing. Hence, we have proved this theorem. ☐

Also, in next theorem we discuss the relationship between Cartan's first curvature tensor S^i_{jkh} and associate tensor C^i_{jk} of the $(h)hv-$torsion tensor C_{ijk}.

Theorem 4.4. *The behavior of Cartan's first curvature tensor S^i_{jkh}, its associative curvature tensor S_{pjkh} and $S-$Ricci tensor S_{jk} as birecurrent in $P^* - G(\mathfrak{B}P) - BRF_n$.*

Proof. Taking $\mathfrak{B}-$covariant derivative for (2.12) twice with respect to x^m and x^l, respectively, we get

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m S^i_{jkh} &= (\mathfrak{B}_l\mathfrak{B}_m C^i_{rk})C^r_{jh} + (\mathfrak{B}_m C^i_{rk})(\mathfrak{B}_l C^r_{jh}) + (\mathfrak{B}_l C^i_{rk})(\mathfrak{B}_m C^r_{jh}) \\ &\quad + C^i_{rk}(\mathfrak{B}_l\mathfrak{B}_m C^r_{jh}) - (\mathfrak{B}_l\mathfrak{B}_m C^i_{rh})C^r_{jk} - (\mathfrak{B}_m C^i_{rh})(\mathfrak{B}_l C^r_{jk}) \\ &\quad - (\mathfrak{B}_l C^i_{rh})(\mathfrak{B}_m C^r_{jk}) - C^i_{rh}(\mathfrak{B}_l\mathfrak{B}_m C^r_{jk}).\end{aligned}$$

Using (2.26) in above equation, then use (2.4), we get

$$\mathfrak{B}_l\mathfrak{B}_m S^i_{jkh} = 2(a_{lm} + \lambda_l\lambda_m)(C^i_{rk}C^r_{jh} - C^i_{rh}C^r_{jk}) + 2\mu_l\mu_m y_j(\delta^i_k y_h - \delta^i_h y_k).$$

Using (2.12) in above equation, we get

$$\mathfrak{B}_l\mathfrak{B}_m S^i_{jkh} = \alpha_{lm} S^i_{jkh}, \tag{4.6}$$

where $\alpha_{lm} = 2(a_{lm} + \lambda_l\lambda_m)$ and $\delta^i_k y_h = \delta^i_h y_k$.

Transvecting (2.12) by g_{ip}, using (2.4) and (2.13), we get

$$S_{pjkh} = C_{prk}C^r_{jh} - C_{prh}C^r_{jk}. \tag{4.7}$$

Taking $\mathfrak{B}-$covariant derivative for (4.7) twice with respect to x^m and x^l, respectively, we get

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m S_{pjkh} &= (\mathfrak{B}_l\mathfrak{B}_m C_{prk})C^r_{jh} + (\mathfrak{B}_m C_{prk})(\mathfrak{B}_l C^r_{jh}) + (\mathfrak{B}_l C_{prk})(\mathfrak{B}_m C^r_{jh}) \\ &\quad + C_{prk}(\mathfrak{B}_l\mathfrak{B}_m C^r_{jh}) - (\mathfrak{B}_l\mathfrak{B}_m C_{prh})C^r_{jk} - (\mathfrak{B}_m C_{prh})(\mathfrak{B}_l C^r_{jk}) \\ &\quad - (\mathfrak{B}_l C_{prh})(\mathfrak{B}_m C^r_{jk}) - C_{prh}(\mathfrak{B}_l\mathfrak{B}_m C^r_{jk}).\end{aligned}$$

Using (2.26) in above equation, then use (2.4), we get

$$\mathfrak{B}_l\mathfrak{B}_m S_{pjkh} = 2(a_{lm} + \lambda_l\lambda_m)(C_{prk}C^r_{jh} - C_{prh}C^r_{jk}) + 2\mu_l\mu_m y_j(y_h g_{pk} - y_k g_{ph}).$$

Using (4.7) in above equation, we get

$$\mathfrak{B}_l\mathfrak{B}_m S_{pjkh} = \alpha_{lm} S_{pjkh}. \tag{4.8}$$

where $\alpha_{lm} = 2(a_{lm} + \lambda_l\lambda_m)$ and $y_h g_{nk} = y_k g_{nh}$.
Contracting the indices i and h in (4.6), using (2.14), we get

$$\mathfrak{B}_l\mathfrak{B}_m S_{jk} = \alpha_{lm} S_{jk}. \tag{4.9}$$

The equations (4.6), (4.8) and (4.9) show that the tensors S^i_{jkh}, S_{pjkh} and S_{jk} behave as birecurrent. Hence, we have proved this theorem. □

5. A $P-$ Reducible$-$Generalized $\mathfrak{B}P-$Birecurrent Space

Definition 5.1. *The generalized $\mathfrak{B}P-$birecurrent space which is $P-$ reducible space i.e. satisfies one of the conditions* (2.17) *or* (2.18)*, will be called a $P-$reducible generalized $\mathfrak{B}P-$birecurrent space and will be denoted briefly by $P-reducible-G(\mathfrak{B}P)-BRF_n$.*

Remark 5.2. *It will be sufficient to call the tensor which satisfies the condition of $P-reducible-G(\mathfrak{B}P)-BRF_n$ as a generalized $\mathfrak{B}-$birecurrent.*

In $P-$reducible space, the associate tensor P_{ijkh} of $hv-$curvature tensor P^i_{jkh} is given by [10]

$$P_{ijkh} = \Big(\Theta_j C_{ikh} + \vartheta_j h_{kh} C_i + E_{kj} h_{ih} + B_{hj} h_{ik} - i/j\Big) - \lambda S_{ijkh}, \tag{5.1}$$

where

$$\begin{cases} a)\ \Theta_j = \lambda_j - \vartheta C_j \\ b)\ E_{kj} = C_k \vartheta_j + \vartheta \partial_j C_k + \vartheta F^{-1}(L_j C_k + L_k C_j) \\ c)\ B_{hj} = C_h \vartheta_j + \vartheta C_{h|j} + \vartheta F^{-1}(L_h C_j + L_j C_h) \\ d)\ \lambda_j = \dot{\partial}_j \lambda, \\ e)\ \vartheta_j = \dot{\partial}_j \vartheta, \\ f)\ F^{-1} = 1/F,\ F \text{ is the fundamental function of Finsler space.} \end{cases}$$

Let us consider a $P-\text{reducible}-G(\mathfrak{B}P)-BRF_n$.
In next theorem we obtain the tensor $g^{ir}\Big[\Big(\Theta_j C_{ikh}+\vartheta_j h_{kh} C_i+E_{kj} h_{ih}+B_{hj} h_{ik}-i/j\Big)-\lambda S_{ijkh}\Big]$ satisfies the generalized birecurrence property.

Theorem 5.3. *In $P-reducible-G(\mathfrak{B}P)-BRF_n$, the tensor $g^{ir}\Big[\Big(\Theta_j C_{ikh}+\vartheta_j h_{kh} C_i+E_{kj} h_{ih}+B_{hj} h_{ik}-i/j\Big)-\lambda S_{ijkh}\Big]$ is a generalized $\mathfrak{B}-$birecurrent.*

Proof. Transvecting (5.1) by g^{ir}, using (2.10), we get

$$P^r_{jkh} = g^{ir}\Big[\Big(\Theta_j C_{ikh} + \vartheta_j h_{kh} C_i + E_{kj} h_{ih} + B_{hj} h_{ik} - i/j\Big) - \lambda S_{ijkh}\Big]. \tag{5.2}$$

Taking $\mathfrak{B}-$covariant derivative for above equation twice with respect to x^m and x^l, respectively, using the condition (2.21), we get

$$\begin{aligned} &\mathfrak{B}_l\mathfrak{B}_m\Big(g^{ir}\Big[\Big(\Theta_j C_{ikh} + \vartheta_j h_{kh} C_i + E_{kj} h_{ih} + B_{hj} h_{ik} - i/j\Big) - \lambda S_{ijkh}\Big]\Big) \\ &= a_{lm} P^i_{jkh} + b_{lm}\Big(\delta^i_j g_{kh} - \delta^i_k g_{jh}\Big) - 2y^t \mu_m \mathfrak{B}_t\Big(\delta^i_j C_{khl} - \delta^i_k C_{jhl}\Big). \end{aligned}$$

Using (5.2) in above equation, we get

$$\begin{aligned}
&\mathfrak{B}_l\mathfrak{B}_m\Big(g^{ir}\Big[\Big(\Theta_j C_{ikh} + \vartheta_j h_{kh} C_i + E_{kj} h_{ih} + B_{hj} h_{ik} - i/j\Big) - \lambda S_{ijkh}\Big]\Big)\\
&= a_{lm}\Big(g^{ir}\Big[\Big(\Theta_j C_{ikh} + \vartheta_j h_{kh} C_i + E_{kj} h_{ih} + B_{hj} h_{ik} - i/j\Big) - \lambda S_{ijkh}\Big]\Big)\\
&\quad + b_{lm}\Big(\delta^i_j g_{kh} - \delta^i_k g_{jh}\Big) - 2y^t \mu_m \mathfrak{B}_t\Big(\delta^i_j C_{khl} - \delta^i_k C_{jhl}\Big).
\end{aligned} \tag{5.3}$$

Hence, we have proved this theorem. □

Now, we infer a corollary related to the previous theorem.

Transvecting (2.17) by g^{ij}, using (2.9) and (2.4), we get

$$P^i_{kh} = \lambda C^i_{kh} + \vartheta(h^i_k C_h + h_{kh} C^i + h^i_h C_k) \tag{5.4}$$

where $h^i_k = g^{ij} h_{jk}$ and $C^i = g^{ij} C_j$.

Taking $\mathfrak{B}-$covariant derivative for (5.4) twice with respect to x^m and x^l, respectively, using (2.22), we get

$$\begin{aligned}
\mathfrak{B}_l\mathfrak{B}_m\Big[\lambda C^i_{kh} + \varphi\Big(h^i_k C_h + h_{kh} C^i + h^i_h C_k\Big)\Big] \quad = \quad & a_{lm} P^i_{kh} + b_{lm}\Big(y^i g_{kh} - \delta^i_k y_h\Big)\\
& -2y^t \mu_m \mathfrak{B}_t\Big(y^i C_{khl}\Big).
\end{aligned}$$

Using (5.4) in above equation, we get

$$\begin{aligned}
&\mathfrak{B}_l\mathfrak{B}_m\Big[\lambda C^i_{kh} + \vartheta\Big(h^i_k C_h + h_{kh} C^i + h^i_h C_k\Big)\Big]\\
&= a_{lm}\Big[\lambda C^i_{kh} + \vartheta\Big(h^i_k C_h + h_{kh} C^i + h^i_h C_k\Big)\Big]\\
&+ b_{lm}(y^i g_{kh} - \delta^i_k y_h\Big) - 2y^t \mu_m \mathfrak{B}_t\Big(y^i C_{khl}\Big).
\end{aligned} \tag{5.5}$$

Also, transvecting (2.18) by g^{ij}, using (2.9), we get

$$P^i_{kh} = \frac{1}{n+1}(h^i_k P_h + h_{kh} P^i + h^i_h P_k), \tag{5.6}$$

where $h^i_h = g^{ij} h_{hj}$ and $P^i = g^{ij} P_j$.

Taking $\mathfrak{B}-$covariant derivative for (5.6) twice with respect to x^m and x^l, respectively, using (2.22), we get

$$\begin{aligned}
\mathfrak{B}_l\mathfrak{B}_m\Big[\frac{1}{n+1}\Big(h^i_k P_h + h_{kh} P^i + h^i_h P_k\Big)\Big] \quad = \quad & a_{lm} P^i_{kh} + b_{lm}\Big(y^i g_{kh} - \delta^i_k y_h\Big)\\
& -2y^t \mu_m \mathfrak{B}_t\Big(y^i C_{khl}\Big).
\end{aligned}$$

Using (5.6) in above equation, we get

$$\begin{aligned}
&\mathfrak{B}_l\mathfrak{B}_m\Big[\frac{1}{n+1}\Big(h^i_k P_h + h_{kh} P^i + h^i_h P_k\Big)\Big]\\
&= a_{lm}\Big[\frac{1}{n+1}\Big(h^i_k P_h + h_{kh} P^i + h^i_h P_k\Big)\Big]\\
&+ b_{lm}\Big(y^i g_{kh} - \delta^i_k y_h\Big) - 2y^t \mu_m \mathfrak{B}_t\Big(y^i C_{khl}\Big).
\end{aligned} \tag{5.7}$$

Thus, we conclude the following corollary:

Corollary 5.4. *$P-$ reducible $-G(\mathfrak{B}P)-BRF_n$, Berwald's covariant derivative of second order for the tensors* $\left[\lambda C^i_{kh}+\vartheta\Big(h^i_k C_h+h_{kh}C^i+h^i_h C_k\Big)\right]$ *and* $\left[\frac{1}{n+1}\Big(h^i_k P_h+h_{kh}P^i+h^i_h P_k\Big)\right]$ *are given by* (5.5) *and* (5.7), *respectively.*

6. **Examples**

Some examples related to the previous mentioned theorems will be discussed to clarify the proved findings.

Example 6.1. *The behavior of Cartan's first curvature tensor S^i_{jkh} as birecurrent if and only if the projection on indicatrix for S^i_{jkh} is birecurrent.*

Firstly, since Cartan's first curvature tensor S^i_{jkh} behaves as birecurrent, then the condition (4.6) is satisfied. In view of (2.19), the projection of Cartan's first curvature tensor S^i_{jkh} on indicatrix is given by

$$p.S^i_{jkh}=S^a_{bcd}h^i_a h^b_j h^c_k h^d_h. \tag{6.1}$$

By using $\mathfrak{B}-$covariant derivative for (6.1) twice with respect to x^m and x^l, respectively, using (4.6) and the fact that h^a_b is covariant constant in above equation, we get

$$\mathfrak{B}_l\mathfrak{B}_m\left(p.S^i_{jkh}\right)=\alpha_{lm}\left(S^a_{bcd}h^i_a h^b_j h^c_k h^d_h\right).$$

Using (6.1) in above equation, we get

$$\mathfrak{B}_l\mathfrak{B}_m\left(p.S^i_{jkh}\right)=\alpha_{lm}\left(p.S^i_{jkh}\right). \tag{6.2}$$

Equation (6.2) refers to the projection on indicatrix for Cartan's first curvature tensor S^i_{jkh} behaves as birecurrent.

Secondly, let the projection on indicatrix for Cartans first curvature tensor S^i_{jkh} is birecurrent i.e. satisfy (6.2). Using (2.19) in (6.2), we get

$$\mathfrak{B}_l\mathfrak{B}_m\left(S^a_{bcd}\;h^i_a h^b_j h^c_k h^d_h\right)=\alpha_{lm}\left(S^a_{bcd}\;h^i_a h^b_j h^c_k h^d_h\right).$$

By using (2.20) in above equation, we get

$$\begin{aligned}
&\mathfrak{B}_l\mathfrak{B}_m\Big[S^i_{jkh} - S^i_{jkd}l^d l_h - S^i_{jch}l^c l_k + S^i_{jcd}l^c l_k l^d l_h - S^i_{bkh}l^b l_j \\
&+S^i_{bkd}l^b l_j l^d l_h + S^i_{bch}l^b l_j l^c l_k - S^i_{bcd}l^b l_j l^c l_k l^d l_h - S^a_{jkh}l^i l_a \\
&+S^a_{jkd}l^i l_a l^d l_h + S^a_{jch}l^i l_a l^c l_k - S^a_{jcd}l^i l_a l^c l_k l^d l_h + S^a_{bkh}l^i l_a l^b l_j \\
&-S^a_{bkd}l^i l_a l^b l_j l^d l_h - S^a_{bch}l^i l_a l^b l_j l^c l_k + S^a_{bcd}l^i l_a l^b l_j l^c l_k l^d l_h\Big] \\
&= \alpha_{lm}\Big[S^i_{jkh} - S^i_{jkd}l^d l_h - S^i_{jch}l^c l_k + S^i_{jcd}l^c l_k l^d l_h - S^i_{bkh}l^b l_j \\
&+S^i_{bkd}l^b l_j l^d l_h + S^i_{bch}l^b l_j l^c l_k - S^i_{bcd}l^b l_j l^c l_k l^d l_h - S^a_{jkh}l^i l_a \\
&+S^a_{jkd}l^i l_a l^d l_h + S^a_{jch}l^i l_a l^c l_k - S^a_{jcd}l^i l_a l^c l_k l^d l_h + S^a_{bkh}l^i l_a l^b l_j \\
&-S^a_{bkd}l^i l_a l^b l_j l^d l_h - S^a_{bch}l^i l_a l^b l_j l^c l_k + S^a_{bcd}l^i l_a l^b l_j l^c l_k l^d l_h\Big].
\end{aligned}$$

In view of (2.5) and if $S^a_{bcd}y_a = S^a_{bcd}y^b = S^a_{bcd}y^c = S^a_{bcd}y^d = 0$, then above equation becomes

$$\mathfrak{B}_l\mathfrak{B}_m S^i_{jkh} = \alpha_{lm}S^i_{jkh}.$$

Above equation means the Cartan's first curvature tensor S^i_{jkh} behaves as birecurrent.

Example 6.2. *The associate curvature tensor S_{pjkh} behaves as birecurrent if and only if satisfies*

$$\mathfrak{B}_l\mathfrak{B}_m\left(p.S_{pjkh}\right) = \alpha_{lm}\left(p.S_{pjkh}\right).$$

Firstly, since the associate curvature tensor S_{pjkh} behaves as birecurrent, then the condition (4.8) is satisfied. In view of (2.19), the projection of associate curvature tensor S_{pjkh} on indicatrix is given by

$$p.S_{pjkh} = S_{abcd}h^a_p h^b_j h^c_k h^d_h. \tag{6.3}$$

Using $\mathfrak{B}-$covariant derivative for (6.3) twice with respect to x^m and x^l, respectively, using (4.8) and the fact that h^a_b is covariant constant in above equation, we get

$$\mathfrak{B}_l\mathfrak{B}_m\left(p.S_{pjkh}\right) = \alpha_{lm}\left(S_{abcd}h^a_p h^b_j h^c_k h^d_h\right).$$

Using (6.3) in above equation, we get

$$\mathfrak{B}_l\mathfrak{B}_m\left(p.S_{pjkh}\right) = \alpha_{lm}\left(p.S_{pjkh}\right). \tag{6.4}$$

Equation (6.4) means the projection on indicatrix for associate curvature tensor S_{pjkh} behaves as birecurrent.

Secondly, let the projection on indicatrix for associate curvature tensor S_{pjkh} is birecurrent i.e. satisfy (6.4). Using (2.19) in (6.4), we get

$$\mathfrak{B}_l\mathfrak{B}_m\left(S_{abcd}\ h^a_p h^b_j h^c_k h^d_h\right) = \alpha_{lm}\left(S_{abcd}\ h^a_p h^b_j h^c_k h^d_h\right).$$

By using (2.20) in above equation, we get

$$\begin{aligned}
&\mathfrak{B}_l\mathfrak{B}_m\Big[S_{pjkh} - S_{pjkd}l^dl_h - S_{pjch}l^cl_k + S_{pjcd}l^cl_kl^dl_h - S_{pbkh}l^bl_j \\
&+S_{pbkd}l^bl_jl^dl_h + S_{pbch}l^bl_jl^cl_k - S_{pbcd}l^bl_jl^cl_kl^dl_h - S_{ajkh}l^al_p \\
&+S_{ajkd}l^al_pl^dl_h + S_{ajch}l^al_pl^cl_k - S_{ajcd}l^al_pl^cl_kl^dl_h + S_{abkh}l^al_pl^bl_j \\
&-S_{abkd}l^al_pl^bl_jl^dl_h - S_{abch}l^al_pl^bl_jl^cl_k + S_{abcd}l^al_pl^bl_jl^cl_kl^dl_h\Big] \\
&= \alpha_{lm}\Big[S_{pjkh} - S_{pjkd}l^dl_h - S_{pjch}l^cl_k + S_{pjcd}l^cl_kl^dl_h - S_{pbkh}l^bl_j \\
&+S_{pbkd}l^bl_jl^dl_h + S_{pbch}l^bl_jl^cl_k - S_{pbcd}l^bl_jl^cl_kl^dl_h - S_{ajkh}l^al_p \\
&+S_{ajkd}l^al_pl^dl_h + S_{ajch}l^al_pl^cl_k - S_{ajcd}l^al_pl^cl_kl^dl_h + S_{abkh}l^al_pl^bl_j \\
&-S_{abkd}l^al_pl^bl_jl^dl_h - S_{abch}l^al_pl^bl_jl^cl_k + S_{abcd}l^al_pl^bl_jl^cl_kl^dl_h\Big].
\end{aligned}$$

In view of (2.5) and if $S_{abcd}y^a = S_{abcd}y^b = S_{abcd}y^c = S_{abcd}y^d = 0$, then above equation can be written as

$$\mathfrak{B}_l\mathfrak{B}_mS_{pjkh} = \alpha_{lm}S_{pjkh}.$$

Last equation refers to the associate curvature tensor S_{rjkh} behaves as birecurrent. Also, we can apply same technique for proving the $S-$Ricci tensor S_{jk} is birecurrent if and only if the projection on indicatrix for it behaves as birecurrent.

7. **Conclusion**

We extended the generalized $\mathfrak{B}P-$birecurrent space by using the properties of $P2-$like space, P^*-space, $P-$reducible space in the above mentioned space to obtain new spaces related to it. Also, the relationship between Cartan's first curvature tensor S^i_{jkh} and associate tensor C^i_{jk} of the $(h)hv-$torsion tensor C_{ijk} has been discussed.

Acknowledgment: The authors are grateful to the editor and anonymous reviewers for their helpful, valuable comments and suggestions in the improvement of this manuscript.

References

1. A.A. Abdallah, A.A. Hamoud, A. Navlekar, K. Ghadle, B. Hardan, H. Emadifar, and M. Khadem, *On Birecurrent for Some Tensors in Various Finsler Spaces*, Journal of Finsler Geometry and its Applications, **4**(1)(2023), 33-44.
2. A.A. Abdallah, A.A. Navlekar, K.P. Ghadle, and B. Hardan, *Fundamentals and recent studies of finsler geometry*, International Journal of Advances in Applied Mathematics and Mechanics, **10**(2)(2022), 27-38.
3. A.A. Abdallah, A.A. Navlekar, and K.P. Ghadle, *On $\mathfrak{B}-$covariant derivative of first order for some tensors in different spaces*, Journal of Mathematical Analysis and Modeling, **2**(2)(2021), 30-37.

4. A.A. Abdallah, A.A. Navlekar, and K.P. Ghadle, *Special types of generalized $\mathfrak{B}P-$recurrent spaces*, Journal of Computer and Mathematical Sciences, **10**(5)(2019), 972-979.
5. A.A. Abdallah, A.A. Navlekar, K.P. Ghadle, and A.A. Hamoud, *Decomposition for Cartan's second curvature tensor of different order in Finsler spaces*, Nonlinear Functional Analysis and Applications, **27**(2)(2022), 433-448.
6. A.M. Al-Qashbari, *Recurrence decompositions in Finsler space*, Journal of Mathematical Analysis and Modeling, **1**(1)(2020), 77-86.
7. F.A. Assallal, *On certain generalized $h-$birecurrent of curvature tensor*, M.Sc. Thesis, University of Aden, (Yemen), (2018).
8. M. Dahl, *An brief introduction to Finsler geometry*, Springer, (2006).
9. A.D. Dubey, *Special Finsler spaces - a review of literature*, International Journal of Research and Development in Applied Science and Engineering, **12**(1)(2017), 6-8.
10. P.K. Dwivedi, *P^*-reducible Finsler spaces and applications*, Int. journal of Math. Analysis, **5**(5)(2011), 223-229.
11. M. Gheorghe, *The indicatrix in Finsler geometry*, Analele Stiintifice Ale Uuiversitatii Matematica. Tomul LIII, (2007), 163-180.
12. W.H. Hadi, *Study of certain types of generalized birecurrent in Finsler spaces*, Ph.D. Thesis, Faculty of Education-Aden, University of Aden, (Yemen), (2016).
13. H. Izumi, *On Finsler space of scalar curvature*, Tensor N.S., **38**(1982), 220-222.
14. M. Matsumoto, *On $h-$isotropic and C^h- recurrent Finsler*, J. Math. Kyoto Univ., **11**(1971), 1-9.
15. M. Matsumoto, *On Finsler spaces with curvature tensor of some special forms*, Tensor N. S., **22**(1971), 201-204.
16. S.K. Narasimhamurthy, P. Kumar, and S.T. Aveesh, *A study of hypersurfaces on special Finsler spaces*, International Journal of Pure and Applied Mathematics, **48**(1)(2008), 67-74.
17. S.I. Ohta, *Comparison Finsler geometry*, Springer International Publishing, (2021).
18. A.M. Otman, *On covariant differentiation for curvature tensor of third order in the sense of Berwald*, M.SC. Thesis, University of Aden, Yemen, (2018).
19. F.Y. Qasem, and W.H. Hadi, *On a generalized $\mathfrak{B}R-$birecurrent affinely connected space*, International Journal of Mathematics and statistics invention, **4**(5)(2016), 30-33.
20. H. Rund, *The differential geometry of Finsler spaces*, Springer-Verlag, Berlin Gttingen, (1959); 2nd Edit. (in Russian), Nauka, (Moscow), 1981.
21. P.S. Saxena, *A study of P^*-reducible Finsler space with Douglas tensor*, Journal of International Academy of physical Science, **17**(3)(2013), 277-285.
22. Y.B. Shen, and S. Zhongmin, *Introduction to modern Finsler geometry*, World Scientific Publishing Company, (2016).
23. B.K. Tripathi, and K.B. Pandey, *On a special form of $h(hv)-$torsion tensor P_{ijk} in Finsler space*, Journal of Mathematics, Hindawi, (2016), 1-5.
24. H. Wosoughi, *On $P3-$Like Finsler spaces*, Journal of Mathematics and Statistics Research, **2**(1)(2020), 1-2.
25. S.M. Zamanzadeh, B. Najafi, and M. Toomanian, *On generalized $P-$reducible Finsler manifolds*, Open Mathematics Research Article, **16**(2018), 718-723.

Received: 13.05.2023
Accepted: 23.06.2023