Journal of Finsler Geometry and its Applications
Vol. 3, No. 1 (2022), pp 141-154
DOI: 10.22098/jfga.2022.10850.1066

On a Special Class of Dually Flat (α, β)-Metrics

Saeedeh Masoumi, Bahman Rezaei, and Mehran Gabrani

Department of Mathematics, Faculty of Science, Urmia University
Urmia, Iran.

E-mail: s.masoumi94@gmail.com
E-mail: b.rezaei@urmia.ac.ir
E-mail: m.gabrani@urmia.ac.ir

Abstract. In this paper, we first study a special class of (α, β)-metrics in the form $F = \alpha + \varepsilon\beta + k\beta^2/\alpha$, where α is Riemannian metric, β is a 1-form, and $\varepsilon, k(\neq 0)$ are constant. We give a complete classification for such metrics to be locally dually flat. By assumption β is a conformal 1-form, we show that the metric is locally dually flat if and only if α is a Euclidean metric and β is a constant 1-form. Further, we classify locally dually flat of a class of Finsler metric in the form $F = \alpha \exp(\alpha/\beta) + \varepsilon\beta$, where ε is constant.

Keywords: Finsler metric, (α, β)- metric, locally dually flat.

1. Introduction

The class of Dually flat Finsler metrics arise from α-flat information structures on Riemann-Finsler manifolds. The notion of dual flatness for Riemannian metrics was first introduced from the study of the information geometry on Riemannian manifolds by Amari and Nagaoka, [1]. Then, Z. Shen have been extended this notion to general Finsler metrics in 2007, [7]. A Finsler metric $F = F(x, y)$ on a manifold M is locally dually flat if at every point there is a coordinate system (x^i) in which the geodesic spray coefficients are in the following from

$$G^i = -\frac{1}{2} g^{ij} H_{y^j},$$

*Corresponding Author

AMS 2020 Mathematics Subject Classification: 53B40, 53C60

where

$$H = H(x,y) := -\frac{1}{6}(F^2)_{x^k}y^k$$

is a C^∞ scalar function on $TM \setminus \{0\}$ satisfying $H(x,\lambda y) = \lambda^3 H(x,y)$ for all $\lambda > 0$. Such a coordinate system is called an adapted coordinate system. In particular, a Riemannian metric is locally dually flat if in an adapted coordinate it is given by

$$g_{ij} = \frac{\partial^2 \varphi}{\partial x^i \partial x^j}(x),$$

where the function $\varphi = \varphi(x)$ is C^∞ on the manifold M (see [3], [4]).

It is well known that the Funk metric

$$F = \frac{\sqrt{1 - \big(|x|^2\ |y|^2 - < x, y >^2\big)}}{1 - |x|^2} \pm \frac{< x, y >}{1 - |x|^2},$$

defined on the unit ball is the first example of locally dually flat non-Riemannian metrics.

In [4], Cheng-Shen-Zhou have characterized locally dually flat Randers metrics. Q. Xia gave the equivalent conditions of locally dually flat (α,β)-metrics on a manifold with dimension $n \geq 3$, [10]. Tayebi-Peyghan-Sadeghi have considered locally dually flat (α,β)-metrics with isotropic S-curvature and found some necessary and sufficient conditions under which these metrics reduce to locally Minkowskian metrics, [9].

In this paper, we prove the following:

Theorem 1.1 (see [8]). *Let $F = \alpha + \varepsilon\beta + k\beta^2/\alpha$ be a Finsler metric on an n-dimensional manifold M, $n \geqslant 2$. If β is a conformal 1-form, then F is dually flat if and if α is a Euclidean metric and β is a constant 1-form.*

Theorem 1.2. *Let $F = \alpha e^s + \varepsilon\beta$ be a Finsler metric on a manifold M. Then F is dually locally flat if in an adapted coordinate system, β and α satisfy*

$$r_{00} = \frac{1}{3}(\psi\alpha^2 + 2\theta\beta) \tag{1.1}$$

$$G^i_\alpha = \frac{1}{3}(2y^i\theta - \varphi^i\alpha^2) \tag{1.2}$$

$$s_{k0} = \frac{1}{3}\Big\{\varphi_k\beta - b_k\theta + \frac{2}{(e^s+\varepsilon)}\big\{\varepsilon\theta_k - e^s\varphi_k\big\}\beta\Big\} \tag{1.3}$$

where $\theta = \theta_i(x)y^i$ is a 1- form and $\varphi = \varphi_i y^i$ are on M, and $\psi = \psi(x)$ and $\varphi^i := a^{ij}\varphi_j$.

2. **Preliminaries**

A Finsler metric on a manifold M is a function $F : TM \to [0, \infty)$ which has the following properties:

(a) F is C^∞ on TM_0;
(b) $F(x, \lambda y) = \lambda F(x, y), \quad \lambda > 0$;
(c) For any tangent vector $y \in T_xM$, the vertical Hessian of $F^2/2$ given by

$$g_{ij}(x, y) = \left[\frac{1}{2}F^2\right]_{y^i y^j}$$

is positive definite.

In Finsler geometry, (α, β)-metrics are an important class of Finsler metrics which are defined by a Riemann metric $\alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ and a 1-form $\beta(x, y) = b_i(x)y^i$ on n-dimensional manifold M. These metrics can be expressed in the following form

$$F = \alpha\phi(s), \qquad s = \frac{\beta}{\alpha},$$

where $\phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$. It is known that $F = \alpha\phi(s)$ is a positive definite Finsler metric for any α and β with $b := \|\beta_x\|_\alpha < b_0$, if and only if, $\phi(s)$ satisfies the following condition, [6]:

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad \big(|s| \leqslant b < b_0\big).$$

In this paper, we are going to study two special classes of (α, β)-metrics on n-dimensional manifold M. Suppose first

$$\phi(s) = 1 + \varepsilon s + ks^2. \tag{2.1}$$

Then,

$$F = \alpha + \varepsilon\beta + k\frac{\beta^2}{\alpha},$$

where $\varepsilon, k(\neq 0)$ are constant and

$$1 + 2kb^2 - 3ks^2 > 0, \quad \big(|s| \leqslant b < b_0\big)$$

where b_0 such that $1 + \varepsilon s + ks^2 > 0$.

We also study the following Finsler metric:

$$F = \alpha exp(\frac{\beta}{\alpha}) + \varepsilon\beta, \tag{2.2}$$

where

$$1 - s + b^2 - s^2 0, \quad |s| \leqslant b < b_0 \tag{2.3}$$

where b_0 depends on ε such that $exp(s) + \varepsilon s > 0$.

It is known that a Finsler metric $F = F(x, y)$ on a manifold M is said to be locally dually flat if and only if

$$\big(F^2\big)_{x^l y^k} y^l = 2\big(F^2\big)_{x^k}.$$

A spray G on a manifold M is a special smooth vector field on $TM \setminus \{0\}$. The spray coefficients are locally defined by

$$G^i(x, y) = \frac{1}{4} g^{ij} \Big\{ [F^2]_{x^k y^j} y^k - [F^2]_{x^j} \Big\}.$$

It can be used to characterize the geodesics of F as

$$\frac{d^2 x^i}{dt^2} + 2G^i \left(x, \frac{dx}{dt} \right) = 0.$$

So G^i are also called geodesic coefficients of F. If F is induced from a Riemannian metric, $F(x, y) = \sqrt{g_{ij}(x) y^i y^j}$, then we have

$$G^i = \frac{1}{2} \Gamma^i_{jk}(x) y^j y^k,$$

where Γ^i_{jk} are second christoffel symbols.

In local coordinates, $\alpha(x, y) = \sqrt{a_{ij}(x) y^i y^j}$ is a positive definite Riemannian metric and $\beta(x, y) = b^i(x) y^i$ is a 1-form with length $b = \|\beta\|_\alpha := a^{ij} b_i b_j$, where $a^{ij} = (a_{ij})^{-1}$, β is a constant 1-form means $b^i(x) = constant$ for each i.

Followings are the regular symbols always used in our discussion:

$$r_{ij} := \frac{1}{2} \left(b_{i|j} + b_{j|i} \right), \quad s_{ij} := \frac{1}{2} \left(b_{i|j} - b_{j|i} \right),$$

where "$|$" denotes the covariant derivative with respect to Levi-Civita connection of α. Denote

$$r^i{}_j := a^{ik} r_{kj}, \quad r_j := b^i r_{ij}, \quad r_0 := r_j y^j = r_{ij} b^i y^j, \quad r_{00} = r_{ij} y^i y^j$$
$$s^i{}_j := a^{ik} s_{kj}, \quad s_j := b^i s_{ij}, \quad s_0 := s_j y^j = s_{ij} b^i y^j,$$

where $b^i := a^{ij} b_j$.

We call β a closed 1-form if $s_{ij} = 0$ and a parallel 1-form if $r_i + s_i = 0$ for any $i, j \in \{1, \ldots, n\}$. Moreover, β is a conformal 1-form means

$$b_{i|j} + b_{j|i} = 2\lambda(x) a_{ij}$$

with some scalar Function $\lambda(x)$.

In our writing G^i and G^i_α denote the geodesic coefficients of F and α respectively.

The spray coefficients of (α,β)-metrics are given in [2], [5].

$$G^i = G^i_\alpha + \frac{\alpha\phi'}{\phi - s\phi\phi'} s^i_0 + \frac{(\phi - s\phi')\phi' - s\phi\phi''}{2\phi\Big[\big(\phi - s\phi'\big) + \big(b^2 - s^2\big)\phi''\Big]} \times\Big(\frac{-2\alpha\phi'}{\phi - s\phi'} s_0 + r_{00}\Big)\Big[\frac{\phi\phi''}{(\phi - s\phi')\phi' - s\phi\phi''} b^i + \frac{y^i}{\alpha}\Big], \tag{2.4}$$

where

$$s^i_0 = s^i_j y^j, \quad s_0 := s_i y^i, \quad r_{00} = r_{ij} y^i y^j, \quad b^2 := a^{ij} b_i b_j.$$

Let $\alpha(x,y) = \sqrt{a_{ij}(x) y^i y^j}$ be a Riemannian metric and $\beta(x) = b_i y^i$ be 1-form on a manifold M. Consider an (α,β)-metric in the following form:

$$F = \alpha + \varepsilon\beta + \kappa\frac{\beta^2}{\alpha}, \tag{2.5}$$

where ε and κ are constants with $\kappa \neq 0$. Then by (2.4), the spray coefficients G^i of F are given by

$$G^i = G^i_\alpha + \frac{(\varepsilon\alpha + 2\kappa\beta)\alpha^2}{\alpha^2 - \kappa\beta^2} s^i_0 + \frac{\varepsilon\alpha^3 - 3\kappa\varepsilon\alpha\beta^2 - 4\kappa^2\beta^3}{2F\big[(1+2\kappa b^2)\alpha^2 - 3\kappa\beta^2\big]}\Big[\frac{-2(\varepsilon\alpha + 2\kappa\beta)\alpha^2}{\alpha^2 - \kappa\beta^2} s_0 + r_{00}\Big] \times\Big[\frac{y^i}{\alpha} + \frac{2\kappa\alpha(\alpha^2 + \varepsilon\alpha\beta + \kappa\beta^2)}{\varepsilon\alpha^3 - 3\kappa\varepsilon\alpha\beta^2 - 4\kappa^2\beta^3} b^i\Big], \tag{2.6}$$

where G^i_α denote the spray coefficients of α. Assume that β satisfies

$$b_{i|j} = \tau\Big\{(1 + 2\kappa b^2)a_{ij} - 3\kappa\kappa b_i b_j\Big\}, \tag{2.7}$$

where $\tau = \tau(x)$ is scalar function on M, then $s^i_0 = 0$, $s_0 = 0$ and

$$r_{00} = \tau\Big\{(1 + 2\kappa b^2)\alpha^2 - 3\kappa\beta^2\Big\}. \tag{2.8}$$

Thus, the Spray coefficients G^i of F are reduced to

$$G^i = G^i_\alpha + \tau\Big\{\frac{\varepsilon\alpha^3 - 3\kappa\varepsilon\alpha\beta^2\beta - 4\kappa^2\beta^3}{2(\alpha^2 + \varepsilon\alpha\beta + \kappa\beta^2)} y^i + \kappa\alpha^2 b^i\Big\}. \tag{2.9}$$

Further, assume that G^i_α are in the following form

$$G^i_\alpha = \theta y^i - \kappa\tau\alpha^2 b^i, \tag{2.10}$$

where $\theta = \theta_i y^i$ is 1- form, then

$$G^i = \Big\{\theta + \tau\frac{\varepsilon\alpha^3 - 3\kappa\varepsilon\alpha\beta^2 - 4\kappa^2\beta^3}{2(\alpha^2 + \varepsilon\alpha\beta + \kappa\beta^2)}\Big\} y^i. \tag{2.11}$$

3. **Locally Dually Flat** (α, β)**-Metrics**

A Finsler metric $F = F(x, y)$ on an open subset $U \subset R^n$ is dually flat if and only if it satisfies the following equations:

$$\{F^2\}_{x^k y^l} y^k - 2\{F^2\}_{x^l} = 0. \tag{3.1}$$

We remark the following key lemma.

Lemma 3.1. [11] *Let $F = \alpha + \varepsilon\beta + k\beta^2/\alpha$ be a Finsler metric on a manifold M, where ε, k are nonzero constants. Then F is locally dually flat if and only if in an adapted coordinate system, α and β satisfy:*

$$\begin{aligned} r_{00} =& \frac{2}{3}[\theta\beta - (b_m\theta^m)\alpha^2], \\ s_{k_0} =& -\frac{\theta b_k - \beta\theta_k}{3}, \\ G^m_\alpha =& \frac{1}{3}(2\theta y^m + \theta^m \alpha^2), \end{aligned}$$

where $\theta = \theta_k y^k$ is a 1-form on M and $\theta^m := a^{im}\theta_i$.

Theorem 3.2 (see [8])**.** *Let $F = \alpha + \varepsilon\beta + k\frac{\beta^2}{\alpha}$ be a Finsler metric on a an n-dimensional manifold M, $n \geqslant 2$. F is dually flat and projectively flat if and only if α is a Euclidean metric and β is a constant 1-form.*

Proof. A Finsler metric $F = \alpha + \varepsilon\beta + k\frac{\beta^2}{\alpha}$ is projectively flat if and only if it is a Douglas metric and the spray coefficients of α are related to the 1-form β by (2.10) we can get

$$\left(\frac{1}{3}\theta^i b_i - 2k\tau b^2\right)\alpha^2 = \frac{1}{3}\theta\beta.$$

Then $\theta = 0$, since both sides of this equation are polynomails of y and α^2 which can not be divided by β since are equivalent to

$$\begin{aligned} \frac{\partial a_{ij}}{\partial x^l} &= \frac{4}{3}\Big(\theta_l a_{ij} + \theta_i a_{lj} + \theta_j a_{li}\Big), \\ b_{i|j} &= \frac{1}{3}\Big\{\theta_j b_i + 3\theta_i b_j - 4a_{ij}(\theta_m b^m)\Big\}, \end{aligned}$$

on the other hand, we can obtain a similar rigidity theorem with some assumption on β. □

4. **Proof of Theorem 1.1**

Now we give the proof of Theorem 1.1.

By assumption β is a conformal 1-form, which means that

$$b_{i|j}+b_{j|i}=2h(x)a_{ij},$$

for some scalar function $h(x)$. The following holds,

$$\frac{1}{3}\big(\theta_j b_i+2\theta_i b_j-4a_{ij}(\theta_m b^m)\big)=h(x)a_{ij}.$$

Contracting it with y^i and y^j, we obtain

$$\beta\theta=\left(\frac{4}{3}(\theta_m b^m)+h(x)\right)\alpha^2,$$

i.e., $\beta=0$ and $h=0$.

5. **Proof of Theorem 1.2**

In this section, we are going to prove Theorem 1.2.

Assume that $F=\alpha e^s+\varepsilon\beta$ is dually flat on a manifold M. First we have the following identities:

$$\alpha_{x^k}=\frac{y_m}{\alpha}\frac{\partial G^m_\alpha}{\partial y^k},\qquad \beta_{x^k}=b_{m|k}y^m+b_m\frac{\partial G^m_\alpha}{\partial y^k},$$

$$\begin{aligned}\big\{F^2\big\}_{x^k} &= 2FF_{x^k}=2(\alpha e^s+\varepsilon\beta)\Big\{(b_{m|k}y^m+b_m\frac{\partial G^m_\alpha}{\partial y^k})(e^s+\varepsilon)\\ &\quad+(1-s)\big(\frac{y_m}{\alpha}\frac{\partial G^m_\alpha}{\partial y^k}e^s\big)\Big\},\end{aligned} \tag{5.1}$$

$$\begin{aligned}\big\{F^2\big\}_{x^l y^k}y^l &= 2\Big(\frac{y_k}{\alpha}e^s+e^s\big(\frac{b_k\alpha^2-y_k\beta}{\alpha^2}+\varepsilon b_k\big)\Big)\Big\{(b_{m|l}y^m y^l\\ &\quad+b_m\frac{\partial G^m_\alpha}{\partial y^l}y^l(e^s+\varepsilon)+(1-s)\Big(\frac{y_m}{\alpha}\frac{\partial G^m_\alpha}{\partial y^k}y^l e^s\Big)\Big\}\\ &\quad+2\Big(\alpha e^s+\varepsilon\beta\Big)\Big\{\Big(b_{k|m}y^m+b_m\frac{\partial G^m_\alpha}{\partial y^l\partial y^k}y^l\Big)(e^s+\varepsilon)\\ &\quad+\Big(b_{m|l}y^m y^l+b_m\frac{\partial G^m_\alpha}{\partial y^l}y^l\Big)e^s\frac{b_k\alpha^2-y_k\beta}{\alpha^2}\\ &\quad-\Big(\frac{b_k\alpha^2-y_k\beta}{\alpha^2}\Big)\Big(\frac{y_m}{\alpha}\frac{\partial G^m_\alpha}{\partial y^l}e^s y^l\Big)\\ &\quad+(1-s)\frac{\alpha^2 a_{mk}-y_k y_m}{\alpha^2}\frac{\partial G^m_\alpha}{\partial y^l}y^l e^s\\ &\quad+\frac{y_m}{\alpha}\frac{\partial G^m_\alpha}{\partial y^l\partial y^k}y^l e^s+\frac{y_m}{\alpha}\frac{\partial G^m_\alpha}{\partial y^l}y^l\Big(\frac{b_k\alpha^2-y_k\beta}{\alpha^2}e^s\Big)\Big\}.\end{aligned} \tag{5.2}$$

Now, we can prove the following.

Proposition 5.1. *Let $F = \alpha exp(\frac{\beta}{\alpha}) + \varepsilon\beta$ be a Finsler metric on a manifold M. If F is locally dually flat, there are 1- form $\theta(x) = \theta_i(x)y^i$ and scalar function $k = k(x)$ on M such that:*

$$y_m G^m_\alpha = \theta\alpha^2, \tag{5.3}$$

$$3r_{00} - 2\theta\beta = \psi\alpha^2. \tag{5.4}$$

Proof. By assumption F is locally dually flat Finsler metrics, it must be satisfy (3.1). By replacing (5.1) and (5.2) into (3.1), removing denominators and sorting by α, one can obtain following equation:

$$\begin{aligned}
0 \;=\; & \alpha^6\Big\{2e^s(e^s+\varepsilon)(3s_{k0} - r_{k0} - b_m\frac{\partial G^m_\alpha}{\partial y^k})\Big\} \\
& +\alpha^5\Big\{2e^s b_k(e^s+\varepsilon)(r_{00} + 2b_m G^m_\alpha) \\
& +2\varepsilon b_k(e^s+\varepsilon)(r_{00} + 2b_m G^m_\alpha) + 2\varepsilon(e^s+\varepsilon)(3s_{k0} - r_{k0} - b_m\frac{\partial G^m_\alpha}{\partial y^k})\beta \\
& +2e^{2s}b_k(r_{00} + 2b_m G^m_\alpha) + 4e^{2s}a_{mk}G^m_\alpha - 2e^{2s}y_m\frac{\partial G^m_\alpha}{\partial y^k}\Big\} \\
& +\alpha^4\Big\{2e^s y_k(e^s+\varepsilon)(r_{00} + 2b_m G^m_\alpha) + 4e^s\varepsilon b_k y_m G^m_\alpha + 2e^s\varepsilon b_k(r_{00} \\
& +2b_m G^m_\alpha)\beta - 4e^{2s}a_{mk}G^m_\alpha\beta + 4e^s\varepsilon a_{mk}G^m_\alpha\beta + 2e^{2s}y_m\frac{\partial G^m_\alpha}{\partial y^k}\beta \\
& -2e^s\varepsilon y_m\frac{\partial G^m_\alpha}{\partial y^k}\beta + 4e^{2s}b_k y_m G^m_\alpha\Big\} \\
& +\alpha^3\Big\{-2e^s y_k(e^s+\varepsilon)(r_{00} + 2b_m G^m_\alpha)\beta - 8e^{2s}b_k y_m G^m_\alpha\beta - 2e^{2s}y_k(r_{00} \\
& +2b_m G^m_\alpha)\beta - 4e^s\varepsilon b_k y_m G^m_\alpha\beta - 4e^s\varepsilon a_{mk}G^m_\alpha\beta^2 + 2e^s\varepsilon y_m\frac{\partial G^m_\alpha}{\partial y^k}\beta^2\Big\} \\
& +\alpha^2\Big\{-2e^s\varepsilon y_k(r_{00} + 2b_m G^m_\alpha)\beta^2 - 4e^s\varepsilon y_k y_m G^m_\alpha\beta - 4e^{2s}y_k y_m G^m_\alpha\beta \\
& -4e^s\varepsilon b_k y_m G^m_\alpha\beta^2\Big\} + \alpha\Big\{8e^{2s}y_k y_m G^m_\alpha\beta^2 + 4e^s\varepsilon y_k y_m G^m_\alpha\beta^2\Big\} \\
& +4e^s\varepsilon y_k y_m G^m_\alpha\beta^3.
\end{aligned} \tag{5.5}$$

Contracting this equation with b^k, yields

$$\begin{aligned}
0 \;=\; & \alpha^6\Big\{2e^s(e^s+\varepsilon)(3s_0 - r_0 - b_m\frac{\partial G^m_\alpha}{\partial y^k}b^k)\Big\} \\
& +\alpha^5\Big\{2e^s b^2(e^s+\varepsilon)(r_{00} + 2b_m G^m_\alpha) \\
& +2\varepsilon b^2(e^s+\varepsilon)(r_{00} + 2b_m G^m_\alpha) + 2\varepsilon(e^s+\varepsilon)(3s_0 - r_0 - b_m\frac{\partial G^m_\alpha}{\partial y^k}b^k)\beta \\
& +2e^{2s}b^2(r_{00} + 2b_m G^m_\alpha) + 4e^{2s}b_m G^m_\alpha - 2e^{2s}y_m\frac{\partial G^m_\alpha}{\partial y^k}b^k\Big\} \\
& +\alpha^4\Big\{2(e^s+\varepsilon)(r_{00} + 2b_m G^m_\alpha)\beta + 4e^s\varepsilon b^2 y_m G^m_\alpha + 2e^s\varepsilon b^2(r_{00}
\end{aligned}$$

$$+2b_mG^m_\alpha)\beta-4e^{2s}b_mG^m_\alpha\beta+4e^s\varepsilon b_mG^m_\alpha\beta+2e^{2s}y_m\frac{\partial G^m_\alpha}{\partial y^k}b^k\beta$$
$$-2e^s\varepsilon y_m\frac{\partial G^m_\alpha}{\partial y^k}b^k\beta+4e^{2s}b^2y_mG^m_\alpha\Big\}$$
$$+\alpha^3\Big\{-2e^s(r_{00}+2b_mG^m_\alpha)\beta^2-8e^{2s}b^2y_mG^m_\alpha\beta-2e^{2s}(r_{00}$$
$$+2b_mG^m_\alpha)\beta^2-4e^s\varepsilon b^2y_mG^m_\alpha\beta-4e^s\varepsilon b_mG^m_\alpha\beta^2+2e^s\varepsilon y_m\frac{\partial G^m_\alpha}{\partial y^k}b^k\beta^2\Big\}$$
$$+\alpha^2\Big\{-2e^s\varepsilon(r_{00}+2b_mG^m_\alpha)\beta^3-4e^s\varepsilon y_mG^m_\alpha\beta^2-4e^{2s}y_mG^m_\alpha\beta^2$$
$$-4e^s\varepsilon b^2y_mG^m_\alpha\beta^2\Big\}+\alpha\Big\{8e^{2s}y_mG^m_\alpha\beta^3+4e^s\varepsilon y_mG^m_\alpha\beta^3\Big\}$$
$$+4e^s\varepsilon y_mG^m_\alpha\beta^4. \tag{5.6}$$

Contracting this equation with b^k, yields

$$\begin{aligned}0 \;=\;& \alpha^6\Big\{2e^s(e^s+\varepsilon)(3s_0-r_0-b_m\frac{\partial G^m_\alpha}{\partial y^k}b^k)\Big\}\\
&+\alpha^5\Big\{2e^sb^2(e^s+\varepsilon)(r_{00}+2b_mG^m_\alpha)\\
&+2\varepsilon b^2(e^s+\varepsilon)(r_{00}+2b_mG^m_\alpha)+2\varepsilon(e^s+\varepsilon)(3s_0-r_0-b_m\frac{\partial G^m_\alpha}{\partial y^k}b^k)\beta\\
&+2e^{2s}b^2(r_{00}+2b_mG^m_\alpha)+4e^{2s}b_mG^m_\alpha-2e^{2s}y_m\frac{\partial G^m_\alpha}{\partial y^k}b^k\Big\}\\
&+\alpha^4\Big\{2(e^s+\varepsilon)(r_{00}+2b_mG^m_\alpha)\beta+4e^s\varepsilon b^2y_mG^m_\alpha+2e^s\varepsilon b^2(r_{00}\\
&+2b_mG^m_\alpha)\beta-4e^{2s}b_mG^m_\alpha\beta+4e^s\varepsilon b_mG^m_\alpha\beta+2e^{2s}y_m\frac{\partial G^m_\alpha}{\partial y^k}b^k\beta\\
&-2e^s\varepsilon y_m\frac{\partial G^m_\alpha}{\partial y^k}b^k\beta+4e^{2s}b^2y_mG^m_\alpha\Big\}\\
&+\alpha^3\Big\{-2e^s(r_{00}+2b_mG^m_\alpha)\beta^2-8e^{2s}b^2y_mG^m_\alpha\beta-2e^{2s}(r_{00}+2b_mG^m_\alpha)\beta^2\\
&-4e^s\varepsilon b^2y_mG^m_\alpha\beta-4e^s\varepsilon b_mG^m_\alpha\beta^2+2e^s\varepsilon y_m\frac{\partial G^m_\alpha}{\partial y^k}b^k\beta^2\Big\}\\
&+\alpha^2\Big\{-2e^s\varepsilon(r_{00}+2b_mG^m_\alpha)\beta^3-4e^s\varepsilon y_mG^m_\alpha\beta^2-4e^{2s}y_mG^m_\alpha\beta^2\\
&-4e^s\varepsilon b^2y_mG^m_\alpha\beta^2\Big\}+\alpha\Big\{8e^{2s}y_mG^m_\alpha\beta^3+4e^s\varepsilon y_mG^m_\alpha\beta^3\Big\}\\
&+4e^s\varepsilon y_mG^m_\alpha\beta^4\end{aligned} \tag{5.7}$$

which implies that there is a 1- form $\theta(x)=\theta_i(x)y^l$ on M such that:

$$y_mG^m_\alpha=\theta\alpha^2, \tag{5.8}$$

then

$$0 \;=\; \alpha^4\Big\{2e^s(e^s+\varepsilon)(3s_0-r_0-b_mb^k\frac{\partial G^m_\alpha}{\partial y^k})+4e^s\varepsilon b^2\theta+4e^{2s}b^2\theta\Big\}$$

$$
\begin{aligned}
&+\alpha^3\Big\{2e^s b^2(e^s+\varepsilon)(r_{00}+2b_mG^m_\alpha)+2\varepsilon b^2(e^s+\varepsilon)(r_{00}+2b_mG^m_\alpha)\\
&+2\varepsilon(e^s+\varepsilon)(3s_0-r_0-b_mb^k\frac{\partial G^m_\alpha}{\partial y^k})\beta+2e^{2s}b^2(r_{00}+2b_mG^m_\alpha)\\
&+4e^{2s}b_mG^m_\alpha-2e^{2s}b^ky_m\frac{\partial G^m_\alpha}{\partial y^k}-8e^{2s}b^2\theta\beta-4e^s\varepsilon b^2\theta\beta\Big\}\\
&+\alpha^2\Big\{2e^s(e^s+\varepsilon)(r_{00}+2b_mG^m_\alpha)\beta\\
&+2e^s\varepsilon b^2\beta(r_{00}+2b_mG^m_\alpha)-4e^{2s}b_mG^m_\alpha\beta+4e^s\varepsilon b_mG^m_\alpha\beta\\
&+2e^{2s}y_m\frac{\partial G^m_\alpha}{\partial y^k}b^k\beta-2e^s\varepsilon y_m\frac{\partial G^m_\alpha}{\partial y^k}b^k\beta-4e^s\varepsilon\theta\beta^2\\
&-4e^{2s}\theta\beta^2-4e^s\varepsilon b^2\theta\beta^2\Big\}+\alpha\Big\{-2e^s(e^s+\varepsilon)(r_{00}+2b_mG^m_\alpha)\beta^2\\
&-2e^{2s}(r_{00}+2b_mG^m_\alpha)\beta^2-4e^s\varepsilon b_mG^m_\alpha\beta^2+2e^s\varepsilon y_mb^k\frac{\partial G^m_\alpha}{\partial y^k}\beta^2\\
&+8e^{2s}\theta\beta^3+4e^s\varepsilon\theta\beta^3\Big\}+\Big\{-2e^s\varepsilon(r_{00}+2b_mG^m_\alpha)\beta^3+4e^s\varepsilon\theta\beta^4\Big\}. \quad (5.9)
\end{aligned}
$$

Differentiating (5.8) with respect to y^k yields

$$a_{mk}G^m_\alpha+y_m\frac{\partial G^m_\alpha}{\partial y^k}=\theta_k\alpha^2+2\theta y_k. \tag{5.10}$$

Contracting (5.10) with b^k, one has

$$b_mG^m_\alpha+y_m\frac{\partial G^m_\alpha}{\partial y^k}b^k=\theta_kb^k\alpha^2+2\theta\beta. \tag{5.11}$$

Also

$$
\begin{aligned}
&(5.9)+2e^{2s}\alpha^3\times(5.11)-2e^{2s}\times\alpha^2\beta(5.11)+2e^s\varepsilon\times\alpha^2\beta\times(5.11)\\
&-2e^s\varepsilon\alpha\beta^2(5.11)
\end{aligned}
$$

yields

$$
\begin{aligned}
0 \;=\;& \alpha^5\Big\{-2e^{2s}\theta_kb^k\Big\}+\alpha^4\Big\{2e^s(e^s+\varepsilon)(3s_0-r_0-b_m\frac{\partial G^m_\alpha}{\partial y^k}b^k)\\
&+4e^s\varepsilon b^2\theta+4e^{2s}b^2\theta+2e^{2s}\theta_kb^k\beta+-2e^s\varepsilon\theta_kb^k\beta\Big\}\\
&+\alpha^3\Big\{2e^sb^2(e^s+\varepsilon)(r_{00}+2b_mG^m_\alpha)+2\varepsilon b^2(e^s+\varepsilon)(r_{00}+2b_mG^m_\alpha)\\
&+2\varepsilon(e^s+\varepsilon)(3s_0-r_0-b_m\frac{\partial G^m_\alpha}{\partial y^k}b^k)\beta+2e^{2s}b^2(r_{00}+2b_mG^m_\alpha)\\
&+4e^{2s}b_mG^m_\alpha-8e^{2s}b^2\theta\beta-4e^s\varepsilon b^2\theta\beta+2e^{2s}b_mG^m_\alpha-4e^{2s}\theta\beta\\
&+2e^s\varepsilon\theta_kb^k\beta^2\Big\}+
\end{aligned}
$$

$$+\alpha^2\Big\{2e^s(e^s+\varepsilon)(r_{00}+2b_mG^m_\alpha)\beta+2e^s\varepsilon b^2(r_{00}+2b_mG^m_\alpha)\beta$$
$$-4e^{2s}b_mG^m_\alpha\beta+4e^s\varepsilon b_mG^m_\alpha\beta-8e^s\varepsilon\theta\beta^2-4e^s\varepsilon b^2\theta\beta^2-2e^{2s}b_mG^m_\alpha\beta$$
$$+2e^s\varepsilon b_mG^m_\alpha\beta\Big\}$$
$$+\alpha\Big\{-2e^s(e^s+\varepsilon)(r_{00}+2b_mG^m_\alpha)\beta^2-2e^{2s}(r_{00}+2b_mG^m_\alpha)\beta^2$$
$$-4e^s\varepsilon b_mG^m_\alpha\beta^2+8e^{2s}\theta\beta^3+8e^s\varepsilon\theta\beta^3-2e^s\varepsilon b_mG^m_\alpha\beta^2\Big\}$$
$$+\Big\{-2e^s\varepsilon(r_{00}+2b_mG^m_\alpha)\beta^3+4e^s\varepsilon\theta\beta^4\Big\}, \tag{5.12}$$

which implies there is a scalar function $k=k(x)$ on manifold M such that:

$$2\theta\beta-(r_{00}+2b_mG^m_\alpha)=k\alpha^2, \tag{5.13}$$

that is:

$$b_mG^m_\alpha=\theta\beta-\frac{r_{00}+k\alpha^2}{2}. \tag{5.14}$$

Diffierentiating (5.14) with respect to y^k yields

$$b_m\frac{\partial G^m_\alpha}{\partial y^k}=\theta_k\beta+b_k\theta-r_{k0}-ky_k. \tag{5.15}$$

Plugging (5.14) and (5.15) into (5.12), we can get (5.4). □

Proof of Theorem 1.2: Plugging (5.8) into (5.5), yields

$$0 = \alpha^4\Bigg\{2e^s\Big(e^s+\varepsilon)(3s_{k0}-r_{k0}-b_m\frac{\partial G^m_\alpha}{\partial y^k}\Big)+4e^s\varepsilon b_k\theta+4e^{2s}b_k\theta\Bigg\}$$
$$+\alpha^3\Bigg\{2e^sb_k(e^s+\varepsilon)(r_{00}+2b_mG^m_\alpha)+2\varepsilon b_k(e^s+\varepsilon)(r_{00}+2b_mG^m_\alpha)$$
$$+2\varepsilon(e^s+\varepsilon)\Big(3s_{k0}-r_{k0}-b_m\frac{\partial G^m_\alpha}{\partial y^k}\Big)\beta+2e^{2s}b_k(r_{00}+2b_mG^m_\alpha)$$
$$+4e^{2s}a_{mk}G^m_\alpha-2e^{2s}y_m\frac{\partial G^m_\alpha}{\partial y^k}-8e^{2s}b_k\theta\beta-4e^s\varepsilon b_k\theta\beta\Bigg\}$$
$$+\alpha^2\Bigg\{2e^sy_k\Big(e^s+\varepsilon\Big)\Big(r_{00}+2b_mG^m_\alpha\Big)+2e^s\varepsilon b_k\Big(r_{00}+2b_mG^m_\alpha\Big)\beta$$
$$-4e^{2s}a_{mk}G^m_\alpha\beta+4e^s\varepsilon a_{mk}G^m_\alpha\beta+2e^{2s}y_m\frac{\partial G^m_\alpha}{\partial y^k}\beta-2e^s\varepsilon y_m\frac{\partial G^m_\alpha}{\partial y^k}\beta$$
$$-4e^s\varepsilon y_k\theta\beta-4e^{2s}y_k\theta\beta-4e^s\varepsilon b_k\theta\beta^2\Bigg\}+\alpha\Bigg\{-2e^sy_k\Big(e^s+\varepsilon\Big)\Big(r_{00}$$
$$+2b_mG^m_\alpha\Big)\beta-2e^{2s}y_k\Big(r_{00}+2b_mG^m_\alpha\Big)\beta-4e^s\varepsilon a_{mk}G^m_\alpha\beta^2$$

$$+2e^s\varepsilon y_m\frac{\partial G^m_\alpha}{\partial y^k}\beta^2+8e^{2s}y_k\theta\beta^2+4e^s\varepsilon y_k\theta\beta^2\Bigg\}+\Bigg\{-2e^s\varepsilon y_k\Big(r_{00}$$
$$+2b_mG^m_\alpha\Big)\beta^2+4e^s\varepsilon y_k\theta\beta^3\Bigg\}. \qquad (5.16)$$

We know:

$$\begin{aligned} y_m\frac{\partial G^m_\alpha}{\partial y^k} &= \frac{\partial(y_mG^m_\alpha)}{\partial y^k}-a_{mk}G^m_\alpha \\ &= \theta_k\alpha^2+2y_k\theta-a_{mk}G^m_\alpha. \end{aligned} \qquad (5.17)$$

Pluging (5.17) into (5.16), yields

$$\begin{aligned} 0 \;=\; & \alpha^5\Big\{-2e^{2s}\theta_k\Big\}+\alpha^4\Bigg\{2e^s(e^s+\varepsilon)(3s_{k0}-r_{k0}-b_m\frac{\partial G^m_\alpha}{\partial y^k})+4e^s\varepsilon b_k\theta \\ & +4e^{2s}b_k\theta+2e^{2s}\theta_k\beta-2e^s\varepsilon\theta_k\beta\Bigg\}+\alpha^3\Bigg\{2e^sb_k(e^s+\varepsilon)(r_{00}+2b_mG^m_\alpha) \\ & +2\varepsilon b_k(e^s+\varepsilon)(r_{00}+2b_mG^m_\alpha)+2\varepsilon(e^s+\varepsilon)(3s_{k0}-r_{k0}-b_m\frac{\partial G^m_\alpha}{\partial y^k})\beta \\ & +2e^{2s}b_k(r_{00}+2b_mG^m_\alpha)+6e^{2s}a_{mk}G^m_\alpha-4e^{2s}\theta y_k-8e^{2s}b_k\theta\beta \\ & -4e^s\varepsilon b_k\theta\beta+2e^s\varepsilon\theta_k\beta^2\Bigg\}+\alpha^2\Bigg\{2e^sy_k(e^s+\varepsilon)(r_{00}+2b_mG^m_\alpha) \\ & +2e^s\varepsilon b_k(r_{00}+2b_mG^m_\alpha)\beta-6e^{2s}a_{mk}G^m_\alpha\beta+6e^s\varepsilon a_{mk}G^m_\alpha\beta \\ & -8e^s\varepsilon\theta y_k\beta-4e^s\varepsilon b_k\theta\beta^2\Bigg\}+\alpha\Bigg\{-2e^sy_k(e^s+\varepsilon)(r_{00}+2b_mG^m_\alpha)\beta \\ & -2e^{2s}y_k(r_{00}+2b_mG^m_\alpha)\beta-6e^s\varepsilon a_{mk}G^m_\alpha\beta^2+8e^{2s}y_k\theta\beta^2+8e^s\varepsilon y_k\theta\beta^2\Bigg\} \\ & +\Bigg\{-2e^s\varepsilon y_k(r_{00}+2b_mG^m_\alpha)\beta^2+4e^s\varepsilon y_k\theta\beta^3\Bigg\}. \end{aligned}$$

Plugging (5.14).(5.15) in the above equation:

$$\begin{aligned} 0 \;=\; & \alpha^4\Bigg\{-2e^sb_k(e^s+\varepsilon)k-2\varepsilon b_k(e^s+\varepsilon)k-2e^{2s}b_kk-2e^{2s}\theta_k\Bigg\} \\ & \alpha^3\Bigg\{6e^s(e^s+\varepsilon)s_{k0}+2e^s\varepsilon b_k\theta+2e^{2s}b_k\theta-4e^s\varepsilon\theta_k\beta-2e^s\varepsilon b_kk\beta\Bigg\} \end{aligned}$$

$$\alpha^2\Big\{2\varepsilon b_k(e^s+\varepsilon)\theta\beta+2\varepsilon(e^s+\varepsilon)(3s_{k0}+ky_k)\beta+6e^{2s}a_{mk}G^m_\alpha$$
$$-4e^{2s}\theta y_k-2\varepsilon^2\theta_k\beta^2+2e^s y_k(e^s+\varepsilon)k\beta+2e^{2s}y_k k\beta\Big\}$$
$$\alpha\Big\{4e^{2s}y_k\theta\beta-6e^{2s}a_{mk}G^m_\alpha\beta+6e^s\varepsilon a_{mk}G^m_\alpha\beta-4e^s\varepsilon\theta y_k\beta+2e^s\varepsilon y_k k\beta^2\Big\}$$
$$+\Big\{-6e^s\varepsilon a_{mk}G^m_\alpha\beta^2+4e^s\varepsilon y_k\theta\beta^2\Big\}, \tag{5.18}$$

which implies there is scalar function $\varphi_k=\varphi_k(x)$ on manifold M such that:

$$2y_k\theta-3a_{mk}G^m_\alpha=\varphi_k\alpha^2. \tag{5.19}$$

Contracting (5.19) with a^{il}, we have

$$G^i_\alpha=\frac{1}{3}(2\theta y^i-\varphi^i\alpha^2). \tag{5.20}$$

By using (5.18) we can obtain two essential equation:

$$Rat+\alpha Irat=0, \tag{5.21}$$

where

$$Irat = \alpha\Big\{\alpha^2\Big[-2e^s(e^s+\varepsilon)b_k k-2\varepsilon b_k(e^s+\varepsilon)k-2e^{2s}b_k k-2e^{2s}\varphi_k \tag{5.22}$$
$$-2e^{2s}\theta_k\Big]+\Big[2\varepsilon b_k(e^s+\varepsilon)\theta\beta+6\varepsilon(e^s+\varepsilon)s_{k0}\beta-2\varepsilon^2\theta_k\beta^2 \tag{5.23}$$
$$+4e^{2s}y_k k\beta+4e^s\varepsilon y_k k\beta+2\varepsilon^2 y_k k\beta+2e^s\varepsilon\varphi_k\beta^2\Big]\Big\}=0, \tag{5.24}$$

and

$$Rat = \alpha^2\Big\{6e^s(e^s+\varepsilon)s_{k0}+2e^s\varepsilon b_k\theta+2e^{2s}b_k\theta-4e^s\varepsilon\theta_k\beta \tag{5.25}$$
$$-2e^s\varepsilon b_k k\beta+2e^{2s}\varphi_k\beta-2e^s\varepsilon\varphi_k\beta\Big\}\Big\{2e^s\varepsilon y_k k\beta^2\Big\}=0. \tag{5.26}$$

Contraction this very equation with b^k, then $k=0$:

$$s_0=\frac{1}{3}\Big\{\varphi_k b^k\beta-b^2\theta+\frac{2}{(e^s+\varepsilon)}\big\{\varepsilon\theta_k b^k-e^s\varphi_k b^k\big\}\beta\Big\}. \tag{5.27}$$

we obtain (1.3). This completes the proof. □

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Received: 10.04.2022
Accepted: 24.06.2022