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On Hyperactions and Lie Hypergroup

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Abstract. Using the action of a Lie group on a hypergroup, the notion of Lie hypergroup is defined. It is proved that tangent space of a Lie hypergroup is a hypergroup and that a differentiable map between two Lie hypergroup is good homomorphism if and only if its differential map is a good homomorphism. The action of a hypergroup on a set is defined. Using this new notion, hypergroup bundle is introduced and some of its basic properties are investigated. In addition, some results on quotient hypergroups are given.

Keywords: Hypergroup, Action, Quotient Space, Sub Hypergroup, Hypergroup Bundle.

1. **Introduction**

The concept of hypergroup arose originally as a generalization of the concept of abstract group. This concept was first introduced by Marty in 1935 [12]. Furthermore, some surveys and papers such as [3, 12, 15, 19] were published in the field of hypergroup and its applications. In [8, 16] the notions of hyperring and hyperfield is introduced to use it as a technical tool in the study of the approximation of valued fields. Then the notion of hypervector space was

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introduced by M. Scafati Tallini in 1988 [14]. M. R Molaei at.el studied semi hypergroups and their properties in [9, 10, 11, 13]. The concept of topological hypergroup is introduced in [6]. In addition in [17] M. Toomanian at.el give a definition for Lie hypergroup by using of convolution map on a hypergroup, which is also a smooth manifold. In the theory of Lie groups many aspects of topology and harmonic analysis become simpler and more natural. Many facts from harmonic analysis and representation theory of groups carry over hypergroups. In this paper, we introduce Lie hypergroup from a geometric point of view by using the action of a Lie group on a hypergroup. The paper has two basic parts. In the first part, the concept of Lie hypergroup is defined generally. Then using some preliminaries hypergroup bundle is introduced in the second part. In the second section, some properties of quotient hypergroups are found. In section 3, a sufficient condition for regularity of the equivalence relation assotiated to the action of a Lie group on a hypergroup is given. In section 4, using of the action of a Lie group on a hypergroup Lie hypergroup is introduced and some basic properties are given. It is proved that if left transformation on a Lie hypergroup be homomorphism then the transformation on its associated Lie hypergroup is homomorphism. In section 5, by using of hyper action some properties of Lie hypergroups are given. For example, if a regular hypergroup be reversible then its associated Lie hypergroup is also reversible. In the last section, hypergroup bundle is defined and fibers in different situations are found.

As follows, some basic notions and examples are reviewed.

Let P be a non-empty set and $p^*(P)$ be the set of all non empty subsets of P. A hyperoperation on P is a map $\circ : P \times P \to p^*(P)$ [5]. The ordered pair $(P, \circ)$ is called a *hypergroupoid*. If A and B are two non empty subsets of P and $x \in P$, then

$$A \circ B = \cup_{a\in A, b\in B} a \circ b, \quad x \circ A = \{x\} \circ A, \quad A \circ x = A \circ \{x\}.$$

The hypergroupoid $(P, \circ)$ with the following properties is called a *hypergroup*.
1) $a \circ (b \circ c) = (a \circ b) \circ c$, for all $a, b, c \in P$,
2) $a \circ P = P \circ a = P$, for all $a \in P$.
If there is an $e \in P$ such that $a \in a \circ e \cap e \circ a$, for all $a \in P$, then e is called *identity*. Let P be a hypergroup with at least one identity then an element $a^{-1} \in P$ is *inverse* of $a \in P$ if $e \in a \circ a^{-1} \cap a^{-1} \circ a$. A hypergroup is called *regular* if it has at least one identity and for any element has at least one inverse.

Remark 1.1. *If every element of p has inverse, so* $(p_1 \circ p_2)^{-1} = p_2^{-1} \circ p_1^{-1}$ *for all* $p_1, p_2 \in P$.

Remark 1.2. *An equivalence relation, $\sim$, on the hypergroup $(P, \circ)$ is regular to the right if*

$$a \sim b \implies \forall u \in P \ \& \ \forall x \in a \circ u \ \exists y \in b \circ u \ : \ x \sim y.$$

and

$$\forall y^{'} \in b \circ u, \ \exists x^{'} \in a \circ u \ : \ x^{'} \sim y^{'}.$$

The concept of regular to the left is defined in the similar way.

Definition 1.3. [3] *A subset S of a hypergroup $(P, \circ)$ is called sub hypergroup if it satisfies the following properties:*
i) $a \circ b \subseteq S$ for all $a, b \in S$.
ii) $a \circ S = S \circ a = S$ for every $a \in S$.

Example 1.4. *(Affine Join Space) Let V be a vector space. Define hyperoperation $\circ : V \times V \longrightarrow P^*(V)$ by*

$$a \circ b = \Big\{ \lambda a + \mu b \ : \ \lambda, \mu > 0 \ , \lambda + \mu = 1 \Big\}, \quad \forall a, b \in V.$$

We can see easily V with this hyperoperation is a hypergroup and every subspace of V is a sub hypergoup.

A sub hypergroup S of hypergroup $(P, \circ)$ is *normal* if $x \circ H = H \circ x$, for all $x \in P$ and is *supernormal* if $x \circ H \circ x^{-1} \subseteq H$, for all $x \in P$. A mapping f from a hypergroup $(P_1, \circ)$ to a hypergroup $(P_2, *)$ is called a
1) *homomorphism* if for all $x, y \in P_1$, $f(x \circ y) \subseteq f(x) * f(y)$.
2) *good homomorphism* if for all $x, y \in P_1$, $f(x \circ y) = f(x) * f(y)$.
Let (T, τ) be a topological space. Then, the family U consisting of all sets $S_V = \{U \in p^*(T) | U \subseteq V, U \in \tau\}$ is a basis for a topology on $p^*(T)$. This topology is denoted by τ^* [7].

2. **Some Results on Quotient Hypergroups**

Let H be a sub hypergroup of hypergroup $(P, \circ)$. We recall that the right coset space is $P/H = \{[x] \ : \ x \in P \ \}$ where $[x] = x \circ H$. Let us consider $\otimes : P/H \times P/H \longrightarrow p^*(P/H)$ where $[x] \otimes [y] = \{[z] \ : \ z \in x \circ y \ \}$ for all $[x], [y] \in P/H$.

Theorem 2.1. *Let $(P, \circ)$ be a regular hypergroup and H be a normal sub hypergroup of P. Then $(P/H, \otimes)$ is a regular hypergroup.*

Proof. Obviously $(x\circ H)\otimes(y\circ H)\subseteq P/H$ and the identity is $e_{P/H}=H$. Since H is a normal sub hypergroup, so $x\circ H\in(x\circ H)\otimes H\bigcap H\otimes(x\circ H)$. Also set $(x\circ H)^{-1}=x^{-1}\circ H$, so

$$\begin{aligned}(x\circ H\otimes y\circ H)^{-1} &= (x\circ y\circ H)^{-1}\\ &= (x\circ y)^{-1}\circ H\\ &= y^{-1}\circ x^{-1}\circ H\\ &= (y\circ H)^{-1}\otimes(x\circ H)^{-1}.\end{aligned}$$

Hence $(P/H,\otimes)$ is a regular hypergroup. □

Theorem 2.2. *Let H be a normal sub hypergroup of a hypergroup $(P,\circ)$. Then the function $f: P\longrightarrow P/H$ that $f(x)=x\circ H$, for all $x\in H$, is a good homomorphism.*

Proof. H is a normal sub hypergroup so, $H\circ H=H$. Let $x,y\in P$ be arbitrary. Then, we get

$$\begin{aligned}f(x\circ y)=x\circ y\circ H &= x\circ H\circ y\\ &= x\circ H\circ H\circ y\\ &= x\circ H\circ y\circ H\\ &= f(x)\otimes f(y).\end{aligned}$$

The proof is complete. □

In the next theorem, it is proved that the homomorphism f which is defined in the above theorem takes a normal sub hypergroup of P to a sub hypergroup of P/H.

Theorem 2.3. *Let $H,K,\ H\subseteq K$, be normal sub hypergroups of hypergroup $(P,\circ)$. Then, K/H is a sub hypergroup of $(P/H,\otimes)$.*

Proof. The following hold:

1 . Let $a,b\in K$ be arbitrary. Hence, we have

$$(a\circ H)\otimes(b\circ H)=\{\ z\circ H\ :\ z\in a\circ b\ \}\subseteq\frac{K}{H}.$$

2 . Let $a\in K$ be arbitrary, so

$$(a\circ H)\otimes\frac{K}{H}=\cup_{k\in K}(a\circ H)\otimes(k\circ H)=\cup_{k\in K}\{\ z\circ H\ :z\in a\circ k\ \}\subseteq K/H.$$

Similarly, we can prove $\frac{K}{H}\otimes(a\circ H)\subseteq\frac{K}{H}$.
On the other hand, let $z\circ H\in(a\circ H)\otimes\frac{K}{H}$ where $z\in a\circ k$ for some $k\in K$. The claim is $z\circ H\in\frac{K}{H}\otimes(a\circ H)$. K is a normal sub hypergroup of P so, $a\circ K=K\circ a$. Thus, there is $k_1\in K$ such that $z\in k_1\circ a$. Hence, $(a\circ H)\otimes\frac{K}{H}=\frac{K}{H}\otimes(a\circ H)=\frac{K}{H}$ for all $a\in K$. □

3. **Action of a Lie Group on a Hypergroup**

In this section some properties of action of a Lie group on a hypergroup are investigated. This kind of action is used to define Lie hypergroup in the next section. Let G be a Lie group and $(P, \circ)$ be a hypergroup. Suppose that G acts on P to the right by $\varphi : P \times G \to P$.

Lemma 3.1. *Let H be a Lie subgroup of G and $\varphi(p_1, g_1) \circ \varphi(p_2, g_2) \subseteq \varphi(p_1 \circ p_2, g_1 g_2)$, for all $p_1, p_2 \in P$ and $g_1, g_2 \in G$. Then the range of the restriction of action on H is a sub hypergroup.*

Proof. Let $\varphi(P \times H) = P_0$, we prove that $(P_0, \circ)$ is closed with respect to the action of hypergroup. If $p'_1, p'_2 \in P_0$ then there are $p_1, p_2 \in P$ and $h_1, h_2 \in H$ such that $p'_1 = \varphi(p_1, h_1)$ and $p'_2 = \varphi(p_2, h_2)$. Thus,

$$p'_1 \circ p'_2 = \varphi(p_1, h_1) \circ \varphi(p_2, h_2) \subseteq \varphi((p_1 \circ p_2), h_1 h_2) \subseteq P_0.$$

Next, we prove that P_0 is a sub hypergroup. Let $\varphi(p, h) \in P_0$ so,

$$\varphi(p, h) \in \varphi(p_1, h_1) \circ \varphi(p_2, h_2) \in \varphi(p_1 \circ p_2, h_1 h_2) \subseteq x \circ P,$$

where $x = \varphi(p_1, h_1)$. □

Theorem 3.2. *Let φ be a transitive action of a Lie group G on a hypergroup $(P, \circ)$ and $\varphi(p_1 \circ p_2, g) \subseteq \varphi(p_1, g) \circ p_2$, $\varphi(p_1 \circ p_2, g) \subseteq p_1 \circ \varphi(p_2, g)$, for every $p_1, p_2 \in P$ and $g \in G$, Then $\sim$ is regular.*

Proof. Let $p_1 \sim p_2$ for $p_1, p_2 \in P$. $u \in P$ and $x \in p_1 \circ u$ are arbitrary. There is $g \in G$ such that $p_2 = \varphi(p_1, g)$ and $\varphi(x, g) \in \varphi((p_1 \circ u), g) \subseteq \varphi(p_1, g) \circ u$. Therefore $\varphi(x, g) \in p_2 \circ u$. Set $y = \varphi(x, g)$, so $x \sim y$.
On the other hand, let $y^{'} \in p_2 \circ u$ be arbitrary. Then, we have

$$\varphi(y^{'}, g^{-1}) \in \varphi((p_2 \circ u), g^{-1}) \subseteq \varphi(p_2, g^{-1}) \circ u = p_1 \circ u.$$

Set $x^{'} = \varphi(y^{'}, g^{-1})$, so $x^{'} \sim y^{'}$. Similarly, we can prove that $\sim$ is regular to the left. □

The converse of above theorem is not true.

Example 3.3. *Let $P = \mathbb{R}$. Then P defined by hyperoperation $\circ : P \times P \longrightarrow p^*(P)$, $p_1 \circ p_2 \mapsto \{p_1, p_2\}$ is a hypergroup.*
Define $p_1 \sim p_2$ if and only if $p_1 \equiv p_2 \pmod m$ where $m \in \mathbb{N}$. (i.e. $p_1 = mk + p_2$ $\exists k \in \mathbb{Z}$).
Consider the Lie group $G = \mathbb{R}$ acts on P to the right such that the equivalence relation is preserved. One can check easily $(P, \sim)$ is a regular equivalence relation. Also $\varphi((p_1 \circ p_2), g) = \{\varphi(p_1, g), \varphi(p_2, g)\}$ and $\varphi(p_1, g) \circ p_2 = \{\varphi(p_1, g), p_2\}$. Clearly, The sets $\varphi((p_1 \circ p_2), g)$ and $\varphi(p_1, g) \circ p_2$ are not subset of each other.

4. **Lie Hypergroup and Tangent Space**

Now to define Lie hypergroup, the quotient of a Lie group on its stabilazer is concidered. Let $(P, \circ)$ be a hypergroup, G be a Lie group and G acts on P to the right by smooth action φ. We recall that $G_{p_0} = \{g \in G : \varphi(p_0, g) = p_0\}$ is called the *stabilizer* of action for $p_0 \in P$. If P be a smooth manifold then $\frac{G}{G_{p_0}} \cong P$ [18]. Consider $f : P \longrightarrow \frac{G}{G_{p_0}}$ be the isomorphism function between hypergroup P and Lie hypergroup $\frac{G}{G_{p_0}}$. $f^{-1} : \frac{G}{G_{p_0}} \longrightarrow P$ is defined by $f^{-1}(gG_{p_0}) = \varphi(g, p_0)$. This isomorphism induces the following hyperoperation on $\frac{G}{G_{p_0}}$.

$$\circ' : \frac{G}{G_{p_0}} \times \frac{G}{G_{p_0}} \longrightarrow p^*(\frac{G}{G_{p_0}})$$

where $gG_{p_0} \circ' g'G_{p_0} = f(p \circ p')$ which $f(p) = gG_{p_0}$ and $f(p') = g'G_{p_0}$.
The identity set is

$$e\Big(\frac{G}{G_{p_0}}\Big) = \Big\{f(e) \ : \ e \in e(P)\Big\}.$$

If the hypergroup P is invertible, then $\frac{G}{G_{p_0}}$ is invertible and the set of all inverses of gG_{p_0} in $\frac{G}{G_{p_0}}$ is

$$i(gG_{p_0}) = \Big\{g'G_{p_0} \ : f(e) \in gG_{p_0} \circ' g'G_{p_0} \cap g'G_{p_0} \circ' gG_{p_0}\Big\}.$$

Hence, $\frac{G}{G_{p_0}}$ is a hypergroup which we call Lie hypergroup.
This definition is different from the one which is given in [17].

Example 4.1. *$O(3)$, the set of orthogonal 3×3 matrices, acts on S^2 in the following way:*
$\varphi(A, x) = Ax$, for $A \in O(3)$ and $x \in S^2$. $\frac{O(3)}{O(2)} \cong S^2$[18]. Using example 1.3 $\frac{O(3)}{O(2)}$ is a Lie hypergroup.

Let us investigate some basic properties of Lie hypergroups. actually, Theses are generalization of Lie group properties.

Remark 4.2. *If P' is a sub hypergroup of P then $f(P')$ is a sub hypergroup of $\frac{G}{G_{p_0}}$.*

Theorem 4.3. *Let G be a Lie group and $p_0 \in P$ be a fixed element of hypergroup P. If $p_0 \in p_0 \circ p_0$ and $\varphi(g_1, p_0) \circ \varphi(g_2, p_0) = \varphi(g_1 g_2, p_0 \circ p_0)$ for all $g_1, g_2 \in G$. Then $g_1 g_2 G_{p_0} \in g_1 G_{p_0} \circ' g_2 G_{p_0}$.*

Proof. Let $p_1 = \varphi(g_1, p_0)$ and $p_2 = \varphi(g_2, p_0)$. By assumptions

$$p_1 \circ p_2 = \varphi(g_1, p_0) \circ \varphi(g_2, p_0) = \varphi(g_1 g_2, p_0 \circ p_0).$$

Hence, $f^{-1}(g_1 g_2 G_{p_0}) = \varphi(g_1 g_2, p_0) \in p_1 \circ p_2$. So, $g_1 g_2 G_{p_0} \in f(p_1 \circ p_2) = g_1 G_{p_0} \circ' g_2 G_{p_0}$. $\square$

Theorem 4.4. *Let* $(P_1, \circ_1)$ *and* $(P_2, \circ_2)$ *be hypergroups. Consider* $\varphi_1 : P_1 \times G \longrightarrow P_1$ *and* $\varphi_2 : P_2 \times G \longrightarrow P_2$ *are good homomorphism actions. Consider* $p_1 \in P_1$ *and* $p_2 \in P_2$ *are arbitrary and fixed points. If* $\psi_0 : \frac{G}{G_{p_1}} \longrightarrow \frac{G}{G_{p_2}}$ *is a good homomorphism then there is a map* $\psi : P_1 \longrightarrow P_2$ *such that the following diagram commutes and* ψ *is a good homomorphism. Where* $f : P_1 \longrightarrow \frac{G}{G_{p_1}}$ *and* $g : P_2 \longrightarrow \frac{G}{G_{p_2}}$ *are isomorphisms which are correspond the actions* φ_1 *and* φ_2.

Proof. Using assumptions $\psi(\omega) = g^{-1} \circ \psi_0 \circ f(\omega) = \varphi_2(p_2, \psi_0(f(\omega)))$ for all $\omega \in P_1$. Therefore,

$$\begin{aligned}
\psi(\omega_1) \circ_2 \psi(\omega_2) &= \varphi_2(p_2, \psi_0(f(\omega_1))) \circ_2 \varphi_2(p_2, \psi_0(f(\omega_2))) \\
&= \varphi_2(p_2, \psi_0(f(\omega_1)) \circ_2' \psi_0(f(\omega_2))) \\
&= \varphi_2(p_2, \psi_0(f(\omega_1)) \circ_2' f(\omega_2))) \\
&= \varphi_2(p_2, \psi_0(f(\omega_1 \circ_1 \omega_2)) \\
&= g^{-1} \circ \psi_0 \circ f(\omega_1 \circ_1 \omega_2) \\
&= \psi(\omega_1 \circ_1 \omega_2).
\end{aligned}$$

As follows, tangent space of a hypergroup is introduced as a hypergroup. Consider T_pP is tangent space on the manifold P at point p and $TP = \cup_{p \in P} T_pP$ is tangent bundle. In addition, suppose that $(P, \circ)$ be a hypergroup. Let $v_1 \in T_{p_1}P$ and $v_2 \in T_{p_2}P$. Define the hyperoperation $\# : TP \times TP \longrightarrow p^*(TP)$ where $v_1 \# v_2 = \{v \;\;:\;\; v \in T_pP \quad p \in p_1 \circ p_2 \}$. □

Theorem 4.5. $(TP, \#)$ *is a hypergroup.*

Proof. Let v_1, v_2 and v_3 are arbitrary tangent vectors. Then, we have

$$\begin{aligned}
v_1 \# (v_2 \# v_3) &= \Big\{ v_1 \# v \;\;:\;\; v \in T_pP \quad p \in p_1 \circ p_2 \Big\} \\
&= \Big\{ v' \;\;:\;\; v' \in T_{p'}P \quad p' \in p_1 \circ p \Big\} \\
&= \Big\{ v' \;\;:\;\; v' \in T_{p'}P \quad p' \in p_1 \circ (p_2 \circ p_3) \Big\} \\
&= \Big\{ v' \;\;:\;\; v' \in T_{p'}P \quad p' \in (p_1 \circ p_2) \circ p_3 \Big\} \\
&= (v_1 \# v_2) \# v_3.
\end{aligned}$$

Also for $v \in T_pP$, we have

$$\begin{aligned}
v \# TP &= \{ v' \;\; : v' \in v \# v_1, \; v_1 \in T_{p_1}P \;\; \exists p_1 \in P \} \\
&= \{ v' \;\;:\;\; v' \in T_{p'}P \;\; p' \in p \circ p_1 \} \\
&= TP.
\end{aligned}$$

Thus, $(TP, \#)$ is a hypergroup. This implies that the tangent bundle of a Lie hypergroup is a hypergroup. □

Let $(P_1, \circ_1)$ and $(P_2, \circ_2)$ be hypergroups which are manifold too. By previous theorem $(TP_1, \#)$ and $(TP_2, \#')$ are hypergroups.

Theorem 4.6. *If $\psi : P_1 \longrightarrow P_2$ is a differentiable function. Then, ψ is good homomorphism if and only if $d\psi : TP_1 \longrightarrow TP_2$ is good homomorphism.*

Proof. Let $\omega_1, \omega_2 \in P_1$ be arbitrary and $v_1 \in T_{\omega_1}P_1$ and $v_2 \in T_{\omega_2}P_1$. Firsly, consider ψ is a good homomorphism. Let $v \in v_1\#v_2$, $d\psi(v) \in d\psi(v_1\#v_2)$ such that $v \in T_\omega P_1$ and $\omega \in \omega_1 \circ_1 \omega_2$. Hence, $d\psi(v) \in T_{\psi(\omega)}P_2$ and $\psi(\omega) \in \psi(\omega_1 \circ_1 \omega_2) = \psi(\omega_1) \circ_2 \psi(\omega_2)$. On the other hand,

$$d\psi(v_1)\#'d\psi(v_2) = \Big\{v' \quad : \quad v' \in T_{\omega'}P_2 \quad , \; \omega' \in \psi(\omega_1) \circ_2 \psi(\omega_2)\Big\}.$$

Thus, $d\psi(v) \in d\psi(v_1)\#'d\psi(v_2)$.

Conversely, if $v \in d\psi(v_1)\#'d\psi(v_2)$ then $v \in T_{\psi(\omega)}P_2$. Since, ψ is onto map so, there is $v' \in T_\omega P_1$ such that $v' = d\psi(v)$ and $v \in v_1\#v_2$. Hence, $v \in d\psi(v_1\#v_2)$.

Secondly, consider $d\psi$ is good homomorphism. Let $\omega' \in \psi(\omega_1) \circ_2 \psi(\omega_2)$ and $v' \in T_{\omega'}P_2$ so, $v' \in d\psi(v_1)\#'d\psi(v_2) = d\psi(v_1\#v_2)$ where $v_1 \in T_{\omega_1}P_1$ and $v_2 \in T_{\omega_2}P_1$. Therefore, there is $v \in v_1\#v_2$ such that $v' = d\psi(v)$ and there exists an $\omega \in \omega_1 \circ_1 \omega_2$ such that $v' \in T_{\psi(\omega)}P_2$. Hence, $\omega' = \psi(\omega) \in \psi(\omega_1 \circ_1 \omega_2)$.

Conversely, let $\omega \in \omega_1 \circ_1 \omega_2$ and $\omega' = \psi(\omega)$. If $v \in T_\omega P_1$ then $d\psi(v) \in T_{\omega'}P_2$. Since, $d\psi$ is good homomorphism, $d\psi(v) \in T_{\psi(\omega)}P_2$. Therefore, $\psi(\omega) \in \psi(\omega_1) \circ_2 \psi(\omega_2)$. □

Remark 4.7. *Let $l_{gG_{p_0}} : \frac{G}{G_{p_0}} \longrightarrow \frac{G}{G_{p_0}}$ be left transformation on $\frac{G}{G_{p_0}}$ where $l_{gG_{p_0}}(g'G_{p_0}) = gg'G_{p_0}$. Then there is a transformation $l_g : P \longrightarrow P$ such that $f \circ l_g = l_{gG_{p_0}} \circ f$.*

Theorem 4.8. *If $l_{gG_{p_0}} : \frac{G}{G_{p_0}} \longrightarrow \frac{G}{G_{p_0}}$ is a homomorphism left transformation. Then $l_g : P \longrightarrow P$ is homomorphism.*

Proof. Let $p_1, p_2 \in P$ be arbitrary and there are $g_1, g_2 \in G$ such that $f(p_1) = g_1G_{p_0}$ and $f(p_2) = g_2G_{p_0}$. Then, we have

$$\begin{aligned}
l_g(p_1 \circ p_2) &= \{l_g(p) \quad : \quad p \in p_1 \circ p_2\} \\
&= \{f^{-1} \circ l_{gG_{p_0}}(g'G_{p_0}) \quad : \quad f(p) = g'G_{p_0},\ p \in p_1 \circ p_2\} \\
&= \{f^{-1}(gg'G_{p_0}) \quad : \quad g'G_{p_0} \in g_1G_{p_0} \circ' g_2G_{p_0}\} \\
&= \{f^{-1}(l_{gG_{p_0}}(g'G_{p_0})) \quad : \quad g'G_{p_0} \in g_1G_{p_0} \circ' g_2G_{p_0}\} \\
&\subseteq f^{-1}(l_{gG_{p_0}}(g_1G_{p_0}) \circ' l_{gG_{p_0}}(g_2G_{p_0})) \\
&= f^{-1}(gg_1G_{p_0} \circ' gg_2G_{p_0}) \\
&= f^{-1}(g_1gG_{p_0}) \circ f^{-1}(gg_2G_{p_0}) \\
&= (f^{-1} \circ l_{gG_{p_0}} \circ f)(p_1) \circ (f^{-1} \circ l_{gG_{p_0}} \circ f)(p_2) \\
&= l_g(p_1) \circ l_g(p_2).
\end{aligned}$$

Hence, left transformation gives a relation between Lie group action and hyperoperation. In the next section using the definition of *hyperaction* which is given in [20] some properties of Lie hypergroup are investigated. □

5. **Hyperaction**

Let us recall the definition of *hyperaction*.
Let X be a nonempty set and $(P, \circ)$ be a hypergroup such that $e(P)$ be a nonempty set. A left hyperaction of P on X is a map $\alpha : P \times X \longrightarrow P^*(X)$ such that
i) $\alpha(a, \alpha(b, x)) = \alpha(a \circ b, x)$, for all $a, b \in P$ and for all $x \in X$,
ii) $x \in \alpha(e, x)$, for all $x \in X$ and for all $e \in e(P)$. By above assumptions the following theorem is established.

Theorem 5.1. *The map* $\alpha : \frac{G}{G_{p_0}} \times P \longrightarrow p^*(P)$ *which is defined by* $\alpha(gG_{p_0}, p) = f^{-1}(gG_{p_0}) \circ p$ *is a hyperaction.*

Proof. . Suppose that $f^{-1}(g_1 G_{p_0}) = p_1$ and $f^{-1}(g_2 G_{p_0}) = p_2$, for g_1 and $g_2 \in G$. Therefore, we get

$$\begin{aligned} i)\alpha(g_1 G_{p_0}, \alpha(g_2 G_{p_0}, p)) &= \alpha(g_1 G_{p_0}, f^{-1}(g_2 G_{p_0}) \circ p) \\ &= f^{-1}(g_1 G_{p_0}) \circ (f^{-1}(g_2 G_{p_0}) \circ p \\ &= p_1 \circ (p_2 \circ p) \\ &= (p_1 \circ p_2) \circ p \\ &= f^{-1}(g_1 G_{p_0} \circ' g_2 G_{p_0}) \circ p \\ &= \alpha(g_1 G_{p_0} \circ' g_2 G_{p_0}, p). \end{aligned}$$

ii) $p \in \alpha(e', p) = f^{-1}(e') \circ p = e \circ p.$ □

Definition 5.2. *A regular hypergroup* $(P, \circ)$ *is called reversible if for all* $p_1, p_2, p_3 \in P$
i) $p_2 \in p_3 \circ p_1$, *then there exists* $p'_3 \in i(p_3)$ *such that* $p_1 \in p'_3 \circ p_2$.
ii) $p_2 \in p_1 \circ p_3$, *then there exists* $p''_3 \in i(p_3)$ *such that* $p_1 \in p_2 \circ p''_3$.

The hypergroup which given in example 1.3 is reversible.

Lemma 5.3. *If the regular hypergroup* $(P, \circ)$ *is reversible then the regular Lie hypergroup* $(G/G_{p_0}, \circ')$ *is reversible.*

Theorem 5.4. *Suppose that* $(P, \circ)$ *is a reversible hypergroup and* $\alpha : \frac{G}{G_{p_0}} \times P \longrightarrow p^*(P)$ *is a hyperaction. If* $\alpha(g_1 G_{p_0}, p_1) \cap \alpha(g_2 G_{p_0}, p_2) \neq \varnothing$ *for all* $g_1, g_2 \in$

G and $p_1, p_2 \in P$, then $p_1 \in \alpha(g_1' G_{p_0} \circ' g_2 G_{p_0}, p_2)$ and $p_2 \in \alpha(g_2' G_{p_0} \circ' g_1 G_{p_0}, p_1)$ for all $g_1' G_{p_0} \in i(g_1 G_{p_0})$ and $g_2' G_{p_0} \in i(g_2 G_{p_0})$.

Proof. We have $g_1 G_{p_0} \circ' g_2 G_{p_0} = f(x \circ y)$ for some $x, y \in P$ where $f^{-1}(g_1 G_{p_0}) = x$ and $f^{-1}(g_2 G_{p_0}) = y$. Since $\alpha(g_1 G_{p_0}, p_1) \cap \alpha(g_2 G_{p_0}, p_2) \neq \varnothing$. Hence, there exist $p \in P$ such that $p \in f^{-1}(g_1 G_{p_0}) \circ p_1 \cap f^{-1}(g_2 G_{p_0}) \circ p_2$ so $p \in x \circ p_1 \cap y \circ p_2$. The hypergroup $(P, \circ)$ is reversible so, $p_1 \in x' \circ p$ and $p_2 \in y' \circ p$ for some $x' \in i(x)$ and $y' \in i(y)$. Therefore,

$$\begin{aligned} p_1 \in x' \circ p \subseteq x' \circ (y \circ p_2) &= (x' \circ y) \circ p_2 \\ &= \alpha(f(x' \circ y), p_2) \\ &= \alpha(g_1' G_{p_0} \circ' g_2 G_{p_0}, p_2). \end{aligned}$$

Similarly, we can prove $p_2 \in \alpha(g_2' G_{p_0} \circ' g_1 G_{p_0}, p_1)$. □

6. **Hypergroup Bundle**

To introduce hypergroup bundle, the action of a hypergroup $(P, \circ)$ on set X is defined.

Definition 6.1. *Let $(P, \circ)$ be a hypergroup and X be a non-empty set. $\alpha : X \times P \longrightarrow X$ is the action of the hypergroup P on the set X if it satisfies:*
i) $\alpha(\alpha(x,p), p') \in \alpha(x, p \circ p')$ for all $p, p' \in P$ and $x \in X$, where $\alpha(x, p \circ p') = \{\alpha(x,u) \ : \ u \in p \circ p'\}$,
ii) $\alpha(x, e) = x$ for all $x \in X$.

Example 6.2. [2] *Let $J \subseteq \mathbb{R}$ be an open interval and $P_{n'}(J)$ the collection of all polynomials of degree n'. Let us consider the set $LA_n(J)$, $n \in \mathbb{N}$, of all linear differential operators of the nth order in the form*

$$D(p_0, ..., p_{n-1}) = \frac{d^n}{dx^n} + \sum_{k=0}^{n-1} p_k(x) \frac{d^k}{dx^k},$$

where

$$p_k \in P_n(J), \quad k = 0, 1, ..., n-1; \qquad D(p_0, ..., p_{n-1}) : C^\infty(J) \longrightarrow C^\infty(J).$$

Hence

$$D(p_0, ..., p_{n-1})(f) = f^{(n)}(x) + p_{n-1}(x) f^{(n-1)}(x) + ... + p_0(x) f(x) \quad f \in C^\infty(J).$$

For any $m \in \{0, 1, ..., n-1\}$ we set

$$LA_n(J)_m = \Big\{ D(p_0, ..., p_{n-1}) \ : \ p_k \in P_{n'}(J) \ , \ p_m > 0 \Big\}.$$

Let us put

$$p = \Big(p_0(x), ..., p_{n-1}(x) \Big), \qquad x \in J.$$

On the set $LA_n(J)_m$ we define a binary operation " $\circ_m$ " in this way:
$D(p) \circ_m D(q) = D(u)$ *where* $u_k(x) = p_m(x)q_k(x) + (1 - \delta_{km})p_k(x)$ *and* $x \in J$.
An inverse for $D(q)$ is

$$D^{-1}(q) = \Big(\frac{-q_0}{q_m}, ..., \frac{1}{q_m}, ..., \frac{-q_{n-1}}{q_m}\Big).$$

We can check $LA_n(J)_m$ is a C^∞- manifold.

Let $(\mathbb{Z}, *)$ be a hypergroup where $* : \mathbb{Z} \times \mathbb{Z} \longrightarrow p^*(\mathbb{Z})$ defined by

$$k * l = \{\ u \in \mathbb{Z}\ :\ k + l \leq u\ \}.$$

For a fixed $D(1) \in LA_n(J)_m$ we define an action $\alpha : LA_n(J)_m \times \mathbb{Z} \longrightarrow LA_n(J)_m$ as fallow:

$$\alpha(D(p), k) = D^k(1) \circ_m D(p),$$

where

$$D^k(q) = \underbrace{D(q) \circ_m D(q) \circ_m ... \circ_m D(q)}_{k-times}, \quad fork > 0.$$

Obviously α is an action.

Theorem 6.3. [20] *If (X, P, α) is an action of hypergroup on a set X, then the set X is a hypergroup with following oprator*
$\odot : X \times X \longrightarrow p^*(X)$ *where* $x \odot y = \alpha(x, P) \cup \alpha(y, P) \cup \{x, y\}$.

Remark 6.4. *Suppose that α be an action of the hypergroup P on the set X. This action induces the following equivalence relation on X:*
$x \sim y$ *if and only if* $\alpha(x, P) = \alpha(y, P)$.

Theorem 6.5. $[x] = \{\alpha(x, p)\ :\ p \in P\ \}$ *is a sub hypergroup of* $(X, \odot)$.

Remark 6.6. $(X, \odot)$ *is a hypergroup, so* $\frac{X}{\sim}$ *is a hypergroup.*

Definition 6.7. [18] *Let $(P, \circ)$ be a hypergroup and a C^∞ manifold. Also X be a C^∞ manifold. The pair (X, P) is called a hypergroup bundle if it satisfies the following conditions*
i) P acts on X with $\alpha : X \times P \longrightarrow X$,
ii) $\frac{X}{\sim} = M$ is a quotient space of X under equivalence relation. ($x, y \in X$, $x \sim y$ if $\exists p \in P$ such that $y = \alpha(x, p)$),
iii) $\pi : X \longrightarrow M$ be a good homomorphism.
The space M is called the base space and the space X is called the total space. For every $m \in M$, the space $\pi^{-1}(m)$ is called the fibre of hypergroup bundle over m. $\pi^{-1}(m)$ is a sub hypergroup which is called a fiber over m.

Example 6.8. $(\mathbb{R} - \{0\}, \circ)$ *is a hypergroup and a C^∞ manifold, where*

$$r \circ r' = \Big\{\ rr', \frac{r}{r'}, \frac{r'}{r}\Big\}.$$

Define

$$\alpha : \mathbb{R}^d - \{0\} \times \mathbb{R} - \{0\} \to \mathbb{R}^d - \{0\}$$

$$\alpha(V, r) = rV.$$

We can see easily that α is an action. The equivalence relation on $\mathbb{R}^d - \{0\}$ is defined as

$$V \sim W \Leftrightarrow \exists r \in \mathbb{R} - \{0\} \ s.t. \ \alpha(V, r) = W.$$

Hence, $[V] = \{ \ rV \ : \ r \in \mathbb{R} - \{0\} \ \}$. The function $\pi : \mathbb{R}^d - \{0\} \to \mathbb{R}^d - \{0\}/\sim$ is a good homomorphism. Therefore, $(\mathbb{R}^d - \{0\}, \mathbb{R} - \{0\})$ is a hypergroup bundle and $\pi^{-1}(V) = \{ \ rV \ : \ r \in \mathbb{R} - \{0\} \ \}$ is fiber over $V \in \mathbb{R}^d - \{0\}$.

Theorem 6.9. *Let (X, P) be a hypergroup bundle. Consider H is a sub hypergroup of P. Then $x \odot H$ is fiber over x for every $x \in X$.*

Proof. Let P acts on X by α.

$$x, y \in X, \ x \sim y \iff \exists p \in P \ s.t. \ y = \alpha(x, p).$$

Therefore, $(X, \odot)$ is a hypergroup. We prove that

$$\frac{X}{\sim} = \frac{X}{H} = \{x \odot H \ : \ x \in X \ \}.$$

If $y \in [x]$, so there is $p \in P$ such that $\alpha(x, p) = y$. Thus, $y \in \alpha(x, P)$ so $y \in x \odot H$. Conversely, if $y \in x \odot H$ be arbitrary, so there is $z \in H$ such that $y \in x \odot z = \alpha(x, P) \cup \alpha(z, P) \cup \{x, z\}$. If $y \in \alpha(x, P)$, then $y \in [x]$. If $y \in \alpha(z, P)$, then there is $p \in P$ such that $y = \alpha(z, p)$. Thus $y \in [z]$.
Hence by new equivalence relation, we have a hypergroup bundle which $x \odot H$ is a fibre over x for every $x \in X$. □

Remark 6.10. *Let (X, P) be a hypergroup bundle. X_e is a sub hypergroup which is constructed by $\{ \ x \circ x^{-1} \ : \ x \in X \ \}$. Therefore, $x \odot X_e$ is a fibre over x for every $x \in X$.*

Theorem 6.11. *Let $(P, \circ)$ be a hypergroup, H be a sub hypergroup of P and H acts on P with action α. Then (P, H) is a hypergroup bundle which fibers are subsets of $p \circ H$, for all $p \in P$.*

Proof. Let $\alpha : P \times H \longrightarrow P$ be an action that H acts on P such that $\alpha(p, h) \in p \circ h$. The action induces an equivalence relation on P such that $p_1 \sim p_2$ if and only if there is $h \in H$ such that $p_2 = \alpha(p_1, h)$. Set $\frac{P}{\sim} = B$ so, the projection $\pi : P \longrightarrow B$ is defined by $\pi(p) = [p]$, for all $p \in P$.The (P, H) is a hypergroup bundle. The fibers are $\pi^{-1}(b) \in b \circ H$, for all $b \in B$. Also, $b \circ H$ is isomorphic to H. □

7. **Conclusion**

At the beginning, quotient hypergroup and sub quotient hypergroup are studied. futhermore, using action of a Lie group on a hypergroup, Lie hypergroup is introduced. Some basic properties of Lie hypergroups are investigated. actually, Theses are generalization of Lie group properties. It is shown that tangent space of a hypergroup is a hypergroup. Finally using the action of a hypergroup on a set, hypergroup bundle is defined. In the future researches other geometric properties such as Lie hyperalgebra and Lie theorems can investigated.

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