Journal of Finsler Geometry and its Applications
Vol. 3, No. 2 (2022), pp 50-56.
DOI: 10.22098/jfga.2022.11680.1071

On the flag curvature of invariant square metrics

Parastoo Habibi

Department of Mathematics,
Islamic Azad University, Astara branch, Astara, Iran

E-mail: p.habibi@iau-astara.ac.ir

Abstract. In this paper, we give an explicit formula for the flag curvature of invariant square metric and Randers change of square metric.

Keywords: (α,β)-metric, flag curvature, square metric, Randers change, invariant metric.

1. Introduction

In 1929, Berwald construct an interesting family of projectively flat Finsler metrics on the unit ball $\mathbb{B}^n$ which as follows

$$F=\frac{\Big(\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}+\langle x,y\rangle\Big)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}}. \tag{1.1}$$

He showed that this class of metrics has constant flag curvature [4]. Berwald's metric can be expressed as

$$F=\frac{(\alpha+\beta)^2}{\alpha}, \tag{1.2}$$

where

$$\alpha=\frac{\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}}{(1-|x|^2)^2},\quad \beta=\frac{\langle x,y\rangle}{(1-|x|^2)^2}.$$

An Finsler metric in the form (1.2) is called a square metric.

The geometry of invariant Finsler metrics on homogeneous manifolds is one of the interesting subjects in Finsler geometry which has been studied by some Finsler geometers, for example see [1, 2, 6, 12]. The object of this paper is

---

AMS 2020 Mathematics Subject Classification: 53C60, 53C30

to give a formula for flag curvature of homogeneous Finsler space with square metric. The square metric belong to the class of (α, β)-metrics. An (α, β)-metric is a Finsler metric of the form $F = \alpha\phi(s), s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is induced by a Riemannian metric $\tilde{a} = a_{ij}dx^i \bigotimes dx^j$ on a connected smooth n-manifold M and $\beta = b_i(x)y^i$ is a 1-form on M [5, 14].

2. Preliminary

Let M be a smooth n-dimensional C^∞ manifold and TM be its tangent bundle. A Finsler metric on a manifold M, is a non-negative function $F : TM \longrightarrow \mathbb{R}$ with the following properties [3]:

(1) F is smooth on the slit tangent bundle $TM^0 := TM \setminus \{0\}$;
(2) $F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M, y \in T_xM$ and $\lambda > 0$;
(3) The $n \times n$ Hessian matrix

$$(g_{ij}) = \Big(\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}\Big)$$

is positive definite at every point $(x, y) \in TM^0$.

The following bilinear symmetric form $g_y : T_xM \times T_xM \to R$ is positive definite

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}F^2(x, y + su + tv)|_{s=t=0}.$$

Definition 2.1. *Let* $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ *be a Riemannian metric and* $\beta(x, y) = b_i(x)y^i$ *be a 1-form on an n-dimensional manifold* M. *Let*

$$\|\beta(x)\|_\alpha := \sqrt{\tilde{a}^{ij}(x)b_i(x)b_j(x)}$$

Now, let the function F *is defined as follows*

$$F := \alpha\phi(s), \quad s = \frac{\beta}{\alpha}, \tag{2.1}$$

where $\phi = \phi(s)$ *is a positive* C^∞ *function on* $-b_0, b_0$ *satisfying*

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, |s| \le b < b_0.$$

Then by lemma 1.1.2 of [5], F *is a Finsler metric if* $\|\beta(x)\|_\alpha < b_0$ *for any* $x \in M$. *A Finsler metric in the form (2.1) is called an* (α, β)*-metric* [1].

Let (M, F) be a Finsler manifolds and $\mathbf{G} = y^i\delta/\delta x^i$ be its induced spray on TM which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where

$$G^i := \frac{1}{4}g^{il}\Big[\frac{\partial^2 F^2}{\partial x^k \partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\Big].$$

Then, for a non-zero vector $y \in T_xM_0$, the Riemann curvature is a family of linear transformation $\mathbf{R}_y : T_xM \to T_xM$ with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2\mathbf{R}_y$, $\forall\lambda > 0$ which is defined by $\mathbf{R}_y(u) := R^i_k(y)u^k\frac{\partial}{\partial x^i}$, where

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The family $\mathbf{R} := \{\mathbf{R}_y\}_{y\in TM_0}$ is called the Riemann curvature.

For a flag $P := \text{span}\{y, u\} \subset T_xM$ with flagpole y, the flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(x, y, P) := \frac{g_y\big(u, \mathbf{R}_y(u)\big)}{g_y(y, y)g_y(u, u) - g_y(y, u)^2}.$$

The flag curvature $\mathbf{K}(x, y, P)$ is a function of tangent planes $P = \text{span}\{y, v\} \subset T_xM$. This quantity tells us how curved the space is at a point. If F is a Riemannian metric, $\mathbf{K}(x, y, P) = \mathbf{K}(x, P)$ is independent of $y \in P \setminus \{0\}$. Thus the flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry.

3. Flag curvature of invariant square metrics on homogeneous spaces

In this section, we are going to study the flag curvature of invariant square metrics on homogeneous spaces. Let M be a smooth manifold. Suppose that $\tilde{a}$ and β are a Riemannian metric and a 1-form on M, respectively. In this case, we can write the square metric on M as follows:

$$F = \frac{(\alpha + \beta)^2}{\alpha} = \alpha\phi(s),$$

where $\phi(s) = 1+s^2+2s$. The Riemannian metric $\tilde{a}$ induce a linear isomorphism between T^*_xM and T_xM. Then the 1-form β corresponds to a vector field X on M, such that

$$\tilde{a}(X_x, y) = \beta(x, y).$$

Therefore we can write the square metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ as follows:

$$F(x, y) = \frac{(\sqrt{\tilde{a}(y, y)} + \tilde{a}(X_x, y))^2}{\sqrt{\tilde{a}(y, y)}} \tag{3.1}$$

Theorem 3.1. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras of the compact Lie group G and its closed subgroup H respectively and $\ll -, - \gg$ a bi-invariant on G. Further let $\widetilde{a}$ be any invariant Riemannian metric on the homogeneous space $\frac{G}{H}$ such that $\widetilde{a}(Y, Z) = \ll \varphi Y, Z \gg$ where $\varphi : \mathfrak{g} \to \mathfrak{g}$ is a positive definite endomorphism and $Y, Z \in \mathfrak{g}$. Also suppose that $\tilde{X}$ is an invariant vector field on $\frac{G}{H}$ where is parallel with respect to $\widetilde{a}$ and $\widetilde{a}(\tilde{X}, \tilde{X}) < 1$, $\tilde{X}_H = X$.*

Assume that $F = \frac{(\alpha+\beta)^2}{\alpha}$ *be the square metric arising from* $\widetilde{a}$ *and* $\widetilde{X}$ *and* (P, Y) *be a flag in* $T_n\frac{G}{H}$ *such that* $\{U, Y\}$ *is an orthonormal basis of* P *with respect to* $\widetilde{a}$. *Then the flag curvature of the flag* (P, Y) *is given by*

$$K(P,Y) = \frac{A\langle U, R(U,Y)Y\rangle + (6r^2+18r+11)\widetilde{a}(X,U)\langle X, R(U,Y)Y\rangle}{B^3((2+r^2+3r)a^2(X,U) - (2r^2+3r))},$$

where $r := \frac{\widetilde{a}(X,Y)}{\sqrt{\widetilde{a}(Y,Y)}}$ *and*

$$A := 1 - r^4 - 3r^3 + 3r, \quad B := 1 + r^2 + 3r.$$

Proof. Since $\widetilde{X}$ is parallel with respect to $\widetilde{a}$, so β is parallel with respect to $\widetilde{a}$. Therefore F is a Berwald metric, i.e. the Chern connection of F coinside with the Riemannian connection of $\widetilde{a}$. Thus the Finsler metric F has the same curvature tensor as that of the Riemannian metric $\widetilde{a}$ and we denote it by R. Using (3.1), we can write

$$F(Y) = \frac{(\sqrt{\widetilde{a}(Y,Y)} + \widetilde{a}(X_p, Y))^2}{\sqrt{\widetilde{a}(Y,Y)}}$$

By using the formula

$$g_Y(U,V) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(Y + sU + tV)]|_{s=t=0},$$

we get

$$\begin{aligned} g_Y(U,V) &= (1+r^2+3r)^2\widetilde{a}(U,V) \\ &+ (2r^3+9r^2+11r+3)\widetilde{a}(Y,U)\left(\frac{\widetilde{a}(X,V)}{\sqrt{\widetilde{a}(Y,Y)}} - \frac{\widetilde{a}(X,Y)\widetilde{a}(Y,V)}{\widetilde{a}(Y,Y)^{\frac{3}{2}}}\right) \\ &+ (6r^2+18r+11)\left(\frac{\widetilde{a}(X,V)}{\sqrt{\widetilde{a}(Y,Y)}} - \frac{\widetilde{a}(X,Y)\widetilde{a}(Y,V)}{\widetilde{a}(Y,Y)^{\frac{3}{2}}}\right) \\ &\times \left(\widetilde{a}(X,U)\sqrt{\widetilde{a}(Y,Y)} - \frac{\widetilde{a}(Y,U)\widetilde{a}(X,Y)}{\sqrt{\widetilde{a}(Y,Y)}}\right) \\ &+ \frac{2r^3+9r^2+11r+3}{\sqrt{\widetilde{a}(Y,Y)}}\left(\widetilde{a}(X,U)\widetilde{a}(Y,V) - \widetilde{a}(U,V)\widetilde{a}(X,Y)\right). \end{aligned} \tag{3.2}$$

From equation (3.2), we get

$$g_Y(U,U) = (1 - r^4 - 3r^3 + 3r) + (6r^2+18r+11)\widetilde{a}^2(X,U), \tag{3.3}$$

$$g_Y(Y,Y) = (1+r^2+3r)^2 \tag{3.4}$$

and

$$g_Y(Y,U) = (2r^3+9r^2+11r+3)\widetilde{a}(X,U). \tag{3.5}$$

Therefore

$$g_Y(Y,Y)g_Y(U,U) - g_Y^2(Y,U) = (1+r^2+3r)^2\Big((r^4+6r^3+13r^2+12r +3)\widetilde{a}(X,U) - (2r^4+9r^3+11r^2+3r)\Big)$$

Also,

$$\begin{aligned} g_Y(R(U,Y)Y,U) &= (1-r^4-3r^3+3r)\widetilde{a}(R(U,Y)Y,U) \\ &+ \Big((2r^3+9r^2+11r+3)\widetilde{a}(X,U) - ((2r+3)^2 \\ &\quad +(2+2r^2+6r))\widetilde{a}(X,U)\Big) \times \widetilde{a}(R(U,Y)Y,Y) \\ &+ \Big(((2r+3)^2+(2+2r^2+6r))\widetilde{a}(X,U)\widetilde{a}(R(U,Y)Y,X)\Big). \end{aligned}$$

The flag curvature is given by

$$K(P,Y) = \frac{g_Y(U,R(U,Y)Y)}{g_Y(Y,Y)g_Y(U,U) - g_Y^2(y,U)} \tag{3.6}$$

Substituting the above relations in (3.6) give us the proof. □

A change of Finsler metric $F \to \bar{F}$ is called a Randers change of F, if

$$\bar{F}(x,y) = F(x,y) + \beta(x,y), \tag{3.7}$$

where $\beta(x,y) = b_i(x)y^i$ is a 1-form on a smooth manifold M. It is easy to see that, if $\sup_{F(x,y)=1}|b_i(x)y^i| < 1$, then $\bar{F}$ is again a Finsler metric. Hashiguchi-Ichijyō showed that if β is closed, then $\bar{F}$ is pointwise projective to F. The notion of a Randers change has been proposed by Matsumoto, named by Hashiguchi-Ichijyō and studied in detail by Shibata [7][11][15]. If F reduces to a Riemannian metric then $\bar{F}$ reduces to a Randers metric. Due to this reason the transformation (3.7) has been called the Randers change of Finsler metric.

Theorem 3.2. *Let G be a compact Lie group and $\ll,\gg$ a bi-invariant metric on G. Also let $\widetilde{X}$ be an invariant vector field on G which is parallel with respect to $\ll,\gg$. Suppose that $F = \frac{(\alpha+\beta)^2}{\alpha} + \beta$ is the Randers changed square Finsler metric arising from $\ll,\gg$ and $\widetilde{X}$ and (P,Y) be a flag in $T_e\frac{G}{H}$ such that $\{U,Y\}$ be an orthonormal basis of P with respect to $\ll,\gg$. Then the flag curvature of the flag (P,Y) is given by*

$$K(P,Y) = \frac{C\|[Y,U]\|^2 + D \ll X,U \gg \ll [X,Y],[U,Y] \gg}{4E^4 + 8E^3 \ll X,U \gg^2 - 4E(2r^2+3r)}, \tag{3.8}$$

where $r = \ll X,Y \gg$ and

$$C := 1 - r^4 - 3r^3 + 3r, \quad D := (2r+3)^2 + 2(1+r^2+3r), \quad E := 1 + r^2 + 3r.$$

Proof. Since $\widetilde{X}$ is parallel with respect to $\ll, \gg$, the Levi-Civita connection of $\ll, \gg$ and the Chern connection of F and therefore their curvature tensor coincide. So we have

$$R(U,Y)Y = \frac{1}{4}[Y,[U,Y]].$$

From equation (3.2), we deduce following equations

$$\begin{aligned} 4g_Y(R(U,Y)Y,U) &= (1-r^4-3r^3+3r) \ll [Y,[U,Y]],U \gg \\ &+ (2r+3)(1+r^2+3r) \ll X,U \gg \\ &- (6r^2+18r+11) \ll X,U \gg\ll Y,[Y,[U,Y]] \gg \\ &+ (6r^2+18r+11) \ll X,U \gg\ll X,[Y,[U,Y]] \gg, \end{aligned}$$

and

$$\begin{aligned} g_Y(U,U) &= (1+r^2+3r)^2 + (6r^2+18r+11) \ll X,U \gg^2 \\ &\quad -(1+r^2+3r)(2r+3)r. \end{aligned}$$

Also, we get

$$g_Y(Y,Y) = (1+r^2+3r)^2,$$

and

$$g_Y(Y,U) = (1+r^2+3r)(2r+3) \ll X,U \gg$$

By substituting the above equations in equation (3.6) and using the equations

$$\ll X,[Y,[U,Y]] \gg = \ll [X,Y],[U,Y] \gg$$

and

$$\ll Y,[Y,[Y,U]] \gg = 0$$

we get (3.8). □

References

1. H. An and S. Deng, *Invariant $(\alpha,\beta)-$ metrics on homogeneous manifolds*, monatsh. Math., **154** (2008), 89-102.
2. P. Bahmandoust and D. Latifi, *Naturally reductive homogeneous (α,β) spaces*, Int. J. Geom. Methods Mod. Phys. **17(8)** (2020), 2050117.
3. D. Bao, S. S. Chern and Z. Shen, *An introduction to Riemann-Finsler geometry*, Springer-Verlag, New-York, 2000.
4. L. Berwald, *Über die n-dimensionalen Geometrien konstanter Krümmung*, in denen die Geraden die kürzesten sind. Math. Z. **30**(1929), 449-469.
5. S. S. Chern and Z. Shen, *Riemann-Finsler geometry*, World Scientific, Nankai Tracts in Mathematics, Vol. 6 (2005).
6. S. Deng and Z. Hou, *Invariant Randers metrics on homogeneous Riemannian manifolds*, J. Phys. A: Math. Gen. **37** (2004), 4353-4360.
7. Hashiguchi, H. and Ichijyo, Y., *Randers spaces with rectilinear geodetics*, Rep. Fac. Sci.e Kagoshima Univ. (Math. Phys. Chem.)., **13** (1980), 33-40.

8. D. Latifi, *Homogeneous geodesics in homogeneous Finsler spaces*, J. Geom. Phys. **57** (2007) 1421-1433.
9. D. Latifi and A. Razavi, *Bi-invariant Finsler metrics on Lie groups*, Austral. J. Basic Appl. Sci., **5(12)** (2011), 507-511.
10. D. Latifi, *Bi-invariant* $(\alpha, \beta)-$ *metrics on Lie groups*, Acta. Uni. Apulensis, **65** (2021), 121-131.
11. Matsumoto, M., *On Finsler spaces with Randers metric and special forms of important tensors*, J. Math Kyoto Univ., **14** (1975), 477-498.
12. M. Parhizkar and D. Latifi, *On the flag curvature of invariant* $(\alpha, \beta)-$ *metrics*, Int. J. Geom. Methods Mod. Phys., **13** (2016), 1650039, 1-11.
13. T. Puttmann, *Optimal pinching constants of odd dimensional homogeneous spaces*, Invent. Math., **138** (1999), 631-684.
14. G. Shanker and K. Kaur, *Naturally reductive homogeneous space with* $(\alpha, \beta)-$ *metric*, Lobachevskii J. of Math., **40(2)** (2019), 210-218.
15. Shibata, C., *On invariant tensors of* β*-changes of Finsler metrics*, J. Math. Kyoto Univ., **24** (1984), 163-188.

Received: 16.10.2022
Accepted: 15.12.2022