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On Einstein Finsler warped product metrics

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Abstract. In this paper, we study the Finsler warped product metric which is Einstein. We find equation that characterize Einstein Finsler warped product metrics with vanishing Douglas curvature. Moreover, we obtain the differential equation that characterizes Einstein Finsler warped product metrics of locally projectively flat.

Keywords: Finsler warped product metrics, Einstein metrics, Douglas curvature, locally projectively flat.

1. **Introduction**

The warped product metrics form an essential and a rich class of metrics in Riemann geometry and Finsler geometry. Recently, Chen, Shen, and Zhao have introduced a new class of Finsler metric which is as an extension to the Finsler geometry by using the concept of the warped product structure on an n-dimensional manifold $M := I \times \breve{M}$ where I is an interval of $\mathbb{R}$ and $\breve{M}$ is an $(n-1)$-dimensional manifold equipped with a Riemannian metric, [1]. In fact,

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it is considered in the following form:

$$F(u,v) = \breve{\alpha}(\breve{u},\breve{v})\phi\left(u^1, \frac{v^1}{\breve{\alpha}(\breve{u},\breve{v})}\right), \tag{1.1}$$

where $u = (u^1, \breve{u}), v = v^1\frac{\partial}{\partial u^1} + \breve{v}$ and ϕ is a suitable function defined on a domain of $\mathbb{R}^2$.

The class of Finsler warped product metrics can be considered as the spherically symmetric Finsler metrics. It is a fact that a Finsler metric F is called to be spherically symmetric if the orthogonal group $O(n)$ acts as isometries on F. The flag curvature and Ricci curvature of Finsler warped product metrics have been obtained by Chen-Shen-Zhao, [1]. Also, they have characterized these metrics to be Einstein metrics, [1]. H. Liu and X. Mo have obtained the differential equation to classify these kind of metrics with vanishing Douglas curvature, [6]. In [3], Gabrani, Rezaei, and Sevim have characterized Finsler warped product metric with isotropic mean Berwald curvature. Moreover, they have studied and classified the Landsberg Finsler warped product metrics [4]. Moreover, Gabrani, Rezaei, and Sevim have studied the volume form dV on n-dimensional Finsler manifold which admits the Finsler warped product metrics to introduce and classified the S-curvature of Finsler warped product metrics (for more details see [2]).

There are several important non-Riemannian quantities in Finsler geometry such as the Cartan torsion $\mathbf{C}$, the Berwald curvature $\mathbf{B}$, the Douglas curvature $\mathbf{D}$, the Landsberg curvature $\mathbf{L}$, the S-curvature $\mathbf{S}$, the χ-curvature χ, the H-curvature $\mathbf{H}$. Since they all vanish in Riemannian manifold, they are called non-Riemannian quantities. In this paper, we study Einstein Finsler warped product metrics with some non-Riemannian quantities. First, we classify these type of metrics with vanishing Douglas curvature. Troughout this paper, our index conventions are as follows:

$$1 \leq A \leq B \leq \ldots \leq n, \qquad 2 \leq i \leq j \leq \ldots \leq n.$$

Theorem 1.1. *Let $F = \breve{\alpha}\phi(r,s), r = u^1, s = \frac{v^1}{\breve{\alpha}}$ be a Douglas warped product metric on an n-dimensional manifold $M := I \times \breve{M}$. Then F has isotropic Ricci curvature*

$$\mathbf{Ric} = (n-1)K(u)F^2$$

if and only if $\breve{\alpha}$ has constant Ricci curvature $(n-2)c, K(u) = K(r)$ and

$$(n-1)\Big\{\Psi^2 - [s\Psi_r - 2(\xi s^2 + \eta)\Psi_s] + c\Big\} + 2(2\eta\xi + \eta^{'}) - c = (n-1)K\phi^2, \tag{1.2}$$

where $\Psi = \frac{s\phi_r}{2\phi} - \frac{\phi_s}{\phi}[\xi(r)s^2 + \eta(r)]$ and $\xi = \xi(r)$ and $\eta = \eta(r)$ are two differential functions.

Moreover, we prove the following theorem:

Theorem 1.2. *Let* $F = \breve{\alpha}\phi(r, s), r = u^1, s = \frac{v^1}{\breve{\alpha}}$ *be a locally projectively flat warped product metric on an* n*-dimensional manifold* $M := I \times \breve{M}$*, where* $\breve{\alpha}$ *is Ricci flat,* $c = 0$*. Then* F *has isotropic Ricci curvature*

$$\mathbf{Ric} = (n-1)K(u)F^2$$

if and only if the function ϕ *satisfies the following PDE:*

$$\Psi^2 - s\Psi_r + \frac{2\eta^2 - \eta' s^2}{\eta}\Psi_s = K\phi^2, \tag{1.3}$$

where

$$\Psi = \frac{s\phi_r}{2\phi} - \frac{\phi_s}{\phi}[\frac{2\eta^2 - \eta' s^2}{2\eta}], \quad \eta = \eta(r)$$

is a differential functions.

2. Preliminaries

In this section, we briefly introduce some geometric quantities and definitions in Finsler geometry to proof the main theorems.

Let M be an n-dimensional manifold. It is known that a Finsler metric is a non-negative function $F(x, y)$ on TM which has the following properties:

(a) $F(u, v)$ is C^∞ on $TM\backslash\{0\}$;
(b) the restriction $F_u := F_{|T_uM}$ is a Minkowski function on T_uM for all $u \in M$.

For $y \in T_xM_0$, define the Berwald curvature $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ and the mean Berwald curvature $\mathbf{E}_y : T_xM \times T_xM \to \mathbb{R}$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x$ and $\mathbf{E}_y(u, v) := E_{jk}(y)u^jv^k$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jk} := \frac{1}{2}B^m{}_{jkm},$$

$u = u^i\frac{\partial}{\partial x^i}|_x$, $v = v^i\frac{\partial}{\partial x^i}|_x$ and $w = w^i\frac{\partial}{\partial x^i}|_x$. Finsler metrics satisfying $\mathbf{B} = 0$ are called Berwald metrics and those satisfying $\mathbf{E} = 0$ called mean Berwald metrics, respectively.

Define $\mathbf{D}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{D}_y(u, v, w) := D^i{}_{jkl}(y)u^iv^jw^k\frac{\partial}{\partial x^i}|_x$ where

$$D^i{}_{jkl} := B^i{}_{jkl} - \frac{2}{n+1}\Big\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk,l}y^i\Big\}.$$

We call $\mathbf{D} := \{\mathbf{D}_y\}_{y\in TM_0}$ the Douglas curvature. A Finsler metric with $\mathbf{D} = 0$ is called a Douglas metric.

For a non-zero vector $y \in T_xM_0$, the Riemann curvature is a family of linear transformation $\mathbf{R}_y : T_xM \to T_xM$ with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2\mathbf{R}_y$, $\forall\lambda > 0$

which is defined by $\mathbf{R}_y(u) := R^i_k(y)u^k \frac{\partial}{\partial x^i}$, where

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The family $\mathbf{R} := \{\mathbf{R}_y\}_{y\in TM_0}$ is called the Riemann curvature.

Let (M, F) be an n-dimensional Finsler manifold. Put

$$\mathbf{Ric} := \sum_{i=1}^{n} g^{ij}\Big(\mathbf{R}_y(b_i), b_j\Big),$$

where $\{b_i\}$ is a basis for T_xM. **Ric** is a well-defined scalar function on TM_0. We call **Ric** the Ricci curvature (or Ricci scalar). In a local coordinate system,

$$\mathbf{Ric} = g^{ij}R_{ij} = R^m_m.$$

Then, F is called of isotropic Ricci curvature if

$$\mathbf{Ric} = (n-1)KF^2,$$

where $K = K(x)$ is a scalar function on M.

3. **Finsler Warped Product Metrics**

Assume that F is a Finsler metric on an n-dimensional manifold M. In local coordinate $u^1, \ldots, u^n$ and $v = v^A \dfrac{\partial}{\partial v^A}$,

$$\mathbf{G} = v^A \frac{\partial}{\partial u^A} - 2G^A \frac{\partial}{\partial v^A}$$

is a spray induced by F. The spray coefficients G^A are locally expressed as follows:

$$G^A := \frac{1}{4} g^{AB}\Big\{[F^2]_{u^C v^B} v^C - [F^2]_{u^B}\Big\},$$

where $g_{AB}(u,v) = \big[\frac{1}{2}F^2\big]_{v^A v^B}$ and $(g^{AB}) = (g_{AB})^{-1}$.

The spray coefficients of Finsler warped product metrics $F = \breve{\alpha}\phi(r,s)$ has been introduced by the following Lemma:

Lemma 3.1. *The spray coefficients G^A of a Finsler warped product metric $F = \breve{\alpha}\phi(r,s)$ are given by* [1]

$$G^1 = \Phi\breve{\alpha}^2, \qquad G^i = \breve{G}^i + \Psi\breve{\alpha}^2\breve{l}^i, \tag{3.1}$$

where $\breve{l}^i = \frac{v^i}{\breve{\alpha}}$ and

$$\Phi = \frac{s^2(\omega_r\omega_{ss} - \omega_s\omega_{rs}) - 2\omega(\omega_r - s\omega_{rs})}{2(2\omega\omega_{ss} - \omega_s^2)}, \tag{3.2}$$

$$\Psi = \frac{s(\omega_r\omega_{ss} - \omega_s\omega_{rs}) + \omega_s\omega_r}{2(2\omega\omega_{ss} - \omega_s^2)}, \tag{3.3}$$

where $\omega = \phi^2$. Φ *and* Ψ *can be rewritten as follows:*

$$\Phi = s\,\Psi + A, \tag{3.4}$$

$$\Psi = \frac{s\phi_r}{2\phi} - \frac{\phi_s}{\phi}A, \tag{3.5}$$

where

$$A := \frac{s\phi_{rs} - \phi_r}{2\,\phi_{ss}}. \tag{3.6}$$

We give the definition of Douglas tensor:

$$\mathbf{D} = D^A_{BCE} du^B \otimes du^C \otimes du^E$$

is a tensor on $TM \setminus \{0\}$, where

$$D^A_{BCE} := \frac{\partial^3}{\partial v^B \partial v^C \partial v^E}\Big(G^A - \frac{1}{n+1}\frac{\partial G^M}{\partial v^M}v^A\Big). \tag{3.7}$$

A Finsler metric F is called Douglas metric if $\mathbf{D} = 0$.

H. Liu and X. Mo have proved that a warped product Finsler metric $F = \breve{\alpha}\phi(r,s)$ is of Douglas type metric if and only if

$$A = \xi(r)s^2 + \eta(r), \tag{3.8}$$

where $\xi = \xi(r)$ and $\eta = \eta(r)$ are differential functions, [6]. Moreover, they have charcterized the Finsler warped product metrics to be Berwaldian. They have obtained the following lemma:

Lemma 3.2. [6] *Let* $F = \breve{\alpha}\phi(r,s)$ *be a Finsler warped product metric, where* $r = u^1$ *and* $s = \frac{v^1}{\breve{\alpha}}$. *Then* F *is a Berwald metric if and only if*

$$\Phi = a(r)s^2 + b(r), \quad \Psi = c(r)s, \tag{3.9}$$

where $a = a(r)$, $b = b(r)$ *and* $c = c(r)$ *are differentiable functions and* Φ *and* Ψ *are defined in (3.4) and (3.5), respectively.*

Lemma 3.3. [1] *For a Finsler warped product metric* $F = \breve{\alpha}\phi(r,s)$, *the Ricci curvature* ***Ric*** *is given by*

$$\boldsymbol{Ric} = \breve{\mathbf{Ric}} + \breve{\alpha}^2[\lambda + (n-1)\mu - \nu], \tag{3.10}$$

where

$$\lambda = (2\Phi_r - s\Phi_{rs}) + (2\Phi\Phi_{ss} - \Phi_s^2) + 2(\Phi_s - s\Phi_{ss})\Psi - (2\Phi - s\Phi_s)\Phi_s, \tag{3.11}$$

$$\mu = \Psi^2 - 2s\Psi\Psi_s - s\Psi_r + 2\Phi\Psi_s, \tag{3.12}$$

$$\tau = 2\Psi_r - s\Psi_{rs} + s(\Psi_s^2 - 2\Psi\Psi_{ss}) + 2\Psi_{ss}\Phi - \Psi_s\Phi_s, \tag{3.13}$$

$$\nu = s\tau + \mu. \tag{3.14}$$

Lemma 3.4. [7] *Let* $F = \breve{\alpha}\phi(r, s), r = u^1, s = \frac{v^1}{\breve{\alpha}}$ *be a warped product metric on an* n*-dimensional manifold* $M := I \times \breve{M}$. *Then* F *has isotropic Ricci curvature*

$$\boldsymbol{Ric} = (n-1)K(u)F^2$$

if and only if $\breve{\alpha}$ *has constant Ricci curvature* $(n-2)c, K(u) = K(r)$ *and*

$$(n-1)[K(r)\phi^2 - \mu] + \nu - \lambda = (n-2)c. \tag{3.15}$$

We recall that for a Finsler metric $F = F(x, y)$, its geodesics curves are characterized by the system of differential equations $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients. Two Finsler metrics F and $\bar{F}$ on a manifold M are called projectively related if any geodesic of the first is also geodesic for the second and the other way around. Hereby, there is a scalar function $P(x, y)$ defined on TM_0 such that

$$G^i = \bar{G}^i + Py^i,$$

where G^i and $\bar{G}^i$ are the geodesic spray coefficients of F and $\bar{F}$, respectively. The problem of projectively related Finsler metrics is quite old in geometry and its origin is formulated in Hilberts Fourth Problem: determine the metrics on an open subset in $\mathbb{R}^n$, whose geodesics are straight lines. Projectively flat Finsler metrics on a convex domain in $\mathbb{R}^n$ are regular solutions to Hilbert's Fourth Problem. A Finsler metric F on an open subset $U \subset \mathbb{R}^n$ is called projectively flat if all geodesics are straight in U. In this case, F and the Euclidean metric on U are projectively related [5]. In this paper, we will study Einstein Finsler warped product metrics of locally projectively flat.

Lemma 3.5. [8] *Let* $F = \breve{\alpha}\phi(r, s), r = u^1, s = \frac{v^1}{\breve{\alpha}}$ *be a warped product metric. Then* F *is locally projectively flat if and only if* $\breve{\alpha}$ *has constant sectional curvature* κ *(* $\breve{\alpha}$ *is locally projectively flat) and* ϕ *satisfies*

$$(\phi - s\phi_s)_r = 2\Big[-\eta + \frac{2\eta' - \kappa}{4\eta}s^2\Big]\phi_{ss}, \tag{3.16}$$

where $\eta = \eta(r)$ *is a differential function.*

4. **Proof of Main Theorems**

In this section, we are going to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let $F = \breve{\alpha}\phi(r, s), r = u^1, s = \frac{v^1}{\breve{\alpha}}$ be a Douglas warped product metric. Then, from Eq. (3.8), we have

$$A = \xi(r)s^2 + \eta(r), \tag{4.1}$$

where $\xi = \xi(r)$ and $\eta = \eta(r)$ are differential functions, [6]. Now, suppose that F has isotropic Ricci curvature. Therefore, by lemma 3.4, $\breve{\alpha}$ has constant Ricci curvature $(n-2)c, K(u) = K(r)$ and

$$(n-1)[K(r)\phi^2 - \mu] + \nu - \lambda = (n-2)c, \tag{4.2}$$

where

$$\begin{aligned}
\lambda &= (2\Phi_r - s\Phi_{rs}) + (2\Phi\Phi_{ss} - \Phi_s^2) + 2(\Phi_s - s\Phi_{ss})\Psi - (2\Phi - s\Phi_s)\Phi_s, \\
\mu &= \Psi^2 - 2s\Psi\Psi_s - s\Psi_r + 2\Phi\Psi_s, \\
\tau &= 2\Psi_r - s\Psi_{rs} + s(\Psi_s^2 - 2\Psi\Psi_{ss}) + 2\Psi_{ss}\Phi - \Psi_s\Phi_s, \\
\nu &= s\tau + \mu,
\end{aligned}$$

where

$$\begin{aligned}
\Phi &= s\,\Psi + A, \\
\Psi &= \frac{s\phi_r}{2\phi} - \frac{\phi_s}{\phi}A.
\end{aligned}$$

Then, by (4.1) and (4.2), we have

$$(n-1)\Big\{\Psi^2 - [s\Psi_r - 2(\xi s^2 + \eta)\Psi_s] + c\Big\} + 2(2\eta\xi + \eta^{'}) - c = (n-1)K\phi^2, \tag{4.3}$$

where

$$\Psi = \frac{s\phi_r}{2\phi} - \frac{\phi_s}{\phi}\Big[\xi(r)s^2 + \eta(r)\Big]$$

and $\xi = \xi(r)$ and $\eta = \eta(r)$ are two differential functions. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let $F = \breve{\alpha}\phi(r,s), r = u^1, s = \frac{v^1}{\breve{\alpha}}$ be a locally projectively flat warped product metric. Then, from Lemma 3.5, we have

$$A = 2\Big[\eta - \frac{2\eta^{'} - \kappa}{4\eta}s^2\Big], \tag{4.4}$$

where $\eta = \eta(r)$ is a differential function. When $c = 0$, by (4.2) and (4.4), we obtain

$$\Psi^2 - s\Psi_r + \frac{2\eta^2 - \eta^{'}s^2}{\eta}\Psi_s = K\phi^2, \tag{4.5}$$

where

$$\Psi = \frac{s\phi_r}{2\phi} - \frac{\phi_s}{\phi}\Big[\frac{2\eta^2 - \eta^{'}s^2}{2\eta}\Big]$$

and $\eta = \eta(r)$ is a differential functions. This completes the proof of Theorem 1.2. □

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