Journal of Finsler Geometry and its Applications
Vol. 4, No. 1 (2023), pp 23-32
DOI: 10.22098/jfga.2023.12271.1084

Angle geometry between Teichmüller geodesic segments

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Abstract. In this paper we discuss some results related to angle between geodesic segments in an infinite dimensional and an asymptotic Teichmller space. Also, we construct a geodesic triangle in Universal Teichmüller space and calculate all of its interior angles.

Keywords: Asymptotic Teichmüller space; Universal Teichmüller space; geodesic segment; angle.

1. **Introduction**

It is known that on a Riemannian manifold angle between two geodesic segments is well defined by an inner product on the tangent space. In [14] author provided a notion of angle cone $\angle_{x,y,z}$ defined on triple points x, y, z in a general metric space. Teichmüller metric is induced by Finsler structure, precisely equals to Kobayashi metric [16]. Since defining angle between geodesic segments in Finsler spaces is not very obvious, hence situation is very ambiguous here. Tamassy [19], introduced Minkowski measure of angle $\alpha = \angle(a, b)$ between two rays $a, b \in T_{p_0}M$ originating from origin p_0 of $T_{p_0}M$. Since Finsler space makes its tangent space into Minkowski space, measuring angle into Finsler space can be reduced onto Minkowski space.

Apart from this, a lot of difference has been observed between angle geometry of finite and infinite dimensional Teichmüller spaces [6, 8]. For instance, as we

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AMS 2020 Mathematics Subject Classification: 30F60; 30F30; 30C75; 53C60

know that Teichmüller theorem affirms the existence of unique geodesic segment between any two points of Teichmüller space, whereas in infinite dimensional space there exists infinitely many geodesic segments.

Recently, Yao [21] gave an approach to define angle between geodesic segments in finite dimensional Teichmüller space. Later, Li and Qi [7] discovered some conditions under which existence of angle between geodesic segments in infinite dimensional space can be confirmed.

An interesting contribution has been made by Fan and Jiang in [4], authors investigate angle geometry on universal Teichmüller space. Their work indicates that universal Teichmüller space shows all the characteristics of Euclidean, hyperbolic and spherical geometry. In this paper, authors showed existence of three geodesic triangles bounded by geodesic segments in which sum of interior angles is π, less than π and greater than π respectively.

Some trivial queries like what can we say about sum of the interior angles of geodesic triangle in infinite dimensional Teichmüller space and how the angle geometry varies when we switch between finite and infinite dimensional Teichmüller spaces have been discussed in [6, 3, 21]. In applied sense, it has been observed that due to its promising geometric environment for non-perturbative version of bosonic string theory, universal Teichmüller space is an important object to study string theory in physics [15, 1]. In [13], author has proved that in hyperbolic surfaces S of finite type the set $A(S)$ of angles between closed geodesics on S contains finitely many rational multiples of π. In [5], Hu and Shen have shown that sum of interiors angles of a geodesic triangle in an infinite dimensional Teichmüller space, lies between π and 3π.

In [9], Liu, Su and Zhong, have shown that in infinite dimensional Teichmüller space, angle between geodesic rays defined by using law of cosines does not always exist which implies infinite dimensional Teichmüller space is not CAT(k) space for any k. Another interesting fact about Teichmüller space is that this space is of non-positive curvature. In [10], author has shown that two Teichmüller geodesic rays starting from a common point are not divergent. Minsky [12], showed that Teichmüller metric near thin regions of Teichmüller spaces reveals characteristics of positive curvature. Masur and Wolf[11] proved that Teichmüller space with Teichmüller metric is not Gromov-hyperbolic.

In this paper we have covered the following objectives:

- We construct a geodesic triangle and calculate sum of interior angles in Universal Teichmüller space.
- In asymptotic Teichmüller space, we find the bounds of angle between two geodesic segments.

2. **Preliminaries**

The notion of angle between two joint curves in general metric space with common end point can be seen as follows:

Let (X, d) be general metric space. Let α and β be two continuous curves in X with common point p. For any $r > 0$, choose two points $x(r)$ and $y(r)$ on curves α and β respectively such that length of sub-curves between p and $x(r)$ is equal to length of sub curves between p and $y(r)$ and equals to r. Then angle $\theta \in [0, \pi]$ between α and β at point p is defined as :

$$2sin\frac{\theta}{2} = \lim_{r\to 0} \frac{d(x(r), y(r))}{r},$$

if limit exists. Also, angle here reduces to 0, if $\alpha \equiv \beta$ in the neighbourhood of p, and it reduces to π when $\alpha \cup \beta$ is a geodesic at p.

Alexandrov proposed a method to define angle between two geodesic segments. Let γ_1 and γ_2 be two geodesic segments such that $\gamma_1(0) = \gamma_2(0)$, then

$$\angle(\gamma_1, \gamma_2) = arccos\left(\lim_{s,t\to 0} \frac{s^2 + t^2 - d^2(\gamma_1(s), \gamma_2(t))}{2st}\right).$$

Using Alexandrov notion of angle, Su and Zhong [18] defined comparison angle between three points of $T(S)$. A marked Riemann surface (X, ϕ) is a conformal structure X on surface S, endowed with orientation preserving homeomorphism $\phi : S \to X$. Teichmüller space $T(S)$ of surface S is a set of equivalence classes of marked Riemann surfaces, where $(X, \phi) \sim (Y, \psi)$, if there exists a conformal map $\rho : X \to Y$ homotopic to $\psi \circ \phi^{-1}$. The space $T(\triangle)$, where $\triangle$ is the unit disc $\{z|0 < z < 1\}$ is known as Universal Teichmüller space, since it contains all the others. A $(-1, 1)$-measurable form $\mu = \mu(z)\frac{d\bar{z}}{dz}$ which satisfies

$$||\mu||_\infty = ess\sup_{z\in X} |\mu(z)| < \infty,$$

is called Beltrami differential. Suppose the space of Beltrami differentials on X is denoted by $B(X)$. A Beltrami coefficient μ is extremal if $||\mu||_\infty = k(f)$. Let $X \in T(S), \mu \in B(X)$ with $||\mu||_\infty < 1$. Consider the Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z}.$$

The solution of above Beltrami equation is quasiconformal deformation of X, denoted by f^μ. The maximal dilatation of a quasi-conformal mapping f with Beltrami differential μ is

$$K(f) = \frac{1 + ||\mu||_\infty}{1 - ||\mu||_\infty}.$$

Definition 2.1. *Teichmüller distance between two points in X and Y in $T(S)$ is defined as*

$$d_T(X, Y) := \frac{1}{2} \inf_{f\simeq f_2\circ f_1^{-1}} logK(f),$$

where infimum is taken over all quasiconformal maps $f : X \to Y$ that are homotopic to $f_2 \circ f_1^{-1}$. Teichmüller metric is induced from Finsler structure.

Definition 2.2. *Let V be an n-dimensional real vector space endowed with smooth norm F on $V\backslash\{0\}$ satisfying the following conditions:*

(1) $F(u) \geq 0 \;\forall\; u \;\in V$,

(2) $F(\lambda u) = \lambda F(u) \;\forall\; \lambda > 0$, *i.e., F is positively homogeneous,*

(3) *The Hessian matrix* $(g_{ij}) := \left(\left[\frac{1}{2}F^2\right]_{y^i y^j}\right)$, *is positive definite at every point of $V\backslash\{0\}$. The pair (V, F) is called Minkowski space and F is called Minkowski norm.*

Definition 2.3. *Let M be a connected (smooth) manifold. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which satisfies:*

(1) *F is smooth on slit tangent bundle $TM\backslash\{0\}$.*

(2) *The restriction of F to any $T_xM, x \in M$ is a Minkowski norm.*

The space (M, F) is called Finsler space.

Definition 2.4. *Let $\gamma : [0, 1] \to M$ be a C^1-curve. Then Finsler length of γ is defined as*

$$L(\gamma) = \int_0^1 F(\gamma(t), \gamma^{'}(t))dt.$$

Further, Finsler distance between two points $p, q \in M$ is defined as

$$d_F(p, q) = inf_\gamma L(\gamma),$$

where infimum is taken over all piecewise C^1 curves joining p and q.

Definition 2.5. *For any quasi conformal mapping $f : R \longrightarrow S$,*

$$K_z(f) = \frac{1 + |f_{\bar{z}}/f_z|}{1 - |f_{\bar{z}}/f_z|}.$$

We say that f is asymptotically conformal if for every $\epsilon > 0$ there is a compact subset E of R such that $K_z(f) \leq 1 + \epsilon$ for z in $R - E$. The definition of this new equivalence is same as Definition 2.1 with conformal being replaced by asymptotically conformal.

Definition 2.6. *Let $P = [\phi]$ and $Q = [\psi]$ be two points in universal Teichmüller space $T(S)$. Let $\eta \in [f^\psi \circ (f^\phi)^{-1}]$ be extremal Beltrami coefficient. Then geodesic segment γ_{PQ} connecting P and Q is defined as*

$$[f^{t\eta} \circ f^\phi | 0 \leq t \leq 1].$$

Definition 2.7. *Let $P = [\phi], Q = [\psi]$ and $R = [\eta]$ be three points in $T(S)$ and γ_{PQ}, γ_{PR} and γ_{QR} be the segments connecting P to Q, P to R and Q to R, respectively. Then γ_{PQ}, γ_{PR} and γ_{QR} form a geodesic triangle $\triangle PQR$, if γ_{PQ} or γ_{QP}, γ_{PR} or γ_{RP} and γ_{QR} or γ_{RQ} are of the form given in definition 2.6.*

Definition 2.8. *Consider a geodesic triangle $\triangle PQR$ as defined in definition 2.7. Then $\angle P$ is defined as*

$$\begin{aligned}\angle P &= 2\arcsin\left\{\frac{1}{2}\lim_{r\to 0}\frac{d_T(\gamma_{PQ}(r),\gamma_{PR}(r))}{d_T(\gamma_{PQ}(r),P)}\right\}\\ &= 2\arcsin\left\{\frac{1}{2}\lim_{r\to 0}\frac{d_T(\gamma_{PQ}(r),\gamma_{PR}(r))}{r}\right\},\end{aligned}$$

if the limit exists. Similarly $\angle Q$ and $\angle R$ can be defined.

3. **Universal Teichmüller Space**

Let $T(S)$ be the universal Teichmüller space. In [4],Fan-Jiang have constructed geodesic triangles to prove that in universal Teichmüller space there exists geodesic triangles with interior angle sum equal to π, less than π and greater than π.

In order to prove our result, let us recall these Lemma from [17].

Lemma 3.1. [17] *Let E be a subset in ∂S. Let $F_K(x,y) = Kx + iy$, where K is a positive scalar. Consider*

$$Q(F_K, E) = \{f|f \textit{ is a quasi conformal mapping of } S \textit{ onto } F_K(x,y),\ f|_K = F_K|_E\}.$$

Then, F_K is extremal in $Q(F_K, E)$.

Using Lemma 3.1, in [4] authors have proved the following Lemma, which we use further.

Lemma 3.2. *Define*

$$f(x,y) = \begin{cases} K_1x + iy, & (x,y)\in S,\ x\geq 0\\ K_2x+iy, & (x,y)\in S,\ x<0\end{cases}$$

where K_1 and K_2 are two positive constants. Then μ_f is extremal in $[f]$.

In this section, we construct an example of a geodesic triangle and determine all of its interior angles. For $0 < k < 1$, consider

$$\phi(x,y) = \begin{cases} -k, & (x,y)\in S,\ x\geq 0\\ k, & (x,y)\in S,\ x<0\end{cases}$$

$$\psi(x,y) = \begin{cases} -k, & (x,y)\in S,\ x\leq 0\\ 0, & (x,y)\in S,\ x<0\end{cases}$$

Clearly, both ϕ and ψ are extremal.

Let $P = [0], Q = [\phi]$ and $R = [\psi]$ be three points of $T(S)$. Then,

$$\alpha_{PQ} = \Big\{[t\phi]\mid 0\leq t\leq 1\},\ \alpha_{PR} = \{[t\psi]\mid 0\leq t\leq 1\Big\}$$

are geodesic segments connecting P to Q and P to R, respectively.

Now, in order to calculate third geodesic segment connecting R to Q, we determine extremal Beltrami coefficient $\eta(t)$ in $[f^\phi\circ(f^\psi)^{-1}]$, defined as

$$\begin{aligned}\eta(t) &= \mu_{f^\phi\circ(f^\psi)^{-1}} = \left(\frac{\phi(t)-\psi(t)}{1-\bar{\psi(t)}\phi(t)}.\frac{\partial_z f^{\psi(t)}}{\partial_z \bar{f}^{\psi(t)}}\right)\circ(f^\psi)^{-1} \\ &= \begin{cases}0, & (x,y)\in S,\ x\geq 0\\ k, & (x,y)\in S,\ x<0\end{cases}.\end{aligned}$$

Since $k>0$, it is obvious that η is extremal. Hence

$$\alpha_{RQ} = \Big\{[f^{tn}\circ f^\psi] \mid 0\leq t\leq 1\Big\},$$

is the geodesic segment connecting R to Q.

Now we are interested in calculating all three interior angles of $\triangle PQR$ with geodesics segments α_{PQ},α_{PR} and α_{RQ} as the edges of triangle.
It is trivial to see that

$$\alpha_{PQ}(s) = \left[\frac{s}{k}\phi\right], \qquad \alpha_{PR}(r) = \left[\frac{s}{k}\psi\right].$$

Recall that

$$\angle P = 2sin^{-1}\left\{\frac{1}{2}\lim_{s\to 0}\frac{d_T(\alpha_{PQ}(s),\alpha_{PR}(s))}{s}\right\}.$$

We define

$$\eta_s = \left(\frac{\frac{s}{k}\phi-\frac{s}{k}\psi}{1-\frac{s}{k}\phi\bar{\frac{s}{k}}\psi}.\frac{f_z^{\frac{s}{k}\psi}}{f_z^{\bar{\frac{s}{k}}\psi}}\right)\circ(f^{\frac{s}{k}\psi})^{-1} = \begin{cases}0, & (x,y)\in S,\ x\geq 0\\ s, & (x,y)\in S,\ x<0\end{cases}, \tag{3.1}$$

as η_s is extremal in $\left[f^{\frac{s}{k}\phi}\circ(f^{\frac{s}{k}\psi})^{-1}\right]$. Now we are ready to calculate Teichmüller distance and $\angle P$.

$$d_T(\gamma_{PR}(s),\gamma_{PQ}(s)) = \frac{1}{2}log\frac{1+s}{1-s},$$

$$\angle P = 2arcsin\left\{\frac{1}{2}\lim_{s\to 0}\frac{\frac{1}{2}log\frac{1+s}{1-s}}{s}\right\}.$$

Using basic calculus, we get that $\angle P = \dfrac{\pi}{3}$. Next, we calculate $\angle R$.
Let us denote

$$s' = \frac{k-s}{1-ks}.$$

It is easy to check that $d_T(\alpha_{RP}(s),P) = d_T(\alpha_{PR}(s'),P)$ and

$$\alpha_{RQ}(s) = \left[f^{\frac{s}{k}\eta}\circ f^\psi\right] = [\eta'],$$

where

$$\eta' = \begin{cases}k, & (x,y)\in S,\ x\geq 0\\ r, & (x,y)\in S,\ x<0\end{cases}. \tag{3.2}$$

Similarly, we can define

$$
\begin{aligned}
\eta'_s &= \mu_{f^{\frac{s'}{k}\psi}\circ(f^{\eta'})^{-1}} = \left(\frac{\frac{s'}{k}\psi-\eta'}{1-\dfrac{s'}{k}\psi\bar{\eta'}}.\frac{f^{\eta'}_z}{\bar{f^{\eta'}_z}}\right)\circ f^{(\eta')^{-1}} \\
&= \begin{cases} \dfrac{-s'+k}{1-s'k}, & (x,y)\in S,\ x\geq 0, \\ -s, & (x,y)\in S,\ x<0 \end{cases}.
\end{aligned}
$$

Now, since

$$
\Big|\frac{-s'+k}{1-s'k}\Big| = s,
$$

we get

$$
d_T(\alpha_{RQ}(s),\alpha_{RP}(S)) = \frac{1}{2}log\frac{1+s}{1-s}.
$$

Hence, it is trivial to see that $\angle R = \dfrac{\pi}{3}$.

Now, in order to calculate $\angle Q$, consider the following geodesic segments

$$
\alpha_{QR}(s) = \alpha_{RQ}(s') = [f^{\frac{s'}{k}\eta\circ f^{\psi}}] = [\eta'']
$$

and

$$
\eta'' = \begin{cases} -k, & (x,y)\in S,\ x\geq 0 \\ s', & (x,y)\in S,\ x<0 \end{cases}, \tag{3.3}
$$

$$
\begin{aligned}
\eta''_s &= \mu_{f^{\frac{s'}{k}\phi}\circ(f^{\eta''})^{-1}} = \left(\frac{\frac{s'}{k}\phi-\eta''}{1-\frac{s'}{k}\phi\bar{\eta''}}.\frac{f^{\eta''}_z}{\bar{f^{\eta''}_z}}\right)\circ f^{(\eta'')^{-1}} \\
&= \begin{cases} \dfrac{-s'+k}{1-s'k}, & (x,y)\in S,\ x\geq 0 \\ 0, & (x,y)\in S,\ x<0 \end{cases}.
\end{aligned}
$$

η''_s is extremal in $[f^{\frac{s'}{k}\phi}\circ(f^{\eta''})^{-1}]$. This implies

$$
d_T(\alpha_{QR}(s),\alpha_{QP}(s)) = \frac{1}{2}log\frac{1+s}{1-s}.
$$

Hence,

$$
\angle Q = \frac{\pi}{3}.
$$

Then, for the constructed triangle sum of interior angles is equal to π.

4. **Infinite dimensional asymptotic Teichmüller space**

In order to prove Theorem 4.2 and Corollary 4.3, let us recall Theorem 4.1, which gives a necessary and sufficient condition for the existence of angle between two extremal Beltrami differentials ϕ and ψ.

Theorem 4.1. [9] *Let $X \in T(S)$. ϕ and ψ be two extremal Beltrami differentials. Let K be the maximal dilatation of the extremal map in the class of $f : X_{s\phi} \to X_{t\psi}$ and $k = \dfrac{K+1}{K-1}$. Then $\angle(\phi, \psi)$ exists if and only if*

$$\lim_{s,t\to 0} \frac{s^2 + t^2 - k^2}{2st}$$

exists. Moreover,

$$cos\angle(\phi, \psi) = \lim_{s,t\to 0} \frac{s^2 + t^2 - k^2}{2st}.$$

Theorem 4.2. *Suppose angle between two geodesics α_μ and α_ν exists as defined in Theorem 4.1 exists in asymptotic Teichmüller space. Then*

$$\frac{t - h(\eta(s,t))}{s} \leq \; cos\angle(\alpha_\mu, \alpha_\nu) \leq \frac{t - h(\eta(s,t))}{s} + \frac{s}{2t}.$$

Proof. In [20], strong angle between two geodesics is defined as

$$\cos\angle(\alpha_\mu, \alpha_\nu) = \frac{s^2 + t^2 - [h(\eta(s,t))]^2}{2st}$$

which can be written as

$$\cos\angle(\alpha_\mu, \alpha_\nu) = \frac{t - h(\eta(s,t))}{s} + \frac{s^2 - (h(\eta(s,t)) - t)^2}{2st}.$$

As triangle inequality gives

$$|h(\eta(s,t)) - t| \leq s,$$

which further implies

$$0 \leq \frac{s^2 - (h(\eta(s,t)) - t)^2}{2st} \leq \frac{s}{2t}$$

which directly implies

$$\frac{t - h(\eta(s,t))}{s} \leq \; cos\angle(\alpha_\mu, \alpha_\nu) \leq \left(\frac{t - h(\eta(s,t))}{s} + \frac{s}{2t}\right).$$

□

Corollary 4.3. *Let K be maximal dilatation of the extremal map in the class of $f : X_{t\phi} \to X_{t\psi}$, and $k = \frac{K+1}{K-1}$. Then $\angle(\phi,\psi)$ exists if and only if*

$$\lim_{t\to 0} \frac{2t^2 - k^2}{2t^2}$$

exists. Moreover,

$$\cos\angle(\phi,\psi) = 1 - \frac{1}{2}\left[\sup_{\theta\in Q^1(X)} \langle \phi-\psi,\theta\rangle^2\right] = 1 - \lim_{t\to 0}\frac{1}{2t^2}(||n||_T + O(||n||_\infty^2))^2$$

Proof. The first part of corollary can be seen as direct consequence of Theorem 4.1. Further,

$$\begin{aligned} cos\angle(\phi,\psi) &= \lim_{t\to 0} cos\angle(f^{t\phi}(X), f^{t\psi}(X)) \\ &= \lim_{t\to 0}\frac{1}{2}\left[2 - \frac{k^2}{t^2}\right] \\ &= \lim_{t\to 0}\frac{1}{2}\left[2 - \sup_{\theta\in Q^1(X)}\langle \phi-\psi,\theta\rangle^2 + O(t^2)\right] \\ &= \ 1 - \frac{1}{2}\left[\sup_{\theta\in Q^1(X)}\langle \phi-\psi,\theta\rangle^2\right]. \end{aligned} \tag{4.1}$$

As we know

$$||\eta||_T = \sup_{\theta\in Q^1(X)}\langle \eta,\theta\rangle,$$

$$O(||\eta||_\infty^2) = O(s+t)^2,$$

and

$$k(t) = ||\eta||_T + O(||\eta||_\infty^2).$$

Using these results in equation 4.1, we get the last part of equality. □

Acknowledgment: First author is very much thankful to UGC for providing financial assistance in terms of JRF fellowship vide letter with UGC-Ref no.:1010/(CSIR-UGC NET DEC 2018).

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Received: 02.02.2023
Accepted: 22.05.2023