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Shen's L-Process on the Chern Connection

Morteza Faghfouri[a*] and Nadereh Jazer[a]

[a]Department of Pure Mathematics. Faculty of Mathematical Sciences. University of Tabriz. Tabriz. Iran.

E-mail: faghfouri@tabrizu.ac.ir
E-mail: n.jazer@tabrizu.ac.ir

Abstract. The notion of Shen's process was introduced by Tayebi-Najafi in order to construct the Shen connection from the Berwald connection. In this paper, we study the connection which obtained by the Shen's L-process on the Chern connection. Let (M, F) be a Finsler manifold. Suppose that D is the linear torsion-free connection obtained by Shen's L-process on Chern's connection. First, we show the existence and uniqueness of D. Then, we prove that their hv-curvature coincides if and only if F is a Riemannian metric

Keywords: Chern connection, Shen connection, Shen's C and L-Processes.

1. Introduction

It is well-know that after the Einstein's formulation of general relativity, Riemannian geometry became fashionable and one of the connections, namely Levi-Civita connection, came to forefront. This connection in Riemannian geometry is both torsion-free and metric-compatible. On the other hand, Finsler geometry can be considered as a natural extension of Riemannian geometry. Likewise, the connections in Finsler geometry can be prescribed on the natural pulled-back bundle π^*TM. Examples of such connections were proposed by Taylor, Berwald, Cartan, Hashiguchi, Chern, Shen and Tayebi (see [8], [12] and [15]). Recently, Tayebi with his collaborators have defined a general class of Finsler connections which leads to a general representation of some Finsler

*Corresponding Author
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connections in Finsler geometry and yields a classification of Finsler connections into the three classes, namely, Berwald-type, Cartan-type and Shen-type connections (see [3], [4] and [15]).

In [9], Matsumoto introduced a satisfactory and truly aesthetical axiomatic description of Cartan's connection in the sixties. After the Cartan connection has been constructed, easy processes, baptized by Matsumoto "L-process" and "C-process" yield the Chern, the Hashiguchi and the Berwald connections. This means that the Chern, Berwald, and Hashiguchi connections are obtained from the Cartan connection by Matsumoto's processes, as depicted in following

$$\begin{array}{ccc} \text{Cartan connection} & \xrightarrow{C-process} & \text{Chern connection} \\ | & & | \\ L-process & & L-process \\ \downarrow & & \downarrow \\ \text{Hashiguchi connection} & \xrightarrow{C-process} & \text{Berwald connection} \end{array}$$

In [12], Shen introduced a new connection in Finsler geometry, which vanishing hv-curvature of this connection characterizes Riemannian metrics [12]. However, the Shen connection can not be constructed by Matsumoto's processes from these well-known connections. In [18], Tayebi-Najafi introduced two new processes on connections called Shen's C and L-processes and showed that the Shen connection is obtained from the Chern connection by Shen's C-process. Recently, we study the connections which obtained by Shen's C- and L-process on Berwald connection[7].

In this paper, we are going to study the connection which obtained by the Shen's L-process on the Chern connection. it is called D. We show the existence and uniqueness of D (see Theorem 2.1). Let (M, F) be a Finsler manifold. Suppose that D is the linear torsion-free connection obtained by Shen's L-process on Chern's connection. Then, we prove that their the hv-curvature coincide if and only if F is a Riemannian metric (see Theorem 3.1).

2. **Preliminaries**

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM := \cup_{x\in M} T_xM$ the tangent bundle of M. Each element of TM has the form (x, y), where $x \in M$ and $y \in T_xM$. Let $TM_0 = TM \setminus \{0\}$. The natural projection $\pi : TM \to M$ is given by $\pi(x, y) := x$.

The pull-back tangent bundle π^*TM is a vector bundle over TM_0 whose fiber π^*_vTM at $v \in TM_0$ is T_xM, where $\pi(v) = x$. Then

$$\pi^*TM = \Big\{ \big(x, y, v\big) \big| \ y \in T_xM_0, \ v \in T_xM \Big\}.$$

Some authors prefer to define connections in the pull-back tangent bundle π^*TM. From geometrical point of view, the construction of these connections on π^*TM seems to be simple because here the fibers are n-dimensional $\big(i.e., \pi^*(TM)_u = T_{\pi(u)}M,\ \forall u \in TM\big)$ thus torsions and curvatures are obtained quickly from the structure equations. When the construction is done on $T(TM)$ many geometrical objects appear twice and one needs to split $T(TM)$ in the vertical and horizontal parts where the latter is called horizontal distribution or non-linear connection. Nevertheless we do not need to split π^*TM. Indeed the connection on $\pi^*(TM)$ is the most natural connection for Physicists. In order to define curvatures, it is more convenient to consider the pull-back tangent bundle than the tangent bundle, because our geometric quantities depend on directions.

For the sake of simplicity, we denote by $\big\{\partial_i|_v := (v, \frac{\partial}{\partial x^i}|_x)\big\}_{i=1}^n$ the natural basis for π_v^*TM. In Finsler geometry, we study connections and curvatures in $(\pi^*TM, \mathbf{g})$, rather than in (TM, F). The pull-back tangent bundle π^*TM is very special tangent bundle.

A (globally defined) Finsler structure on a manifold M is a function $F : TM \to [0, \infty)$, with the following properties:
(i) F is a differentiable function on the manifold TM_0 and is continuous on the null section of the projection $\pi : TM \to M$;
(ii) $F : TM \to [0, \infty)$ is a positive scalar function;
(iii) F is positively 1-homogeneous on the fibers of tangent bundle TM;
(iv) The Hessian of F^2 with elements

$$(g_{ij}) := \left(\left[\frac{1}{2}F^2\right]_{y^iy^j}\right)$$

is positively defined on TM_0. Given a manifold M and a Finsler structure F on M, the pair (M, F) is called a Finsler manifold. F is called Riemannian if $g_{ij}(x, y)$ are independent of $y \neq 0$.

The Finsler structure F defines a fundamental tensor $\mathbf{g} : \pi^*TM \otimes \pi^*TM \to [0, \infty)$ by the formula $\mathbf{g}(\partial_i|_v, \partial_j|_v) = g_{ij}(x, y)$, where $v = y^i \frac{\partial}{\partial x^i}|_x$. Let

$$g_{ij}(x, y) := FF_{y^iy^j} + F_{y^i}F_{y^j},$$

where $F_{y^i} = \frac{\partial F}{\partial y^i}$. Then $(\pi^*TM, \mathbf{g})$ becomes a Riemannian vector bundle over TM_0.

Put

$$A_{ijk}(x, y) = \frac{1}{2}F(x, y)\frac{\partial g_{ij}}{\partial y^k}(x, y).$$

Clearly, A_{ijk} is symmetric with respect to i, j, k. The Cartan tensor $A : \pi^*TM \otimes \pi^*TM \otimes \pi^*TM \to R$ is defined by

$$A(\partial_i|_v, \partial_j|_v, \partial_k|_v) = A_{ijk}(x, y),$$

where $v = y^i \frac{\partial}{\partial x^i}|_x$ (see [20, 16]). In some literature $C_{ijk} = \frac{A_{ijk}}{F}$ is called Cartan tensor. Riemannian manifolds are characterized by $A \equiv 0$. The homogeneity condition (iii) holds in particular for positive λ. Therefore, by Euler's theorem we see that

$$y^i \frac{\partial g_{ij}}{\partial y^k}(x,y) = y^j \frac{\partial g_{ij}}{\partial y^k}(x,y) = y^k \frac{\partial g_{ij}}{\partial y^k}(x,y) = 0.$$

We recall that the canonical section ℓ is defined by:

$$\ell = \ell(x,y) = \frac{y^i}{F(x,y)} \frac{\partial}{\partial x^i} = \frac{y^i}{F} \frac{\partial}{\partial x^i} := \ell^i \frac{\partial}{\partial x^i}.$$

Put $\ell_i := g_{ij}\ell^j = F_{y^i}$. Thus the canonical section ℓ satisfies

$$g(\ell, \ell) = g_{ij} \frac{y^i}{F} \frac{y^j}{F} = 1$$

and

$$\ell^i A_{ijk} = \ell^j A_{ijk} = \ell^k A_{ijk} = 0.$$

Thus $A(X, Y, \ell) = 0$.

Given an n-dimensional Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y) \frac{\partial}{\partial y^i},$$

where $G^i = G^i(x,y)$ are called spray coefficients and given by the following

$$G^i = \frac{1}{4} g^{il} \Big[\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l} \Big]. \tag{2.1}$$

$\mathbf{G}$ is called the spray associated to F.

Define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i}|_x$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l} = \frac{\partial^2 N^i_j}{\partial y^k \partial y^l}.$$

$\mathbf{B}_y(u, v, w)$ is symmetric in u, v and w. From the homogeneity of spray coefficients, we have $\mathbf{B}_y(y, v, w) = 0$. $\mathbf{B}$ is called the Berwald curvature. Indeed, L. Berwald first discovered that the third order derivatives of spray coefficients give rise to an invariant for Finsler metrics. F is called a Berwald metric if $\mathbf{B} = \mathbf{0}$ [17]. In this case, G^i are quadratic in $y \in T_xM$ for all $x \in M$, i.e., there exists $\Gamma^i{}_{jk} = \Gamma^i{}_{jk}(x)$ such that

$$G^i = \Gamma^i{}_{jk} y^j y^k.$$

There is another equal definition for a Berwald metric as follows. A Finsler metric F is called a Berwald metric if the Cartan torsion of F satisfies the following

$$A_{ijk|l} = 0.$$

where the "$|$ " and "$,$ " denote the horizontal and vertical covariant derivatives with respect to the Berwald connection.

For $y \in T_xM$, define the Landsberg curvature $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{L}_y(u, v, w) := -\frac{1}{2}\mathbf{g}_y\big(\mathbf{B}_y(u, v, w), y\big).$$

In local coordinates, $\mathbf{L}_y(u, v, w) := L\ _{ijk}(y)u^i v^j w^k$, where

$$L_{ijk} := -\frac{1}{2}y_l B^l{}_{ijk}.$$

$\mathbf{L}_y(u, v, w)$ is symmetric in u, v and w and $\mathbf{L}_y(y, v, w) = 0$. $\mathbf{L}$ is called the Landsberg curvature. A Finsler metric F is called a Landsberg metric if $\mathbf{L}_y = 0$ [13]. Equivalently, a Finsler metric F is called a Landsberg metric if the Cartan torsion of F satisfies the following

$$A_{ijk|m}y^m = 0.$$

It is easy to see that, every Berwald metric is a Landsberg metric.

2.1. **The Bundle Maps.** In [1], Akbar-Zadeh developed the modern theory of global Finsler geometry by establishing a global definition of Cartan connection. For this aim, he introduced two bundle maps ρ and μ. Here, we give a short introduction of these bundle maps. Let TTM be the tangent bundle of TM and ρ the canonical linear mapping

$$\begin{cases} \rho : TTM_0 \to & \pi^*TM \\ \qquad \hat{X} \longmapsto & \big(z, \pi_*(\hat{X})\big), \end{cases}$$

where $\hat{X} \in T_zTM_0$ and $z \in TM_0$. The bundle map ρ satisfies

$$\rho\Big(\frac{\partial}{\partial x^i}\Big) = \partial_i, \quad \rho\Big(\frac{\partial}{\partial y^i}\Big) = 0. \tag{2.2}$$

Let V_zTM be the set of vertical vectors at z, that is, the set of vectors tangent to the fiber through z, or equivalently $V_zTM = ker\rho$, called the *vertical space*.

By means of these considerations, one can see that the following sequence is exact

$$0 \to VTM \xrightarrow{i} TTM \xrightarrow{\rho} \pi^*TM \longrightarrow 0\ , \tag{2.3}$$

where i is the natural inclusion map.

Let ∇ be a linear connection on π^*TM, that is $\nabla : T_zTM_0 \times \pi^*TM \to \pi^*TM$ such that $\nabla : (\hat{X}, Y) \mapsto \nabla_{\hat{X}}Y$. Let us define the linear mapping

$$\begin{cases} \mu_z : T_zTM_0 \to & T_{\pi z}M \\ \qquad \hat{X} \longmapsto & \nabla_{\hat{X}}F\ell, \end{cases}$$

where $\hat{X} \in T_zTM_0$. For a torsion-free connection ∇ the bundle map μ satisfies

$$\mu\Big(\frac{\partial}{\partial x^i}\Big) = N_i^k \partial_k, \quad \mu\Big(\frac{\partial}{\partial y^i}\Big) = \nabla_{\frac{\partial}{\partial y^i}} F\ell = \rho\Big([\frac{\partial}{\partial y^i}, y^k \frac{\partial}{\partial x^k}]\Big) = \partial_i. \tag{2.4}$$

where $N_i^k = F\Gamma_{ij}^k \ell^j$ and Γ_{ij}^k are Christoffel symbols of ∇.

Let us put

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_i^k \frac{\partial}{\partial y^k}.$$

Then

$$\mu\Big(\frac{\delta}{\delta x^i}\Big) = 0.$$

The connection ∇ is called a *Finsler connection* if for every $z \in TM_0$, μ_z defines an isomorphism of V_zTM_0 onto $T_{\pi z}M$. Therefore, the tangent space TTM_0 in z is decomposed as

$$T_zTM_0 = H_zTM \oplus V_zTM,$$

where $H_zTM = \ker \mu_z$ is called the *horizontal space* defined by ∇. Indeed any tangent vector $\hat{X} \in T_zTM_0$ in z decomposes to

$$\hat{X} = H\hat{X} + V\hat{X},$$

where $H\hat{X} \in H_zTM$ and $V\hat{X} \in V_zTM$. Thus ρ restricted to HTM is an isomorphism onto π^*TM, and μ restricted to VTM is the bundle isomorphism onto π^*TM.

The structural equations of the Finsler connection ∇ are

$$\mathcal{T}_\nabla(\hat{X}, \hat{Y}) := \nabla_{\hat{X}} Y - \nabla_{\hat{Y}} X - \rho[\hat{X}, \hat{Y}], \tag{2.5}$$

$$\Omega(\hat{X}, \hat{Y})Z := \nabla_{\hat{X}} \nabla_{\hat{Y}} Z - \nabla_{\hat{Y}} \nabla_{\hat{X}} Z - \nabla_{[\hat{X}, \hat{Y}]} Z, \tag{2.6}$$

where $X = \rho(\hat{X})$, $Y = \rho(\hat{Y})$ and $Z = \rho(\hat{Z})$. The tensors $\mathcal{T}_\nabla$ and Ω are called respectively the *torsion and curvature tensors of* ∇. They determine two torsion tensors defined by

$$\mathcal{S}(X, Y) := \mathcal{T}_\nabla(H\hat{X}, H\hat{Y}), \qquad \mathcal{T}(\dot{X}, Y) := \mathcal{T}_\nabla(V\hat{X}, H\hat{Y})$$

and three curvature tensors defined by

$$R(X, Y) := \Omega(H\hat{X}, H\hat{Y}),$$
$$P(X, \dot{Y}) := \Omega(H\hat{X}, V\hat{Y}),$$
$$Q(\dot{X}, \dot{Y}) := \Omega(V\hat{X}, V\hat{Y}),$$

where $\dot{X} = \mu(\hat{X})$ and $\dot{Y} = \mu(\hat{X})$.

Chern Connection: Let (M, F) be an n-dimensional Finsler manifold. Then the Chern connection $\mathfrak{D}$ is a linear connection in π^*TM, which has the following properties:

(i): $\mathfrak{D}$ is torsion-free, i.e., $\forall \hat{X}, \hat{Y} \in C^\infty\big(T(TM_0)\big)$

$$\mathcal{T}(\hat{X}, \hat{Y}) := \mathfrak{D}_{\hat{X}} \rho(\hat{Y}) - \mathfrak{D}_{\hat{Y}} \rho(\hat{X}) - \rho\big([\hat{X}, \hat{Y}]\big) = 0. \tag{2.7}$$

(ii): $\mathfrak{D}$ is almost compatible with F in the following sence

$$\begin{aligned}(\mathfrak{D}_{\hat{Z}}g)(X,Y) &:= \hat{Z}g(X,Y) - g(\mathfrak{D}_{\hat{Z}}X,Y) - g(X,\mathfrak{D}_{\hat{Z}}Y) \\ &= 2F^{-1}A(\mu(\hat{Z}),X,Y),\end{aligned} \tag{2.8}$$

where $X,Y \in C^\infty(\pi^*TM)$ and $\hat{Z} \in T_v(TM_0)$.

Theorem 2.1. *Let (M,F) be an n-dimensional Finsler manifold. Then there is a unique linear connection D in π^*TM, which has the following properties:*
(i) D is torsion-free in the sense of (2.7);
(ii) D is almost compatible with the Finsler structure in the following sense: for all $X,Y \in C^\infty(\pi^*TM)$ and $\hat{Z} \in T_v(TM_0)$,

$$(D_{\hat{Z}}g)(X,Y) := 2F^{-1}\big[A\big(\mu(\hat{Z}),X,Y\big) + \dot{A}\big(\mu(\hat{Z}),X,Y\big)\big] \tag{2.9}$$

Proof. In a standard local coordinate system (x^i,y^i) in TM_0, we write

$$D_{\frac{\partial}{\partial x^i}}\partial_j = \Gamma^k_{ij}\partial_k \quad , \quad D_{\frac{\partial}{\partial y^i}}\partial_j = F^k_{ij}\partial_k.$$

Clearly, (2.7) and (2.9) are equivalent to the following

$$\Gamma^k_{ij} = \Gamma^k_{ji}, \tag{2.10}$$

$$F^k_{ij} = 0, \tag{2.11}$$

$$\frac{\partial(g_{ij})}{\partial x^k} = \Gamma^l_{ki}g_{lj} + \Gamma^l_{kj}g_{il} + 2\Gamma^l_{km}l^m(A_{lij} + \dot{A}_{lij}), \tag{2.12}$$

$$\frac{\partial(g_{ij})}{\partial y^k} = F^s_{ik}g_{sj} + F^s_{kj}g_{is} + 2F^{-1}(A_{ijk} + \dot{A}_{ijk}) + 2F^s_{mk}l^mA_{ijs}. \tag{2.13}$$

Note that (2.11) and (2.13) are just the definition of A_{ijk}. We must compute Γ^k_{ij} from (2.10) and (2.12). Then making a permutation to i,j,k in (2.12), and using (2.10) and $A^k_{ij} = g^{kl}A_{ijl}$. on obtains

$$\Gamma^k_{ij} = \gamma^k_{ij} + g^{k\ell}\Big\{\Gamma^m_{lb}(A_{mij} + \dot{A}_{mij}) - \Gamma^m_{ib}(A_{mlj} + \dot{A}_{mlj}) - \Gamma^m_{jb}(A_{mil} + \dot{A}_{mil})\Big\}\ell^b \tag{2.14}$$

where we have put

$$\gamma^k_{ij} = \frac{1}{2}g^{kl}\left\{\frac{\partial}{\partial x^i}(g_{jl}) + \frac{\partial}{\partial x^j}(g_{il}) - \frac{\partial}{\partial x^l}(g_{ij})\right\}.$$

and $A^k_{ij} = g^{kl}A_{ijl}$. Multiplying (2.14) by ℓ^i implies that

$$\Gamma^k_{ib}\ell^b = \gamma^k_{ib}\ell^b - (A^k_{im} + \dot{A}^k_{im})\Gamma^m_{lb}\ell^l\ell^b. \tag{2.15}$$

Contracting (2.15) with ℓ^j yields

$$\Gamma^k_{ab}\ell^a\ell^b = \gamma^k_{ab}\ell^a\ell^b. \tag{2.16}$$

By putting (2.16) in (2.15), one can obtain

$$\Gamma^k_{ib}\ell^b = \gamma^k_{ib}\ell^b - \ell^a\ell^b\gamma^m_{ab}(A^k_{mi} + \dot{A}^k_{mi}). \tag{2.17}$$

Putting (2.17) in (2.14) give us the following

$$\Gamma^k_{ij} = \gamma^k_{ij} + g^{kl}\left\{\gamma^m_{lb}(A_{mij} + \dot{A}_{mij}) - \gamma^m_{ib}(A_{mlj} + \dot{A}_{mlj}) - \gamma^m_{jb}(A_{mil} + \dot{A}_{mil})\right\}\ell^b$$
$$+\gamma^s_{ab}\ell^a\ell^b\Big\{(A^m_{sj} + \dot{A}^m_{sj})(A^k_{mi} + \dot{A}^k_{mi}) + (A^m_{si} + \dot{A}^m_{si})(A^k_{mj} + \dot{A}^k_{mj})$$
$$-(A^k_{sm} + \dot{A}^k_{sm})(A^m_{ij} + \dot{A}^m_{ij})\Big\}.$$

This proves the uniqueness of D. The set $\{\Gamma^k_{ij}, F^k_{ij} = 0\}$, where $\{\Gamma^k_{ij}\}$ are given by (2.7), defines a linear connection D satisfying (2.7) and (2.9).

3. **Curvatures of the Connection D**

The curvature tensor Ω of D is defined by

$$\Omega(\hat{X}, \hat{Y})Z = D_{\hat{X}}D_{\hat{Y}}Z - D_{\hat{Y}}D_{\hat{X}}Z - D_{[\hat{X},\hat{Y}]}Z, \tag{3.1}$$

where $\hat{X}, \hat{Y} \in C^\infty(T(TM_0))$ and $Z \in C^\infty(\pi^*TM)$. Let $\{e_i\}^n_{i=1}$ be a local orthonormal (with respect to g) frame field for the vector bundle π^*TM such that $g(e_i, e_n) = 0, i = 1, ..., n-1$ and $e_n := \frac{y}{F} = \frac{y^i}{F(x,y)}\frac{\partial}{\partial x^i} = \ell$. Let $\{\omega^i\}^n_{i=1}$ be its dual co-frame field. These are local sections of dual bundle π^*TM. One readily finds that $\omega^n := \frac{\partial F}{\partial y^i}dx^i = \ell_i dx^i = \omega$, which is the *Hilbert form*. It is obvious that $\omega(\ell) = 0$. Now, let us put $\rho = \omega^i \otimes e_i, \quad De_i = {\omega_i}^j \otimes e_j, \quad \Omega e_i = 2{\Omega_i}^j \otimes e_j$. $\{{\Omega_i}^j\}$ and $\{{\omega_i}^j\}$ are called the *curvature forms* and *connection forms* of D with respect to $\{e_i\}$. We have $\mu := DF\ell = F\{{\omega_n}^i + d(logF)\delta^i_n\} \otimes e_i$. Put $\omega^{n+i} := {\omega_n}^i + d(logF)\delta^i_n$. It is easy to see that $\{\omega^i, \omega^{n+i}\}^n_{i=1}$ is a local basis for $T^*(TM_0)$. By definition $\rho = \omega^i \otimes e_i, \quad \mu = F\omega^{n+i} \otimes e_i$. Use the above formula for Theorem 2.1, then it re-express the structure equation of the new connection D

$$d\omega^i = \omega^j \wedge {\omega_j}^i \tag{3.2}$$

$$dg_{ij} = g_{kj}{\omega_i}^k + g_{ki}{\omega_j}^k + 2(A_{ijk} + \dot{A}_{ijk})\omega^{n+k}. \tag{3.3}$$

Define $g_{ij.k}$ and $g_{ij|k}$ by

$$dg_{ij} - g_{kj}{\omega_i}^k - g_{ik}{\omega_j}^k = g_{ij|k}\omega^k + g_{ij.k}\omega^{n+k}. \tag{3.4}$$

where $g_{ij.k}$ and $g_{ij|k}$ are respectively the vertical and horizontal covariant derivative of g_{ij} with respect to the connection D. This gives

$$g_{ij|k} = 0, \tag{3.5}$$

$$g_{ij.k} = 2(A_{ijk} + \dot{A}_{ijk}), \tag{3.6}$$

It can be shown that $\delta^i_{j|s} = 0$ and $\delta^i_{j.s} = 0$, thus $(g^{ij}g_{jk})_{|s} = 0$ and $(g^{ij}g_{jk})_{.s} = 0$. So

$$g^{ij}_{|s} = g^{ij}, \qquad g^{ij}_{.s} = -2(\dot{A}^{ij}_s + A^{ij}_s). \tag{3.7}$$

Moreover torsion freeness is equivalent to the absent of dy^k in $\{{\omega_j}^i\}$ namely

$$ {\omega_j}^i = \Gamma^i_{jk}(x, y)dx^k \tag{3.8}$$

is equivalent to

$$d\omega_i^{\ j} - \omega_i^{\ k} \wedge \omega_k^{\ j} = \Omega_i^{\ j}. \tag{3.9}$$

Since the $\Omega_j^{\ i}$ are 2-forms on the manifold TM_0, they can be generally expanded as

$$\Omega_i^{\ j} = \frac{1}{2} R_i^{\ j}{}_{kl}\omega^k \wedge \omega^l + P_i^{\ j}{}_{kl}\omega^k \wedge \omega^{n+l} + \frac{1}{2} Q_i^{\ j}{}_{kl}\omega^{n+k} \wedge \omega^{n+l}. \tag{3.10}$$

The objects R, P and Q are respectively the hh-, hv- and vv-curvature tensors of the connection D. Let $\{\bar{e}_i, \dot{e}_i\}_{i=1}^n$ be the local basis for $T(TM_0)$, which is dual to $\{\omega^i, \omega^{n+i}\}_{i=1}^n$, i.e., $\bar{e}_i \in HTM,\ \dot{e}_i \in VTM$ such that $\rho(\bar{e}_i) = e_i,\ \mu(\dot{e}_i) = Fe_i$. Let us put

$$R(\bar{e}_k, \bar{e}_l)e_i = R_i^{\ j}{}_{kl}e_j, \qquad P(\bar{e}_k, \dot{e}_l)e_i = P_i^{\ j}{}_{kl}e_j, \qquad Q(\dot{e}_k, \dot{e}_l)e_i = Q_i^{\ j}{}_{kl}e_j.$$

The connection defined in Theorem 2.1 is torsion-free. Then we have $Q = 0$. First Bianchi identity for R is given by

$$R_i^{\ j}{}_{kl} + R_k^{\ j}{}_{li} + R_l^{\ j}{}_{ik} = 0 \tag{3.11}$$

and

$$P_i^{\ j}{}_{kl} = P_k^{\ j}{}_{il}. \tag{3.12}$$

Exterior differentiation of (3.9) gives the second Bianchi identity:

$$d\Omega_i^{\ j} - \omega_i^{\ k} \wedge \Omega_k^{\ j} + \omega_k^{\ j} \wedge \Omega_i^{\ k} = 0. \tag{3.13}$$

We decompose the covariant derivative of the Cartan tensor on TM

$$dA_{ijk} - A_{ljk}\omega_i^{\ l} - A_{ilk}\omega_j^{\ l} - A_{ijl}\omega_k^{\ l} = A_{ijk|l}\omega^l + A_{ijk.l}\omega^{n+l}. \tag{3.14}$$

Similarly, for $\dot{A}_{ijk}$ we get

$$d\dot{A}_{ijk} - \dot{A}_{ljk}\omega_i^{\ l} - \dot{A}_{ilk}\omega_j^{\ l} - \dot{A}_{ijl}\omega_k^{\ l} = \dot{A}_{ijk|l}\omega^l + \dot{A}_{ijk.l}\omega^{n+l}. \tag{3.15}$$

It is easy to see that, $A_{ijk|l}$, $A_{ijk.l}$, $\dot{A}_{ijk|l}$ and $\dot{A}_{ijk.l}$ are symmetric with respect to indices i, j and k.

Put $\dot{A}_{ijk} = \dot{A}(e_i, e_j, e_k)$. Then

$$A_{ijk|n} = \dot{A}_{ijk}. \tag{3.16}$$

By (3.14) and (3.15), we get

$$A_{njk|l} = 0, \ \text{and} \quad A_{njk.l} = -A_{jkl}. \tag{3.17}$$

$$\dot{A}_{njk|l} = 0, \ \text{and} \quad \dot{A}_{njk.l} = -\dot{A}_{jkl}. \tag{3.18}$$

Theorem 3.1. *Let (M, F) be a Finsler manifold. Suppose that D is the linear torsion-free connection obtained by Shen's L-process on Chern's connection. Then their hv-curvature coincides if and only if F is a Riemannian metric.*

Proof. Let $\tilde{\nabla}$ be obtained from ∇ by Shen's L-process

$$\tilde{\omega}^i_j = \omega^i_j + A^i_{kj}\omega^k - \dot{A}^i_{jk}\omega^{n+k} \tag{3.19}$$

Taking exterior differential from (3.19) yields

$$d\tilde{\omega}^i_j = d\omega^i_j + dA^i_{kj}\omega^k + A^i_{kj}d\omega^k - d\dot{A}^i_{jk}\omega^{n+k} - \dot{A}^i_{kj}d\omega^{n+k} \tag{3.20}$$

Substituting (3.15) in (3.20) and using (3.9) and (3.3), we get

$$\begin{aligned}\tilde{\Omega}^i_j = \Omega^i_j &+ A^k_{uj}\dot{A}^k_{km}\omega^u \wedge \omega^m + A^k_{uj}\dot{A}^k_{km}\omega^u \wedge \omega^{n+m} + \dot{A}^k_{ju}A^i_{km}\omega^{n+u} \wedge \omega^m \\ &- \dot{A}^k_{ju}\dot{A}^i_{km}\omega^{n+u} \wedge \omega^{n+m} + A^i_{jk|s}\omega^s \wedge \omega^k + A^i_{jk.s}\omega^{n+s} \wedge \omega^k - \dot{A}^i_{jk}\Omega^k_n \\ &- \dot{A}^i_{jk|s}\omega^s \wedge \omega^{n+k} - \dot{A}^i_{jk.s}\omega^{n+s} \wedge \omega^{n+k} \end{aligned} \tag{3.21}$$

By decomposing $\tilde{\Omega}^i_j$ and Ω^i_j as in (3.10), one can obtain:

$$\tilde{R}^i_{jkl} = R^i_{jkl} + 2A^s_{kj}\dot{A}^i_{sl} - 2A^i_{jk|l} + \dot{A}^i_{js}R^s_{nkl}, \tag{3.22}$$

$$\tilde{P}^i_{jkl} = P^i_{jkl} + A^s_{kj}\dot{A}^i_{sl} - \dot{A}^s_{jl}A^i_{sk} - A^i_{jk.l} - \dot{A}^i_{js}P^s_{nlk} - \dot{A}^i_{jl|k}, \tag{3.23}$$

$$\dot{A}^s_{jk}\dot{A}^i_{sl} + \dot{A}^i_{jk.l} = 0. \tag{3.24}$$

By (3.23), it is easy to see that if F is Riemannian then $\tilde{P} = P$.

Conversely, suppose that $\tilde{P} = P$. Then, we have

$$A^s_{kj}\dot{A}^i_{sl} - \dot{A}^s_{jl}A^i_{sk} - A^i_{jk.l} - \dot{A}^i_{js}P^s_{nlk} - \dot{A}^i_{jl|k} = 0. \tag{3.25}$$

Contracting (3.25) with y^j yields

$$A_{ijk.l}y^j + \dot{A}_{ijl|k}y^j = 0. \tag{3.26}$$

Since ${y^j}_{|k} = 0$, then $\dot{A}_{ijl|k}y^j = (\dot{A}_{ijl}y^j)_{|k}$ which implies that $\dot{A}_{ijl|k}y^j = 0$. Then (3.26) reduces to following

$$A_{ijk.l}y^j = 0. \tag{3.27}$$

On the other hand, the following holds

$$0 = (y^j A_{ijk})_{,l} = A_{ilk} + A_{ijk,l}y^j.$$

Thus

$$A_{ijk,l}\ y^j = -A_{ikl}.$$

Therefore $A_{ikl} = 0$ and F reduces to a Riemannian metric. This completes the proof. $\square$

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