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On Birecurrent for Some Tensors in Various Finsler Spaces

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Abstract. The $\mathfrak{B}C-$ recurrent Finsler space introduced by Alaa et al. [1]. Now in this paper, we introduce and extend $\mathfrak{B}C-$ birecurrent Finsler space by using some properties of different spaces. We study the relationship between Cartan's second curvature tensor P^i_{jkh} and $(h)hv$ torsion tensor C^i_{jk} in sense of Berwald. Additionally, the necessary and sufficient condition for some tensors which satisfy birecurrence property will be discuss in different spaces. Four theorems have been established and proved.

Keywords: $\mathfrak{B}C-$ birecurrent space, birecurrence property, $P2-$like space, P^*-space, generalized P-reducible space.

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1. **Introduction**

The tensors which satisfy a birecurrence property in Finsler spaces has been discussed by the Finslerian geometers. The concept of $C-$birecurrent space in sense of Cartan and Berwald were studied by Pandey and Verma [20] and Sarangi and Goswami [13], respectively. Saleem [6] discussed C^h-generalized birecurrent space and C^h-special generalized birecurrent space. Pandey and Verma [20], Otman [9], Hanballa [8], Alqufail et al. [14] and Dikshit [23] introduced C^h- birecurrnt space, $\mathfrak{B}P-$ birecurrent space, $\mathfrak{B}K-$birecurrent space, K^h- birecurrent space and R^h- birecurrent space, respectively. Also, Qasem and Hanballa [10] studied K^h-generalized birecurrent space.

In the same vein, Saleem and Abdallah [7] introduced the U^h- birecurrent Finsler space and discussed the necessary and sufficient condition for some tensors which satisfy the birecurrence property.

Regarding to special spaces of Finsler space, Pandey and Dikshit [21] discussed P^*- and $P-$reducible Finsler space of recurrent curvature tensor, Otman [9] studied the properties of $P2-$like space and P^*-space in P^h-birecurrent space. In addition, Saleem [6] studied $P2-$like$-$generalized birecurrent space and $P2-$like$-C^h-$special generalized birecurrent. Further, Saxena and Swaroop [22] used $P-$reducibility condition in spacial Finsler spaces. Recently, the properties of $P2-$like space, P^*-space and generalized P-reducible space in generalized $\mathfrak{B}P-$ recurrent space have been studied by [2, 3]. The main idea of this paper to concentrate on obtaining the necessary and sufficient condition for P^i_{jkh}, P_{ijkh}, P^i_{kh}, P_{jk}, P_k and P which satisfy birecurrence property in various spaces.

2. **Preliminaries**

In this section, important concept of Finsler geometry will be given in this paper. An $n-$dimensional space X_n equipped with a function $F(x, y)$ that denoted by $F_n = (X_n, F(x, y))$ called a Finsler space if the function $F(x, y)$ satisfying the request conditions [5, 12, 15, 24].

Matsumoto [18] introduced the $(h)hv-$torsion tensor C_{ijk} that is positively homogeneous of degree -1 in y^i and defined by

$$C_{ijk} = \frac{1}{2}\dot{\partial}_i\ g_{jk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k\ F^2.$$

By using Euler's theorem on homogeneous function, we get

$$a)\ C_{ijk}\ y^i = C_{kij}\ y^i = C_{jki}\ y^i = 0 \text{ and } b)\ C^i_{jk}\ y^j = C^i_{kj}\ y^j = 0, \tag{2.1}$$

where C^i_{jk} is called associate tensor of the tensor C_{ijk}, these tensors are defined by

$$a)\ C^h_{ik} = C_{ijk}g^{hj},\ b)\ C^i_{ji} = C_j \text{ and } c)\ C_k y^k = C, \tag{2.2}$$

The unit vector l^i and the associative vector l_i with the direction of y^i are given by

$$a)\ l^i = \frac{y^i}{F} \text{ and } b)\ l_i = \frac{y_i}{F}. \tag{2.3}$$

Berwald covariant derivative $\mathfrak{B}_k T^i_j$ of an arbitrary tensor field T^i_j with respect to x^k is given by [12]

$$\mathfrak{B}_k T^i_j = \partial_k T^i_j - (\dot{\partial}_r T^i_j) G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}.$$

Berwald covariant derivative of the vector y^i vanish identically, i.e.

$$\mathfrak{B}_k y^i = 0. \tag{2.4}$$

The tensor P^i_{jkh} is called $hv-$curvature tensor (Cartan's second curvature tensor) is positively homogeneous of degree -1 in y^i and defined by [12]

$$P^i_{jkh} = C^i_{kh|j} - g^{ir} C_{jkh|r} + C^r_{jk} P^i_{rh} - P^r_{jh} C^i_{rk}, \tag{2.5}$$

which satisfies the relation

$$P^i_{jkh} y^j = \Gamma^{*i}_{jkh} y^j = P^i_{kh} = C^i_{kh|r} y^r, \tag{2.6}$$

where P^i_{kh} is $(v)hv-$torsion tensor which satisfies

$$P^i_{kh} = P_{rkh} g^{ir}, \tag{2.7}$$

where P_{rkh} is called associative tensor for $v(hv)-$torsion tensor.
$P-$ Ricci tensor P_{jk} , curvature vector P_k and curvature scalar P of Cartan's second curvature tensor are given by

$$a)\ P_{jk} = P^i_{jki},\ b)\ P^i_{ki} = P_k \text{ and } c)\ P = P_k y^k, \tag{2.8}$$

respectively.

Definition 2.1. *A Finsler space F_n is called a $P2-$like space if the Cartan's secend curvature tensor P^i_{jkh} is characterized by the condition* [18]

$$P^i_{jkh} = \varphi_j C^i_{kh} - \varphi^i C_{jkh}, \tag{2.9}$$

where φ_j and φ^i are non - zero covariant and contravariant vectors field, respectively.

Definition 2.2. *A Finsler space F_n is called a P^*-Finsler space if the $(v)hv-$ torsion tensor P^i_{kh} is characterized by the condition* [11]

$$P^i_{kh} = \varphi C^i_{kh}, \tag{2.10}$$

where $P^i_{jkh} y^j = P^i_{kh} = C^i_{kh|s} y^s$.

Definition 2.3. *A Finsler space F_n is called a generalized P-reducible space if the associate tensor P_{jkh} of $(v)hv-$torsion tensor P^i_{kh} is characterized by the condition* [19, 25]

$$P_{jkh} = \lambda C_{jkh} + \vartheta(h_{jk}C_h + h_{kh}C_j + h_{hj}C_k), \tag{2.11}$$

where λ and ϑ are scalar vectors positively homogeneous of degree one in y^j and h_{jk} is the angular metric tensor.

Definition 2.4. *Let the current coordinates in the tangent space at the point x_0 be x^i, then the indicatrix I_{n-1} is a hypersurface defined by* [12] *$F\left(x_0, x^i\right) = 1$ or by the parametric form defined by $x^i = x^i\left(u^a\right)$, $a = 1, 2, \ . \ . \ . \ , \ n-1$.*

Definition 2.5. *The projection of any tensor T^i_j on indicatrix I_{n-1} given by* [12, 16]

$$p.T^i_j = T^a_b h^i_a h^b_j, \tag{2.12}$$

where

$$h^i_c = \delta^i_c - l^i l_c. \tag{2.13}$$

The projection of the vector y^i, the unit vector l^i and the metric tensor g_{ij} on the indicatrix are given by $p.y^i = 0$, $p.l^i = 0$ and $p.g_{ij} = h_{ij}$, where $h_{ij} = g_{ij} - l_i l_j$.

3. On $\mathfrak{B}C-$Birecurrent Space

In this section, we find the condition for different tensors which behave as birecurrent in $\mathfrak{B}C-$birecurrent space. Matsumoto [17] introduced a Finsler space which the $(h)hv-$torsion tensor C_{ijk} and its associate tensor C^i_{jk} satisfy the recurrence property in sense of Cartan. This space is called C^h-recurrent space and characterized by the conditions

$$a)\ C^i_{kh|m} = \lambda_m C^i_{kh} \text{ and } b)\ C_{jkh|m} = \lambda_m C_{jkh}. \tag{3.1}$$

Alaa et al. [1] introduced $\mathfrak{B}C - RF_n$ which is characterized by the conditions

$$a)\ \mathfrak{B}_m C^i_{kh} = \lambda_m C^i_{kh} \text{ and } b)\ \mathfrak{B}_m C_{jkh} = \lambda_m C_{jkh}. \tag{3.2}$$

Sarangi and Goswami [13] introduced a Finsler space which the $(h)hv-$ torsion tensor C_{ijk} and its associate tensor C^i_{jk} satisfy the birecurrence property in sense of Berwald and called it $C-$birecurrent space. Let us denote to this space briefly by a $\mathfrak{B}C - BRF_n$. This space characterized by the conditions

$$a)\ \mathfrak{B}_l\mathfrak{B}_m C^i_{kh} = a_{lm} C^i_{kh} \text{ and } b)\ \mathfrak{B}_l\mathfrak{B}_m C_{jkh} = a_{lm} C_{jkh}, \tag{3.3}$$

where $a_{lm} = \mathfrak{B}_l \lambda_m + \lambda_l \lambda_m$. Using eq. (3.1) in (2.5), we get

$$P^i_{jkh} = \lambda_j C^i_{kh} - \lambda^i C_{jkh} + C^r_{jk} P^i_{rh} - C^i_{rk} P^r_{jh}, \tag{3.4}$$

where $\lambda^i = \lambda_r g^{ir}$.

In next theorem we get the necessary and sufficient condition for some tensors which behave as birecurrent tensor in $\mathfrak{B}C-BRF_n$.

Theorem 3.1. *In $\mathfrak{B}C-BRF_n$, Cartan's second curvature tensor P^i_{jkh}, torsion tensor P^i_{kh}, $P-$Ricci tensor P_{jk}, curvature vector P_k and curvature scalar P satisfy the birecurrence property if and only if*

$$\begin{aligned}&\{(\mathfrak{B}_l\mathfrak{B}_m\lambda_j)+(\mathfrak{B}_m\lambda_j)\lambda_l+(\mathfrak{B}_l\lambda_j)\lambda_m\}C^i_{kh}\\&-\{(\mathfrak{B}_l\mathfrak{B}_m\lambda^i)-(\mathfrak{B}_m\lambda^i)\lambda_l-(\mathfrak{B}_l\lambda^i)\lambda_m\}C_{jkh}\\&+\{\lambda_m(\mathfrak{B}_lP^i_{rh})+\lambda_l(\mathfrak{B}_mP^i_{rh})+(\mathfrak{B}_l\mathfrak{B}_mP^i_{rh})\}C^r_{jk}\\&-\{\lambda_m(\mathfrak{B}_lP^r_{jh})+\lambda_l(\mathfrak{B}_mP^r_{jh})+(\mathfrak{B}_l\mathfrak{B}_mP^r_{jh})\}C^i_{rk}=0,\end{aligned}\tag{3.5}$$

$$\begin{aligned}&\{(\mathfrak{B}_l\mathfrak{B}_m\lambda_j)+(\mathfrak{B}_m\lambda_j)\lambda_l+(\mathfrak{B}_l\lambda_j)\lambda_m\}C^i_{kh}y^j\\&-\{\lambda_m(\mathfrak{B}_lP^r_{jh})+\lambda_l(\mathfrak{B}_mP^r_{jh})+(\mathfrak{B}_l\mathfrak{B}_mP^r_{jh})\}C^i_{rk}y^j=0,\end{aligned}\tag{3.6}$$

$$\begin{aligned}&\{(\mathfrak{B}_l\mathfrak{B}_m\lambda_j)+(\mathfrak{B}_m\lambda_j)\lambda_l+(\mathfrak{B}_l\lambda_j)\lambda_m\}C_k\\&-\{(\mathfrak{B}_l\mathfrak{B}_m\lambda^i)-(\mathfrak{B}_m\lambda^i)\lambda_l-(\mathfrak{B}_l\lambda^i)\lambda_m\}C_{jki}\\&+\{\lambda_m(\mathfrak{B}_lP_r)+\lambda_l(\mathfrak{B}_mP_r)+(\mathfrak{B}_l\mathfrak{B}_mP_r)\}C^r_{jk}\\&-\{\lambda_m(\mathfrak{B}_lP^r_{ji})+\lambda_l(\mathfrak{B}_mP^r_{ji})+(\mathfrak{B}_l\mathfrak{B}_mP^r_{ji})\}C^i_{rk}=0,\end{aligned}\tag{3.7}$$

$$\begin{aligned}&\{(\mathfrak{B}_l\mathfrak{B}_m\lambda_j)+(\mathfrak{B}_m\lambda_j)\lambda_l+(\mathfrak{B}_l\lambda_j)\lambda_m\}C_ky^j\\&-\{\lambda_m(\mathfrak{B}_lP^r_{ji})+\lambda_l(\mathfrak{B}_mP^r_{ji})+(\mathfrak{B}_l\mathfrak{B}_mP^r_{ji})\}C^i_{rk}y^j=0,\end{aligned}\tag{3.8}$$

and

$$\{(\mathfrak{B}_l\mathfrak{B}_m\lambda_j)+(\mathfrak{B}_m\lambda_j)\lambda_l+(\mathfrak{B}_l\lambda_j)\lambda_m\}Cy^j=0,\tag{3.9}$$

respectively.

Proof. Taking $\mathfrak{B}-$ covariant derivative for eq. (3.4) twice with respect to x^m and x^l, respectively, using eqs. (3.2) and (3.3) in the resulting equation, we get

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_mP^i_{jkh}\;=\;&a_{lm}(\lambda_jC^i_{kh}-\lambda^iC_{jkh}+C^r_{jk}P^i_{rh}-C^i_{rk}P^r_{jh})\\&+\{(\mathfrak{B}_l\mathfrak{B}_m\lambda_j)+(\mathfrak{B}_m\lambda_j)\lambda_l+(\mathfrak{B}_l\lambda_j)\lambda_m\}C^i_{kh}\\&-\{(\mathfrak{B}_l\mathfrak{B}_m\lambda^i)-(\mathfrak{B}_m\lambda^i)\lambda_l-(\mathfrak{B}_l\lambda^i)\lambda_m\}C_{jkh}\\&+\{\lambda_m(\mathfrak{B}_lP^i_{rh})+\lambda_l(\mathfrak{B}_mP^i_{rh})+(\mathfrak{B}_l\mathfrak{B}_mP^i_{rh})\}C^r_{jk}\\&-\{\lambda_m(\mathfrak{B}_lP^r_{jh})+\lambda_l(\mathfrak{B}_mP^r_{jh})+(\mathfrak{B}_l\mathfrak{B}_mP^r_{jh})\}C^i_{rk}.\end{aligned}$$

Using eq. (3.4) in above equation, we get

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_mP^i_{jkh}\;=\;&a_{lm}P^i_{jkh}+\{(\mathfrak{B}_l\mathfrak{B}_m\lambda_j)+(\mathfrak{B}_m\lambda_j)\lambda_l+(\mathfrak{B}_l\lambda_j)\lambda_m\}C^i_{kh}\\&-\{(\mathfrak{B}_l\mathfrak{B}_m\lambda^i)-(\mathfrak{B}_m\lambda^i)\lambda_l-(\mathfrak{B}_l\lambda^i)\lambda_m\}C_{jkh}\\&+\{\lambda_m(\mathfrak{B}_lP^i_{rh})+\lambda_l(\mathfrak{B}_mP^i_{rh})+(\mathfrak{B}_l\mathfrak{B}_mP^i_{rh})\}C^r_{jk}\\&-\{\lambda_m(\mathfrak{B}_lP^r_{jh})+\lambda_l(\mathfrak{B}_mP^r_{jh})+(\mathfrak{B}_l\mathfrak{B}_mP^r_{jh})\}C^i_{rk}.\end{aligned}\tag{3.10}$$

This shows that

$$\mathfrak{B}_l\mathfrak{B}_m P^i_{jkh} = a_{lm} P^i_{jkh}. \tag{3.11}$$

if and only if eq. (3.5) holds.

Transvecting eq. (3.10) by y^j, using (2.1), (2.4) and (2.6), we get

$$\begin{aligned} \mathfrak{B}_l\mathfrak{B}_m P^i_{kh} &= a_{lm} P^i_{kh} + \{(\mathfrak{B}_l\mathfrak{B}_m\lambda_j) + (\mathfrak{B}_m\lambda_j)\lambda_l + (\mathfrak{B}_l\lambda_j)\lambda_m\} C^i_{kh} y^j \\ &\quad -\{\lambda_m(\mathfrak{B}_l P^r_{jh}) + \lambda_l(\mathfrak{B}_m P^r_{jh}) + (\mathfrak{B}_l\mathfrak{B}_m P^r_{jh})\} C^i_{rk} y^j \end{aligned} \tag{3.12}$$

This shows that

$$\mathfrak{B}_l\mathfrak{B}_m P^i_{kh} = a_{lm} P^i_{kh}. \tag{3.13}$$

if and only if eq. (3.6) holds.

Contracting the indices i and h in eq. (3.10), using (2.2) and (2.8), we get

$$\begin{aligned} \mathfrak{B}_l\mathfrak{B}_m P_{jk} &= a_{lm} P_{jk} + \{(\mathfrak{B}_l\mathfrak{B}_m\lambda_j) + (\mathfrak{B}_m\lambda_j)\lambda_l + (\mathfrak{B}_l\lambda_j)\lambda_m\} C_k \\ &\quad -\{(\mathfrak{B}_l\mathfrak{B}_m\lambda^i) - (\mathfrak{B}_m\lambda^i)\lambda_l - (\mathfrak{B}_l\lambda^i)\lambda_m\} C_{jki} \\ &\quad +\{\lambda_m(\mathfrak{B}_l P_r) + \lambda_l(\mathfrak{B}_m P_r) + (\mathfrak{B}_l\mathfrak{B}_m P_r)\} C^r_{jk} \\ &\quad -\{\lambda_m(\mathfrak{B}_l P^r_{ji}) + \lambda_l(\mathfrak{B}_m P^r_{ji}) + (\mathfrak{B}_l\mathfrak{B}_m P^r_{ji})\} C^i_{rk}. \end{aligned} \tag{3.14}$$

This shows that

$$\mathfrak{B}_l\mathfrak{B}_m P_{jk} = a_{lm} P_{jk}. \tag{3.15}$$

if and only if eq. (3.7) holds.

Contracting the indices i and h in eq. (3.12), using (2.2) and (2.8), we get

$$\begin{aligned} \mathfrak{B}_l\mathfrak{B}_m P_k &= a_{lm} P_k + \{(\mathfrak{B}_l\mathfrak{B}_m\lambda_j) + (\mathfrak{B}_m\lambda_j)\lambda_l + (\mathfrak{B}_l\lambda_j)\lambda_m\} C_k y^j \\ &\quad -\{\lambda_m(\mathfrak{B}_l P^r_{ji}) + \lambda_l(\mathfrak{B}_m P^r_{ji}) + (\mathfrak{B}_l\mathfrak{B}_m P^r_{ji})\} C^i_{rk} y^j \end{aligned} \tag{3.16}$$

This shows that

$$\mathfrak{B}_l\mathfrak{B}_m P_k = a_{lm} P_k \tag{3.17}$$

if and only if eq. (3.8) holds.

Transvecting eq. (3.16) by y^k, using (2.2), (2.4) and (2.8), we get

$$\mathfrak{B}_l\mathfrak{B}_m P = a_{lm} P + \{(\mathfrak{B}_l\mathfrak{B}_m\lambda_j) + (\mathfrak{B}_m\lambda_j)\lambda_l + (\mathfrak{B}_l\lambda_j)\lambda_m\} C y^j \tag{3.18}$$

This shows that

$$\mathfrak{B}_l\mathfrak{B}_m P = a_{lm} P. \tag{3.19}$$

if and only if eq. (3.9) holds.

Consequently, from eqs. (3.11), (3.13), (3.15), (3.17) and (3.19), we deduce that the behavior of P^i_{jkh}, P^i_{kh}, P_{jk}, P_k and P in $\mathfrak{B}C - BRF_n$ as birecurrent if and only if eqs. (3.5), (3.6), (3.7), (3.8) and (3.9), respectively hold. Hence, we have proved this theorem.

4. Special Spaces of $\mathfrak{B}C-$Birecurrent Space

In this section, we merge the $\mathfrak{B}C-$ birecurrent space with particular spaces of Finsler space to get new spaces.

4.1. A $P2-$Like $\mathfrak{B}C-$Birecurrent Space.

Definition 4.1. *The $\mathfrak{B}C-$birecurrent space which is $P2-$like space, i.e. satisfies the condition (2.9), will be called a $P2-$like $\mathfrak{B}C-$birecurrent space and will be denoted briefly by $P2-$like$-\mathfrak{B}C-BRF_n$.*

In next theorem we get the necessary and sufficient condition for some tensors which behave as birecurrent tensor in $P2-$like$-\mathfrak{B}C-BRF_n$.

Theorem 4.2. *In $P2-$like$-\mathfrak{B}C-BRF_n$, Cartans second curvature tensor P^i_{jkh}, torsion tensor P^i_{kh}, $P-$Ricci tensor P_{jk} and curvature vector P_k satisfy the birecurrence property if and only if*

$$\begin{aligned}&\{(\mathfrak{B}_l\mathfrak{B}_m\vartheta_j)+(\mathfrak{B}_m\vartheta_j)\lambda_l+(\mathfrak{B}_l\vartheta_j)\lambda_m\}C^i_{kh}\\&-\{(\mathfrak{B}_l\mathfrak{B}_m\vartheta^i)-(\mathfrak{B}_m\vartheta^i)\lambda_l-(\mathfrak{B}_l\vartheta^i)\lambda_m\}C_{jkh}=0,\end{aligned}\tag{4.1}$$

$$\{(\mathfrak{B}_l\mathfrak{B}_m\vartheta_j)+(\mathfrak{B}_m\vartheta_j)\lambda_l+(\mathfrak{B}_l\vartheta_j)\lambda_m\}C^i_{kh}y^j=0,\tag{4.2}$$

$$\begin{aligned}&\{(\mathfrak{B}_l\mathfrak{B}_m\vartheta_j)+(\mathfrak{B}_m\vartheta_j)\lambda_l+(\mathfrak{B}_l\vartheta_j)\lambda_m\}C_k\\&-\{(\mathfrak{B}_l\mathfrak{B}_m\vartheta^i)-(\mathfrak{B}_m\vartheta^i)\lambda_l-(\mathfrak{B}_l\vartheta^i)\lambda_m\}C_{jki}=0\end{aligned}\tag{4.3}$$

and

$$\{(\mathfrak{B}_l\mathfrak{B}_m\vartheta_j)+(\mathfrak{B}_m\vartheta_j)\lambda_l+(\mathfrak{B}_l\vartheta_j)\lambda_m\}C_ky^j=0.\tag{4.4}$$

respectively.

Proof. Taking $\mathfrak{B}-$ covariant derivative for the condition (2.9) twice with respect to x^m and x^l, respectively, using eqs. (3.2) and (3.3) in the resulting equation, we get

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_mP^i_{jkh} &= a_{lm}(\vartheta_jC^i_{kh}-\vartheta^iC_{jkh})\\&+\{(\mathfrak{B}_l\mathfrak{B}_m\vartheta_j)+(\mathfrak{B}_m\vartheta_j)\lambda_l+(\mathfrak{B}_l\vartheta_j)\lambda_m\}C^i_{kh}\\&-\{(\mathfrak{B}_l\mathfrak{B}_m\lambda^i)-(\mathfrak{B}_m\vartheta^i)\lambda_l-(\mathfrak{B}_l\vartheta^i)\lambda_m\}C_{jkh}.\end{aligned}$$

Using the condition (2.9) in above equation, we get

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_mP^i_{jkh} &= a_{lm}P^i_{jkh}+\{(\mathfrak{B}_l\mathfrak{B}_m\vartheta_j)+(\mathfrak{B}_m\vartheta_j)\lambda_l+(\mathfrak{B}_l\vartheta_j)\lambda_m\}C^i_{kh}\\&-\{(\mathfrak{B}_l\mathfrak{B}_m\vartheta^i)-(\mathfrak{B}_m\vartheta^i)\lambda_l-(\mathfrak{B}_l\vartheta^i)\lambda_m\}C_{jkh}.\end{aligned}\tag{4.5}$$

This shows that $\mathfrak{B}_l\mathfrak{B}_mP^i_{jkh}=a_{lm}P^i_{jkh}$ if and only if eq. (4.1) holds.

Transvecting eq. (4.5) by y^j using (2.1), (2.4) and (2.6), we get

$$\mathfrak{B}_l\mathfrak{B}_mP^i_{kh}=a_{lm}P^i_{kh}+\{(\mathfrak{B}_l\mathfrak{B}_m\vartheta_j)+(\mathfrak{B}_m\vartheta_j)\lambda_l+(\mathfrak{B}_l\vartheta_j)\lambda_m\}C^i_{kh}y^j.\tag{4.6}$$

This shows that $\mathfrak{B}_l\mathfrak{B}_mP^i_{kh}=a_{lm}P^i_{kh}$ if and only if eq. (4.2) holds.

Contracting the indices i and h in eq. (4.5), using (2.2) and (2.8), we get

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m P_{jk} &= a_{lm}P_{jk} + \{(\mathfrak{B}_l\mathfrak{B}_m\vartheta_j) + (\mathfrak{B}_m\vartheta_j)\lambda_l + (\mathfrak{B}_l\vartheta_j)\lambda_m\}C_k \\ &\quad -\{(\mathfrak{B}_l\mathfrak{B}_m\vartheta^i) - (\mathfrak{B}_m\vartheta^i)\lambda_l - (\mathfrak{B}_l\vartheta^i)\lambda_m\}C_{jki}. \end{aligned} \tag{4.7}$$

This shows that $\mathfrak{B}_l\mathfrak{B}_m P_{jk} = a_{lm}P_{jk}$ if and only if eq. (4.3) holds.

Contracting the indices i and h in eq. (4.6), using (2.2) and (2.8), we get

$$\mathfrak{B}_l\mathfrak{B}_m P_k = a_{lm}P_k + \{(\mathfrak{B}_l\mathfrak{B}_m\vartheta_j) + (\mathfrak{B}_m\vartheta_j)\lambda_l + (\mathfrak{B}_l\vartheta_j)\lambda_m\}C_k y^j \tag{4.8}$$

This shows that $\mathfrak{B}_l\mathfrak{B}_m P_k = a_{lm}P_k$ if and only if eq. (4.4) holds.

Consequently, from previous equations we proved that the behavior of P^i_{jkh}, P^i_{kh}, P_{jk} and P_k in $P2-$like$-\mathfrak{B}C - BRF_n$ as birecurrent if and only if eqs. (4.1), (4.2), (4.3) and (4.4), respectively hold. Hence, we have proved this theorem.

4.2. A P^*- $\mathfrak{B}C-$Birecurrent Space.

Definition 4.3. *The $\mathfrak{B}C-$birecurrent space which is P^*- space, i.e. satisfies the condition* (2.10)*, will be called a $P^* - \mathfrak{B}C-$birecurrent space and will be denoted briefly by $P^* - \mathfrak{B}C - BRF_n$.*

In next theorem we get the necessary and sufficient condition for some tensors which behave as recurrent tensor in $P^*- \mathfrak{B}C - BRF_n$.

Theorem 4.4. *In $P^*- \mathfrak{B}C - BRF_n$, the torsion tensor P^i_{kh}, curvature vector P_k and curvature scalar P satisfy the birecurrence property if and only if*

$$\left[\mathfrak{B}_l\mathfrak{B}_m\vartheta + \lambda_l\left(\mathfrak{B}_m\vartheta\right) + \lambda_m\left(\mathfrak{B}_l\vartheta\right)\right]C^i_{kh} = 0, \tag{4.9}$$

$$\left[\mathfrak{B}_l\mathfrak{B}_m\vartheta + \lambda_l\left(\mathfrak{B}_m\vartheta\right) + \lambda_m\left(\mathfrak{B}_l\vartheta\right)\right]C_k = 0 \tag{4.10}$$

and

$$\left[\mathfrak{B}_l\mathfrak{B}_m\vartheta + \lambda_l\left(\mathfrak{B}_m\vartheta\right) + \lambda_m\left(\mathfrak{B}_l\vartheta\right)\right]C = 0, \tag{4.11}$$

respectively.

Proof. Taking $\mathfrak{B}-$ covariant derivative for the condition (2.10) twice with respect to x^m and x^l, respectively, using eqs.(3.2) and (3.3) in the resulting equation, we get

$$\mathfrak{B}_l\mathfrak{B}_m P^i_{kh} = \vartheta a_{lm}C^i_{kh} + \left[\mathfrak{B}_l\mathfrak{B}_m\vartheta + \lambda_l\left(\mathfrak{B}_m\vartheta\right) + \lambda_m\left(\mathfrak{B}_l\vartheta\right)\right]C^i_{kh}.$$

Using the condition (2.10) in above equation, we get

$$\mathfrak{B}_l\mathfrak{B}_m P^i_{kh} = a_{lm}P^i_{kh} + \left[\mathfrak{B}_l\mathfrak{B}_m\vartheta + \lambda_l\left(\mathfrak{B}_m\vartheta\right) + \lambda_m\left(\mathfrak{B}_l\vartheta\right)\right]C^i_{kh} \tag{4.12}$$

This shows that $\mathfrak{B}_l\mathfrak{B}_m P^i_{kh} = a_{lm}P^i_{kh}$ if and only if eq. (4.9) holds.

Contracting the indices i and h in eq. (4.12), using (2.2) and (2.8), we get

$$\mathfrak{B}_l\mathfrak{B}_m P_k = a_{lm}P_k + \left[\mathfrak{B}_l\mathfrak{B}_m\vartheta + \lambda_l\left(\mathfrak{B}_m\vartheta\right) + \lambda_m\left(\mathfrak{B}_l\vartheta\right)\right]C_k \tag{4.13}$$

This shows that $\mathfrak{B}_l\mathfrak{B}_m P_k = a_{lm}P_k$ if and only if eq. (4.10) holds.

Transvecting eq. (4.13) by y^k, using (2.2) and (2.8), we get

$$\mathfrak{B}_l\mathfrak{B}_m P = a_{lm}P + [\mathfrak{B}_l\mathfrak{B}_m\vartheta + \lambda_l(\mathfrak{B}_m\vartheta) + \lambda_m(\mathfrak{B}_l\vartheta)]\,C \tag{4.14}$$

This shows that $\mathfrak{B}_l\mathfrak{B}_m P = a_{lm}P$ if and only if eq. (4.11) holds.

Consequently, from previous equations we proved that the behavior of P^i_{kh}, P_k and P in $P^*-\ \mathfrak{B}C - BRF_n$ as birecurrent if and only if eqs. (4.9), (4.10) and (4.11), respectively hold. Hence, we have proved this theorem.

4.3. A $P-$Reducible $-\mathfrak{B}C-$Birecurrent Space.

Definition 4.5. *The $\mathfrak{B}C-$birecurrent space which is generalized P-reducible space, i.e. satisfies the condition (2.11), will be called a $P-$reducible $-\mathfrak{B}C-$birecurrent space and will be denoted briefly by $P-$reducible $-\mathfrak{B}C - BRF_n$.*

In next theorem we get the necessary and sufficient condition for some tensors which be non-vanishing in $P-$reducible $-\mathfrak{B}C - BRF_n$.

Theorem 4.6. *In $P-$reducible$-\mathfrak{B}C - BRF_n$, Berwalds covariant derivative of the second order for the tensors $\vartheta(h^i_kC_h + h_{kh}C^i + h^i_hC_k)$ and $\vartheta(h_{jk}C_h + h_{kh}C_j + h_{hj}C_k)$ are given by*

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m\left[\vartheta(h^i_kC_h + h_{kh}C^j + h^i_hC_k)\right] &= a_{lm}\vartheta(h^i_kC_h + h_{kh}C^i + h^i_hC_k)\\ &- [\mathfrak{B}_l\mathfrak{B}_m\lambda + (\mathfrak{B}_m\lambda)\,\lambda_l + (\mathfrak{B}_l\lambda)\,\lambda_m]\,C^i_{kh}\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}&\mathfrak{B}_l\mathfrak{B}_m[\vartheta(h_{jk}C_h + h_{kh}C_j + h_{hj}C_k)] = a_{lm}\vartheta(h_{jk}C_h + h_{kh}C_j + h_{hj}C_k)\\ &-[\mathfrak{B}_l\mathfrak{B}_m\lambda + (\mathfrak{B}_m\lambda)\lambda_l + (\mathfrak{B}_l\lambda)\lambda_m]C_{jkh}\end{aligned} \tag{4.16}$$

if and only if the torsion tensor P^i_{kh} and associate torsion tensor P_{jkh} satisfy the birecurrence property, respectively.

Proof. Transvecting the condition (2.11) by g^{ij}, using (2.7) and (2.2), we get

$$P^i_{kh} = \lambda C^i_{kh} + \vartheta(h^i_kC_h + h_{kh}C^i + h^i_hC_k), \tag{4.17}$$

where $h^i_k = g^{ij}h_{jk}$ and $C^i = g^{ij}C_j$.

Taking $\mathfrak{B}-$ covariant derivative for the condition (4.17) twice with respect to x^m and x^l respectively, using eqs. (3.2) and (3.3) in the resulting equation, we get

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_mP^i_{kh} &= \lambda a_{lm}C^i_{kh} + [\mathfrak{B}_l\mathfrak{B}_m\lambda + (\mathfrak{B}_m\lambda)\,\lambda_l + (\mathfrak{B}_l\lambda)\,\lambda_m]\,C^i_{kh}\\ &+\mathfrak{B}_l\mathfrak{B}_m\left[\vartheta(h^i_kC_h + h_{kh}C^i + h^i_hC_k)\right].\end{aligned}$$

Using the condition (4.17) in above equation, we get

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_mP^i_{kh} &= a_{lm}P^i_{kh} - a_{lm}\vartheta(h^i_kC_h + h_{kh}C^i + h^i_hC_k) + [\mathfrak{B}_l\mathfrak{B}_m\lambda + (\mathfrak{B}_m\lambda)\lambda_l\\ &+ (\mathfrak{B}_l\lambda)\,\lambda_m]C^i_{kh} + \mathfrak{B}_l\mathfrak{B}_m[\vartheta(h^i_kC_h + h_{kh}C^j + h^i_hC_k)].\end{aligned}$$

Then Berwalds covariant derivative of the second order for the tensor $\varphi\Big(h^i_k C_h + h_{kh}C^i + h^i_h C_k\Big)$ satisfies eq. (4.15) if and only if

$$\mathfrak{B}_l\mathfrak{B}_m P^i_{kh} = a_{lm}P^i_{kh}.$$

The above equation refer to P^i_{kh} satisfies the birecurrence property.

Taking $\mathfrak{B}-$ covariant derivative for the condition (2.11) twice with respect to x^m and x^l, respectively, using eqs. (3.2) and (3.3) in the resulting equation, we get

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m P_{jkh} &= \lambda a_{lm}C_{jkh} + [\mathfrak{B}_l\mathfrak{B}_m\lambda + (\mathfrak{B}_m\lambda)\lambda_l + (\mathfrak{B}_l\lambda)\lambda_m]C_{jkh}\\ &\quad +\mathfrak{B}_l\mathfrak{B}_m[\vartheta(h_{jk}C_h + h_{kh}C_j + h_{hj}C_k)].\end{aligned}$$

Using the condition (2.11) in above equation, we get

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m P_{jkh} &= a_{lm}P_{jkh} - a_{lm}\vartheta(h_{jk}C_h + h_{kh}C_j + h_{hj}C_k)\\ &\quad + \left[\mathfrak{B}_l\mathfrak{B}_m\lambda + (\mathfrak{B}_m\lambda)\,\lambda_l + (\mathfrak{B}_l\lambda)\,\lambda_m\right]C_{jkh}\\ &\quad +\mathfrak{B}_l\mathfrak{B}_m\left[\vartheta(h_{jk}C_h + h_{kh}C_j + h_{hj}C_k)\right].\end{aligned}$$

Then Berwalds covariant derivative of the second order for the tensor $\varphi\Big(h_{jk}C_h + h_{kh}C_j + h_{hj}C_k\Big)$ satisfies eq. (4.16) if and only if

$$\mathfrak{B}_l\mathfrak{B}_m P_{jkh} = a_{lm}P_{jkh}.$$

The above equation refer to P_{jkh} satisfies the birecurrence property. Hence, we have proved this theorem.

5. **An Example**

In this section, we give an example to clarify the proved findings.

Example 5.1. *The behavior of the torsion tensor P^i_{kh} as birecurrent if and only if the projection on indicatrix for it is also birecurrent.*

Firstly, since the torsion tensor P^i_{kh} behaves as birecurrent, then the condition (3.13) is satisfied. In view of (2.12), the projection of the torsion tensor P^i_{kh} on indicatrix is given by

$$p.P^i_{kh} = P^a_{bc}h^i_a h^b_k h^c_h. \tag{5.1}$$

Using $\mathfrak{B}-$covariant derivative for eq. (5.1) twice with respect to x^m and x^l, respectively, using the condition (3.13) and the fact that h^a_b is covariant constant in above equation, we get

$$\mathfrak{B}_l\mathfrak{B}_m\left(p.P^i_{kh}\right) = a_{lm}\left(P^a_{bc}h^i_a h^b_k h^c_h\right).$$

Using eq. (5.1) in above equation, we get

$$\mathfrak{B}_l\mathfrak{B}_m\left(p.P^i_{kh}\right) = a_{lm}\left(p.P^i_{kh}\right). \tag{5.2}$$

Equation (5.2) refers to the projection on indicatrix for the torsion tensor P^i_{kh} behaves as birecurrent.

Secondly, let the projection on indicatrix for the torsion tensor P^i_{kh} is birecurrent, i.e. satisfy eq. (5.2). Using (2.12) in eq. (5.2), we get

$$\mathfrak{B}_l \mathfrak{B}_m \left(P^a_{bc}\ h^i_a h^b_k h^c_h \right) = a_{lm} \left(P^a_{bc}\ h^i_a h^b_k h^c_h \right).$$

By using (2.13) in above equation, we get

$$\begin{aligned} &\mathfrak{B}_l \mathfrak{B}_m \big[P^i_{kh} - P^i_{kc} l^c l_h - P^i_{bh} l^b l_k + P^i_{bc} l^b l_k l^c l_h \\ &- P^a_{kh} l^i l_a + P^a_{kc} l^i l_a l^c l_h + P^a_{bh} l^i l_a l^b l_k - P^a_{bc} l^i l_a l^b l_k l^c l_h \big] \\ &= a_{lm} \big[P^i_{kh} - P^i_{kc} l^c l_h - P^i_{bh} l^b l_k + P^i_{bc} l^b l_k l^c l_h \\ &- P^a_{kh} l^i l_a + P^a_{kc} l^i l_a l^c l_h + P^a_{bh} l^i l_a l^b l_k - P^a_{bc} l^i l_a l^b l_k l^c l_h \big]. \end{aligned}$$

In view of (2.3) and if

$$P^a_{bc} y_a = P^a_{bc} y^b = P^a_{bc} y^c = 0,$$

then above equation can be written as

$$\mathfrak{B}_l \mathfrak{B}_m P^i_{kh} = a_{lm} P^i_{kh}.$$

The above equation means the torsion tensor P^i_{kh} behaves as birecurrent.

6. **Conclusion**

We obtained the necessary and sufficient condition for Cartan's second curvature tensor P^i_{jkh}, associate curvature tensor P_{ijkh}, torsion tensor P^i_{kh}, $P-$Ricci tensor P_{jk}, curvature vector P_k and scalar curvature P which satisfy birecurrence property in $\mathfrak{B}C - BRF_n$, $P2 -$ like $- \mathfrak{B}C - BRF_n$, $P^*- \mathfrak{B}C - BRF_n$ and $P-$reducible $-\mathfrak{B}C - BRF_n$. Furthermore, the relationship between Cartan's second curvature tensor P^i_{jkh} and $(h)hv$torsion tensor C^i_{jk} in sense of Berwald has been discussed.

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