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Gradient Estimates for Positive Global Solutions of Heat Equation under Closed Finsler-Ricci Flow

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Abstract. In this paper, we establish first order gradient estimates for positive global solutions of the heat equation under closed Finsler-Ricci flow with weighted Ricci curvature Ric^N bounded below, where $N \in (n, \infty)$. As an application, we derive the corresponding Harnack inequality. Our results are the generalizations and the supplements of the previous known related results.

Keywords: Finsler-Ricci flow, gradient estimate, heat equation, weighted Ricci curvature, Ricci curvature tensor.

1. Introduction

In Riemannian geometry, gradient estimates for solutions of the heat equation have become very powerful tools in geometric analysis. They originated from the pioneering work of P. Li and S.-T. Yau in [8]. After that, the gradient estimates of the solutions of the heat equation have been studied extensively and many important results have been obtained. Further, it is natural to consider the heat equation together with Ricci flow as a system. The study of system of the heat equation together with Ricci flow arose from R. Hamilton's paper [7]. The original idea in [7] was to investigate the Ricci flow combined with the heat flow of harmonic maps. Later, the study of the heat equation

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together with Ricci flow was pursued in many works. For examples, S. Liu obtained the first and the second order gradient estimates for positive solutions of the heat equation under Ricci flow ([10]). In addition, M. Bailesteanu, X. Cao and A. Pulemotov considered a manifold M evolving under the Ricci flow and establishes a series of gradient estimates for positive solutions of the heat equation on M ([2]).

In Finsler geometry, gradient estimates become more complicated because the solutions of nonlinear heat equations lack higher-order regularity and Finsler Laplacian is a nonlinear elliptic differential operator of the second order and has no definition at the maximum point of the function. However, in spite of these, some important and interesting progresses for gradient estimates on Finsler manifolds have been made in recent years. In [12], Ohta and Sturm derived a Li-Yau gradient estimate as well as parabolic Harnack inequalities on compact Finsler manifolds, all of which depend on lower bounds for the weighted Ricci curvature Ric^N. Furthermore, Q. Xia proved that the Li-Yau gradient estimate and the Harnack inequalities still hold if Finsler measurable space (M, F, m) admits a convex boundary ([15]). On the other hand, there are also a few studies on gradient estimates for positive solutions of the heat equation under Finsler-Ricci flow. Lakzian proved first order differential Harnack estimates for positive solutions of the heat equation (in the sense of distributions) under closed Finsler-Ricci flows with Ricci curvature bounds ([9]). Shortly afterwards, F. Zeng and Q. He generalized and corrected Lakzian's results under some curvature constraints ([16]). It should be point out that the authors in both of [9] and [16] all assume that the non-Riemannian quantity S-curvature vanishes. Recently, the first author established first order gradient estimates and the corresponding Harnack inequality for positive solutions of the heat equation under closed Finsler-Ricci flow with the condition that the weighted Ricci curvature Ric^∞ has a non-positive lower bound ([5]). Different from the assupmtions in [9] and [16], there is no any constraint condition about S-curvature in [5].

In this paper, we will mainly derive first order gradient estimates for positive global solutions of the nonlinear heat equation under closed Finsler-Ricci flow with the condition that the weighted Ricci curvature Ric^N has a non-positive lower bound, where $N \in (n, \infty)$. Then, we derive the corresponding Harnack inequality. The researches in this paper can provide important support for further discussions of the gradient estimates of the positive global solution of the nonlinear heat equation under Finsler-Ricci flow. Our results are the natural generalizations and the supplements of the previous known related results. More precisely, we prove the following results.

Theorem 1.1. *Let $(M, F(t), m)_{t\in[0,T]}$ be an n-dimensional closed Finsler-Ricci flow equipped with a measure m on M and evolving by*

$$\frac{\partial g_{ij}}{\partial t} = -2Ric_{ij}. \tag{1.1}$$

Suppose that there exist four constants K_1, K_2, K_3, $K_4 \geq 0$ such that

$$-K_1 g_y \leq \mathcal{R}ic(x,y,t) \leq K_2 g_y, \quad \mathrm{Ric}^N \geq -K_3, \tag{1.2}$$

$$|D\mathcal{R}ic| \leq K_4, \tag{1.3}$$

where $N \in (n,\infty)$, $D\mathcal{R}ic$ denotes the covariant differential of Ricci curvature tensor $\mathcal{R}ic$.

Let $u = u(x,t)$ be a positive global solution of the heat equation $\partial_t u(x,t) = \Delta u(x,t)$ and $\mu := \inf_{(x,y,t)\in TM\times(0,T]} \left\{\frac{\mathbf{S}^2(x,y,t)}{F^2(x,y,t)}\right\}$. Then for any $(x,t) \in M \times (0,T]$, $\alpha > 1$ and $0 < \varepsilon < 1$, we have

$$\begin{aligned} & F^2\left(\nabla(\log u)(x,t)\right) - \alpha\partial_t(\log u)(x,t) \\ & \leq \frac{N\alpha^2}{2(1-\varepsilon)t} + \frac{N\alpha^2}{2(1-\varepsilon)}\left\{\frac{2(\alpha-1)\sqrt{C_1} + 2K_3 + \left|1-\frac{2\mu}{N-n}\right|\varepsilon}{4(\alpha-1)}\right. \\ & \left. + \sqrt{\frac{2(1-\varepsilon)}{N\varepsilon}\left(nC_1 + 2K_4^2\right)}\right\}. \end{aligned} \tag{1.4}$$

Here, $C_1 := \max\{K_1^2, K_2^2\}$ and $\mathbf{S}(x,y,t)$ is the S-curvature of $F(t)$.

From Theorem 1.1, we can derive the following Harnack type inequality.

Corollary 1.2. *Under the same assumptions as in Theorem 1.1, the following estimate holds: for any pair of points $(x_1,t_1), (x_2,t_2)$ in $M\times(0,T]$ such that $t_1 < t_2$ and for any $\alpha > 1$, $0 < \varepsilon < 1$, we have*

$$\begin{aligned} u(x_1,t_1) \quad \leq \quad & u(x_2,t_2)\left(\frac{t_2}{t_1}\right)^{\frac{N\alpha}{2(1-\varepsilon)}} \exp\left\{\frac{\alpha}{4}\int_0^1 \frac{F^2\left(\dot\gamma(s)\right)|_{\tau(s)}}{t_2-t_1}ds\right. \\ & + \frac{(t_2-t_1)N\alpha}{2(1-\varepsilon)}\left[\frac{2(\alpha-1)\sqrt{C_1} + 2K_3 + \left|1-\frac{2\mu}{N-n}\right|\varepsilon}{4(\alpha-1)}\right. \\ & \left.\left. + \sqrt{\frac{2(1-\varepsilon)}{N\varepsilon}\left(nC_1 + 2K_4^2\right)}\right]\right\}. \end{aligned} \tag{1.5}$$

Here, $\gamma = \gamma(s)$ is a smooth curve joining x_2 to x_1 with $\gamma(0) = x_2$ and $\gamma(1) = x_1$, and $F\left(\dot\gamma(s)\right)|_{\tau(s)} = F\left(\gamma(s), \dot\gamma(s), \tau(s)\right)$ is the length of the vector $\dot\gamma(s)$ at time $\tau(s) = (1-s)t_2 + st_1$.

The paper is organized as follows. In Section 2, we will give some necessary definitions and notations. In Section 3, we will give some necessary and important lemmas firstly. Then we will give the proofs of Theorem 1.1 and Corollary 1.2 in Section 3.

2. Preliminaries

In this section, we will recall some necessary definitions and notations in Finsler geomtry. For more details, we refer to [1, 3, 11–13].

Let (M, F) be an n-dimensional smooth Finsler manifold. For any $y \in T_xM\backslash\{0\}$, $F(x, -y) \neq F(x, y)$ in general. The *reversibility* of F is defined by

$$\Lambda_F := \sup_{y\in TM\backslash\{0\}} \frac{F(x,-y)}{F(x,y)}. \tag{2.1}$$

Obviously, $\Lambda_F \geq 1$ and $\Lambda_F = 1$ if and only if F is reversible. In our discussions, we always assume that $\Lambda_F < \infty$.

Let $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$, where $G^i = G^i(x, y)$ are defined by

$$G^i = \frac{1}{4} g^{il} \left\{ \frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right\} y^j y^k. \tag{2.2}$$

The *Riemann curvature* $\mathbf{R}_y = R^i{}_k(x, y) dx^k \otimes \frac{\partial}{\partial x^i} : T_xM \to T_xM$ is defined by ([1])

$$R^i{}_k(x,y) := 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{2.3}$$

Further, the *Ricci curvature* $\mathrm{Ric}(y)$ is defined as the trace of $\mathbf{R}_y$. In a local coordinate system,

$$\mathrm{Ric}(y) = R^k{}_k(x, y),\ y \in T_xM \setminus \{0\}. \tag{2.4}$$

The *Ricci curvature tensor* of a Finsler metric F is defined by

$$Ric_{ij}(x,y) := \left(\frac{1}{2}\mathrm{Ric}\right)_{y^i y^j}(x,y). \tag{2.5}$$

Usually, we denote Ricci curvature tensor by

$$\mathcal{R}ic(x,y) = Ric_{ij}(x,y) dx^i \otimes dx^j. \tag{2.6}$$

Let $\pi : TM_0 \to M$ be the natural projective map, where $TM_0 := TM\backslash\{0\}$. π pulls back TM to a vector bundle π^*TM over TM_0. In other words, $\pi^*TM|_{(x,y)}$ is just a copy of T_xM. Similarly, we define the pull-back cotangent bundle π^*T^*M over TM_0 whose fiber at (x, y) is a copy of T^*_xM. Define

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}, \quad N^i_j := \frac{\partial G^i}{\partial y^j}. \tag{2.7}$$

Then $HTM := span\left\{\frac{\delta}{\delta x^i}\right\}$ is a well-defined subbundle of $T(TM_0)$ and is called the horizontal tangent bundle of M. Furthermore, the vertical tangent bundle of M is defined by $VTM := span\{\frac{\partial}{\partial y^i}\}$. Thus we obtain a decomposition $T(TM_0) = HTM \oplus VTM$ and we have a natural frame $\left\{\frac{\delta}{\delta x^i}, F\frac{\partial}{\partial y^i}\right\}$ for $T(TM_0)$ and its dual frame $\left\{dx^i, \frac{\delta y^i}{F}\right\}$ for $T^*(TM_0)$, where $\delta y^i := dy^i + N^i_j dx^j$.

The vector bundle π^*TM over TM_0 admits a unique linear connection, which is called the *Chern connection*. The Chern connection D is determined by the following equations

$$D_X^V Y - D_Y^V X = [X,Y], \tag{2.8}$$

$$Zg_V(X,Y) = g_V(D_Z^V X, Y) + g_V(X, D_Z^V Y) + 2C_V(D_Z^V V, X, Y) \tag{2.9}$$

for $V \in TM \setminus \{0\}$ and $X, Y, Z \in TM$, where

$$C_V(X,Y,Z) := C_{ijk}(x,V)X^iY^jZ^k = \frac{1}{4}\frac{\partial^3 F^2(x,V)}{\partial V^i \partial V^j \partial V^k} X^iY^jZ^k$$

is the Cartan tensor of F and $D_X^V Y$ is the covariant derivative with respect to the reference vector V ([1,3]). Torsion freeness (2.8) is equivalent to the absence of dy^k terms in Chern connection forms ω_j^i, namely $\omega_j^i = \Gamma^i{}_{jk}(x,V)dx^k$, together with the symmetry $\Gamma^i{}_{jk}(x,V) = \Gamma^i{}_{kj}(x,V)$.

Let $T := T^j_i \frac{\partial}{\partial x^j} \otimes dx^i$ be an arbitrary smooth local section of $\pi^*TM \otimes \pi^*T^*M$. Its *covariant differential* is ([1])

$$DT := (DT)^j{}_i \frac{\partial}{\partial x^j} \otimes dx^i,$$

where

$$(DT)^j{}_i := dT^j_i + T^k_i \omega^j_k - T\,^j_k \omega^k_i.$$

The $(DT)^j{}_i$ are 1-forms on TM_0. They can therefore be expanded in terms of the natural basis $\left\{dx^s, \frac{\delta y^s}{F}\right\}$:

$$(DT)^j{}_i = T^j_{i|s}dx^s + T^j_{i;s}\frac{\delta y^s}{F}, \tag{2.10}$$

where

$$T^j_{i|s} = \frac{\delta T^j_i}{\delta x^s} + T^k_i \Gamma^j{}_{ks} - T^j_k \Gamma^k{}_{is}, \tag{2.11}$$

$$T^j_{i;s} = F\frac{\partial T^j_i}{\partial y^s}. \tag{2.12}$$

On Finsler manifold (M,F), the *Legendre transformation* $\mathcal{L} : TM \to T^*M$ is defined by

$$\mathcal{L}(y) := \begin{cases} g_y(y,\cdot), & y \neq 0, \\ 0, & y = 0. \end{cases}$$

It is known that $F(x,y) = F^*(x, \mathcal{L}(y))$ ([13]).

Given a smooth function u on M, we define the *gradient vector* $\nabla u(x)$ of u at $x \in M$ by $\nabla u(x) := \mathcal{L}^{-1}(du(x)) \in T_xM$. In a local coordinate system, we can express ∇u as

$$\nabla u(x) = \begin{cases} g^{ij}(x,\nabla u)\frac{\partial u}{\partial x^i}\frac{\partial}{\partial x^j}, & x \in M_u, \\ 0, & x \in M \setminus M_u, \end{cases} \tag{2.13}$$

where $M_u = \{x \in M \mid du(x) \neq 0\}$.

Furthermore, the *Hessian* $\nabla^2 u$ of u is defined by

$$\nabla^2 u(X) := D_X^{\nabla u} \nabla u \tag{2.14}$$

for any $X \in T_xM$ and $x \in M_u$ ([4, 11, 14]). In a local coordinate system, let $X = X^i \frac{\partial}{\partial x^i}$. We have

$$D_X^{\nabla u} \nabla u = \left\{ \frac{\partial (\nabla u)^i}{\partial x^j} + N^i_j(x, \nabla u) \right\} X^j \frac{\partial}{\partial x^i},$$

where $(\nabla u)^i := g^{ij}(x, \nabla u) \frac{\partial u}{\partial x^j}$. From this, the components of $\nabla^2 u$ are given by

$$\left(\nabla^2 u\right)^i_j = \frac{\partial (\nabla u)^i}{\partial x^j} + N^i_j(x, \nabla u). \tag{2.15}$$

Let (M, F, m) be a Finsler manifold equipped with a measure m on M. Write the volume form dm of m as $dm = \sigma(x) dx^1 dx^2 \cdots dx^n$. Define

$$\tau(x, y) := \ln \frac{\sqrt{\det\left(g_{ij}(x, y)\right)}}{\sigma(x)}. \tag{2.16}$$

We call τ the *distortion* of F. It is natural to study the rate of change of the distortion along geodesics. For a vector $y \in T_xM \backslash \{0\}$, let $\sigma = \sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. Set

$$\mathbf{S}(x, y) := \frac{d}{dt} \left[\tau(\sigma(t), \dot{\sigma}(t))\right] |_{t=0}. \tag{2.17}$$

$\mathbf{S}$ is called the *S-curvature* of F ([3, 13]).

We decompose the volume form dm of m as $dm = \mathrm{e}^{\Phi} dx^1 dx^2 \cdots dx^n$. Then the *divergence* of a differentiable vector field V on M is defined by

$$\mathrm{div}_m V := \frac{\partial V^i}{\partial x^i} + V^i \frac{\partial \Phi}{\partial x^i}, \quad V = V^i \frac{\partial}{\partial x^i}. \tag{2.18}$$

One can also define $\mathrm{div}_m V$ in the weak form by following divergence formula:

$$\int_M \phi \ \mathrm{div}_m V \ dm = -\int_M d\phi(V) \ dm \tag{2.19}$$

for all $\phi \in \mathcal{C}^\infty_c(M)$, where $\mathcal{C}^\infty_c(M)$ denotes the set of $\mathcal{C}^\infty$-functions on M with compact support.

Now we define the *Finsler Laplacian* Δu of $u \in H^1_{\mathrm{loc}}(M)$ by

$$\Delta u := \mathrm{div}_m(\nabla u). \tag{2.20}$$

Equivalently, we can define Laplacian Δu on the whole M in the weak sense by

$$\int_M \phi \ \Delta u \ dm := -\int_M d\phi(\nabla u) dm \tag{2.21}$$

for all $\phi \in \mathcal{C}^\infty_c(M)$.

The following result is important for our discussions.

Lemma 2.1. ([13] [14]) *On $M_u = \{x \in M \mid \nabla u(x) \neq 0\}$, we have*

$$\Delta u = \sum_a D^2 u\,(e_a, e_a) - \mathbf{S}(\nabla u) = \mathrm{tr}_{g_{\nabla u}} \nabla^2 u - \mathbf{S}(\nabla u), \tag{2.22}$$

where $D^2 u\,(e_a, e_a) := g_{\nabla u}(\nabla^2 u(e_a), e_a)$ and $e_1, \ldots, e_n$ is a local $g_{\nabla u}$-orthonormal frame on M_u.

For any $v \in T_x M\backslash\{0\}$, the *weighted Ricci curvature* of (M, F, m) is defined by

$$\begin{cases} \mathrm{Ric}^n(v) := \begin{cases} \mathrm{Ric}(v) + \dot{\mathbf{S}}(x,v) & \text{if } \mathbf{S}(x,v) = 0, \\ -\infty & \text{if } \mathbf{S}(x,v) \neq 0, \end{cases} \\ \mathrm{Ric}^N(v) := \mathrm{Ric}(v) + \dot{\mathbf{S}}(x,v) - \dfrac{\mathbf{S}^2(x,v)}{N-n}, \\ \mathrm{Ric}^\infty(v) := \mathrm{Ric}(v) + \dot{\mathbf{S}}(x,v). \end{cases} \tag{2.23}$$

Here, $N \in R \setminus \{n\}$. For $K \in R$, we say that $\mathrm{Ric}^N \geq K$ means that $\mathrm{Ric}^N(v) \geq KF^2(x,v)$ for all $v \in T_x M$ and any $x \in M$.

We say that an L^2-continuous function $u = u(x,t)$ in $H_0^1(M)$ is a *global solution* of the nonlinear heat equation $\partial_t u(x,t) = \Delta u(x,t)$ on $(M, F(t), m)$, if $u = u(x,t)$ satisfies the following ([11]):

(1) $u \in L^2\left([0,T], H_0^1(M)\right) \cap H^1\left([0,T], H^{-1}(M)\right)$;

(2) For any $\phi \in \mathcal{C}_c^\infty(M)$ and almost all $t \in [0,T]$, we have

$$\int_M \phi \cdot \partial_t u dm = -\int_M d\phi\,(\nabla u)\,dm.$$

Here we remark that $u \in L^2\left([0,T], H_0^1(M)\right) \cap H^1\left([0,T], H^{-1}(M)\right)$ implies that $u \in \mathcal{C}\left([0,T], L^2(M)\right)$ (also see [6]).

We summarize the existence and regularity of heat flow from [11] as the following

Lemma 2.2. ([11,12], existence and regularity)

(1) *For each $u_0 \in H_0^1(M)$ and $T > 0$, there exists a unique global solution $u = u(x,t)$ to the heat equation on $M \times [0,T]$, and the distributional Laplacian $\Delta u(x,t)$ is absolutely continuous with respect to m for all $t \in (0,T)$.*

(2) *One can take the continuous version of a global solution $u = u(x,t)$ such that $u \in H^2_{\mathrm{loc}}(M) \cap C^{1,\beta}(M \times [0,T])$. Furthermore, the distributional time derivative $\partial_t u$ lies in $H^1_{\mathrm{loc}}(M) \cap C(M)$. Here, $0 \leq \beta < 1$.*

3. **Proofs of main results**

Let $(M, F(t), m)_{t\in[0,T]}$ be an n-dimensional Finsler-Ricci flow equipped with a measure m on M for each $t \in [0,T]$ and evolving by

$$\frac{\partial g_{ij}}{\partial t} = -2Ric_{ij}. \tag{3.1}$$

In order to prove the main results in this paper, we first introduce some necessary lemmas that we will need later.

Lemma 3.1. ([9]) *Let $(M, F(t), m)_{t\in[0,T]}$ be an n-dimensional Finsler-Ricci flow. Then for any smooth function f on $M\times[0,T]$, we have*

$$\begin{aligned}\partial_t\left[F^2(\nabla f)\right] &= 2g^{ij}(df)\left[\partial_t f\right]_i f_j + 2Ric^{ij}(df)f_if_j\\ &= 2d\left(\partial_t f\right)(\nabla f)+2\mathrm{Ric}(\nabla f)\end{aligned}\tag{3.2}$$

on M_f. Here, $f_i := \frac{\partial f}{\partial x^i}$ and $Ric^{ij} := g^{ir}g^{js}Ric_{rs}$.

Let $u = u(x,t)$ in $H^1_0(M)$ be a positive global solution of the nonlinear heat equation $\partial_t u(x,t) = \Delta u(x,t)$ on $(M, F(t), m)$. In the what follows, the Laplacian and gradient vectors are all determined with respect to $V := \nabla u$ and are valid on $M_u := \{x\in M \mid \nabla u(x)\neq 0\}$.

Let $f(x,t) := \log u(x,t)$. By Lemma 2.2, we have $f(x,t)\in H^2_{\mathrm{loc}}(M)\cap C^{1,\beta}(M\times[0,T])$. Then, we have $g_{\nabla f} = g_{\nabla u}$ on M_u and

$$\partial_t f = e^{-f}\partial_t u,\quad \nabla f = e^{-f}\nabla u,\quad \Delta f = e^{-f}\Delta u - F^2(\nabla f).$$

Hence the heat equation for u implies

$$\partial_t f = \Delta f + F^2(\nabla f).\tag{3.3}$$

Let $\sigma(x,t) := t\partial_t f(x,t)$ and

$$\mathcal{F}(x,t) := t\left\{F^2(\nabla f(x,t)) - \alpha\partial_t f(x,t)\right\} = tF^2(\nabla f(x,t)) - \alpha\sigma(x,t).\tag{3.4}$$

Lemma 3.2. ([5]) *In the sense of distributions, $\sigma(x,t)$ satisfies the following parabolic differential equality,*

$$\begin{aligned}&\Delta^{\nabla f}\sigma - \partial_t\sigma + \frac{\sigma}{t} + 2d\sigma(\nabla f)\\ &= t\left\{-2Ric(\nabla f) - 2Ric^{ij}(\nabla f)f_{ij} - 2f_j\frac{\partial}{\partial x^i}\left[Ric^{ij}(\nabla f)\right]\right\}.\end{aligned}\tag{3.5}$$

Here, $f_{ij} := \frac{\partial^2 f}{\partial x^i\partial x^j}$.

From Lemma 3.2 and (2.23), we can derive a parabolic partial differential inequality for $\mathcal{F}$. For more details, also see Lemma 3.3 and (3.10) in [5].

Lemma 3.3. ([5]) *In the sense of distributions, $\mathcal{F}(x,t)$ satisfies*

$$\Delta^{\nabla f}\mathcal{F} + 2d\mathcal{F}(\nabla f) - \partial_t\mathcal{F} + \frac{\mathcal{F}}{t} = \mathcal{B}\tag{3.6}$$

where

$$\begin{aligned}\mathcal{B} = \alpha&\left(2tRic^{ij}(\nabla f)f_{ij} + 2tRic^{ij}_{\ \ |i}(\nabla f)\,f_j + 2t\frac{f_j}{F(\nabla f)}Ric^{ij}_{\ \ ;k}(\nabla f)\left(\nabla^2 f\right)^k_i\right)\\ &+ 2t(\alpha-1)\mathrm{Ric}(\nabla f) + 2t\mathrm{Ric}^N(\nabla f) + 2t\frac{\mathbf{S}(\nabla f)^2}{N-n} + 2t\left\|\nabla^2 f\right\|^2_{HS(\nabla f)}\end{aligned}\tag{3.7}$$

for any $N \in (n, \infty)$, *where* $\left\|\nabla^2 u\right\|^2_{HS(\nabla u)}$ *denotes the Hilbert-Schmidt norm with respect to* $g_{\nabla u}$.

We must say that (3.7) is the starting point for the proof of Theorem 1.1. Now we are in the position to give a proof of Theorem 1.1.

Proof of Theorem 1.1. We choose a local orthonormal frame with respect to $g_{\nabla u}$ at any $x \in M_u$, such that the Chern connection coefficients $\Gamma^i_{jk}(\nabla u) = 0$ for all i, j, k. Then we have

$$Ric^{ij}(\nabla f) = Ric_{ij}(\nabla f), \quad \mathrm{tr}_{g_{\nabla u}} \nabla^2 f = \sum_{i=1}^{n} f_{ii} = \Delta f + \mathbf{S}(\nabla f), \tag{3.8}$$

$$\left|\nabla^2 f\right|^2_{HS(\nabla f)} = \sum_{i,j=1}^{n} f_{ij}^2 \geq \frac{1}{n}\left(\sum_{i=1}^{n} f_{ii}\right)^2 = \frac{1}{n}\left(\Delta f + \mathbf{S}(\nabla f)\right)^2. \tag{3.9}$$

Firstly, for any $a, b \in R$ and $\lambda > 0$, the inequality $\left(\sqrt{\frac{\lambda}{\lambda+1}}a + \sqrt{\frac{\lambda+1}{\lambda}}b\right)^2 \geq 0$ implies

$$(a+b)^2 \geq \frac{1}{\lambda+1}a^2 - \frac{1}{\lambda}b^2.$$

By taking $a = \Delta f$, $b = \mathbf{S}(\nabla f)$ and $\lambda = (N-n)/n$, we get

$$\frac{1}{n}\left(\Delta f + \mathbf{S}(\nabla f)\right)^2 \geq \frac{(\Delta f)^2}{N} - \frac{\mathbf{S}^2(\nabla f)}{N-n}. \tag{3.10}$$

Note that (3.10) is very important for our estimates which will be used in the following.

On the other hand, by Young's inequality, we have

$$|\alpha Ric^{ij}(\nabla f) f_{ij}| \quad \leq \quad \sum_{i,j=1}^{n} \frac{\alpha^2}{2\varepsilon}\left(Ric_{ij}(\nabla f)\right)^2 + \sum_{i,j=1}^{n} \frac{\varepsilon}{2} f_{ij}^2.$$

By our assumptions, $Ric_{ij}(\nabla f)^2 \leq C_1 g_{ij}(\nabla f)^2$, where $C_1 := \max\{K_1^2, K_2^2\}$. Then we have

$$\sum_{i,j=1}^{n}\left(Ric_{ij}(\nabla f)\right)^2 \leq nC_1,$$

from which, we obtain

$$|\alpha Ric^{ij}(\nabla f) f_{ij}| \leq \frac{n\alpha^2}{2\varepsilon}C_1 + \frac{\varepsilon}{2}\sum_{i,j=1}^{n} f_{ij}^2.$$

Hence, we get

$$2t\alpha Ric^{ij}(\nabla f) f_{ij} \geq -\frac{tn\alpha^2}{\varepsilon}C_1 - t\varepsilon \sum_{i,j=1}^{n} f_{ij}^2. \tag{3.11}$$

Similarly, we have the following:

$$2t\alpha Ric^{ij}_{\ |i}(\nabla f)f_j \geq -t\sum_{j=1}^{n}\frac{\alpha^2}{\varepsilon}\left(Ric^{ij}_{\ |i}(\nabla f)\right)^2 - t\sum_{j=1}^{n}\varepsilon f_j^2 \geq -\frac{t\alpha^2}{\varepsilon}K_4^2 - t\varepsilon F^2(\nabla f), \tag{3.12}$$

$$\begin{aligned} &2t\alpha\frac{f_j}{F(\nabla f)}Ric^{ij}_{\ ;k}(\nabla f)\left(\nabla^2 f\right)_i^k \\ &\quad \geq -t\sum_{i,k=1}^{n}\frac{\alpha^2}{\varepsilon}\left(\frac{f_j}{F(\nabla f)}Ric^{ij}_{\ ;k}(\nabla f)\right)^2 - t\sum_{i,k=1}^{n}\varepsilon\left[(\nabla^2 f)_i^k\right]^2 \\ &\quad \geq -\frac{t\alpha^2}{\varepsilon}K_4^2 - t\varepsilon\sum_{i,j=1}^{n}f_{ij}^2. \end{aligned} \tag{3.13}$$

Then, substituting (3.11) - (3.13) into (3.7) and by (3.9), we can get

$$\begin{aligned} \mathcal{B} \ \geq\ & -t[2(\alpha-1)\sqrt{C_1}+2K_3+\varepsilon]F^2(\nabla f) + \frac{2t\mathbf{S}^2(\nabla f)}{N-n} \\ & -\frac{t\alpha^2}{\varepsilon}\left(nC_1+2K_4^2\right) + 2t(1-\varepsilon)\sum_{i,j=1}^{n}f_{ij}^2 \\ \geq\ & -t[2(\alpha-1)\sqrt{C_1}+2K_3+\varepsilon]F^2(\nabla f) + \frac{2t\mathbf{S}^2(\nabla f)}{N-n} \\ & -\frac{t\alpha^2}{\varepsilon}\left(nC_1+2K_4^2\right) + 2t(1-\varepsilon)\times\frac{1}{n}\left(\Delta f+\mathbf{S}(\nabla f)\right)^2. \end{aligned}$$

Further, by (3.6) and (3.10),

$$\begin{aligned} &\Delta^{\nabla f}\mathcal{F} + 2d\mathcal{F}(\nabla f) - \partial_t\mathcal{F} + \frac{\mathcal{F}}{t} \\ &\quad \geq -t\Big[2(\alpha-1)\sqrt{C_1}+2K_3+\varepsilon\Big]F^2(\nabla f) + \frac{2\varepsilon t\mathbf{S}^2(\nabla f)}{N-n} \\ &- \frac{t\alpha^2}{\varepsilon}\left(nC_1+2K_4^2\right) + \frac{2t(1-\varepsilon)}{N}(\Delta f)^2 \\ &\quad \geq -t\left[2(\alpha-1)\sqrt{C_1}+2K_3+\left(1-\frac{2\mu}{N-n}\right)\varepsilon\right]F^2(\nabla f) \\ &- \frac{t\alpha^2}{\varepsilon}\left(nC_1+2K_4^2\right) + \frac{2t(1-\varepsilon)}{N}(\Delta f)^2 \\ &\quad \geq -t\left[2(\alpha-1)\sqrt{C_1}+2K_3+\left|1-\frac{2\mu}{N-n}\right|\varepsilon\right]F^2(\nabla f) \\ &- \frac{t\alpha^2}{\varepsilon}\left(nC_1+2K_4^2\right) + \frac{2t(1-\varepsilon)}{N}(\Delta f)^2. \end{aligned} \tag{3.14}$$

From (3.3), we get

$$\Delta f = \partial_t f - F^2(\nabla f) = -\frac{1}{\alpha}\left(\frac{\mathcal{F}}{t} + (\alpha-1)F^2(\nabla f)\right).$$

Let $\eta(x,t) := \frac{tF^2(\nabla f)}{\mathcal{F}(x,t)}$, and then $F^2(\nabla f) = \frac{\mathcal{F}}{t}\eta$. If $\eta(x,t) \le 0$ for some $(x,t) \in M \times (0,T]$, then $\mathcal{F}(x,t) \le 0$ and (1.4) holds obviously. Hence we can always assume that $\eta(x,t) \ge 0$. In this case, we have the following

$$\Delta f = -\left[1 + (\alpha - 1)\eta\right] \frac{\mathcal{F}}{\alpha t}. \tag{3.15}$$

Furthermore, by (3.14) and (3.15), we have

$$\begin{aligned}&\Delta^{\nabla f}\mathcal{F} + 2d\mathcal{F}(\nabla f) - \partial_t \mathcal{F}\\ &\ge \frac{2(1-\varepsilon)}{N\alpha^2 t}\left[1 + (\alpha-1)\eta\right]^2 \mathcal{F}^2 - \left\{\left[2(\alpha-1)\sqrt{C_1} + 2K_3 + \left|1 - \frac{2\mu}{N-n}\right|\varepsilon\right]\eta + \frac{1}{t}\right\}\mathcal{F}\\ &\quad - \frac{t\alpha^2}{\varepsilon}\left(nC_1 + 2K_4^2\right) := \mathcal{G}(x,t).\end{aligned} \tag{3.16}$$

Now, fix arbitrarily a value $t \in (0,T]$. Because $(M, F(t), m)$ is a closed Finsler manifold for each $t \in [0,T]$, we can assume that $\mathcal{F}$ takes its maximum at some point $(x_0,t_0) \in M \times [0,t]$. Since the assertion (1.4) is obvious if $\mathcal{F}(x_0,t_0) \le 0$, we can assume that $\mathcal{F}(x_0,t_0) > 0$ and hence $0 < t_0 \le t$. In this case, if $\mathcal{G}(x_0,t_0) > 0$, we have $\mathcal{G} > 0$ on a neighborhood of (x_0,t_0). Then, because of (3.16), on such a neighborhood $\mathcal{F}$ is a strict sub-solution to the linear parabolic operator

$$\Delta^{\nabla f}\mathcal{F} + 2d\mathcal{F}(\nabla f) - \partial_t \mathcal{F}.$$

Therefore $\mathcal{F}(x_0,t_0)$ is strictly less than the supremum of $\mathcal{F}$ on the boundary of a parabolic cylinder $[t_0-\delta, t_0] \times B_M^+(x_0,\delta)$ for sufficiently small $\delta > 0$, where $B_M^+(x_0,\delta) := \{x \in M | d_F(x_0,x) < \delta\}$. In particular, $\mathcal{F}$ cannot be maximal at (x_0,t_0). This is a contradiction. Thus we conclude that $\mathcal{G}(x_0,t_0) \le 0$. Therefore, at (x_0,t_0), we have

$$\begin{aligned}&\frac{2(1-\varepsilon)}{N\alpha^2 t_0}\left[1 + (\alpha-1)\eta\right]^2 \mathcal{F}^2 - \left\{\left[2(\alpha-1)\sqrt{C_1} + 2K_3 + \left|1 - \frac{2\mu}{N-n}\right|\varepsilon\right]\eta + \frac{1}{t_0}\right\}\mathcal{F}\\ &- \frac{\alpha^2 t_0}{\varepsilon}\left(nC_1 + 2K_4^2\right) \le 0.\end{aligned} \tag{3.17}$$

Let

$$\begin{aligned}A &:= \frac{2(1-\varepsilon)}{N\alpha^2 t_0}\left[1 + (\alpha-1)\eta\right]^2,\\ B &:= \left[2(\alpha-1)\sqrt{C_1} + 2K_3 + \left|1 - \frac{2\mu}{N-n}\right|\varepsilon\right]\eta + \frac{1}{t_0},\\ C &:= \frac{\alpha^2 t_0}{\varepsilon}\left(nC_1 + 2K_4^2\right).\end{aligned}$$

Obviously, A, B, $C \ge 0$ for any $0 < \varepsilon < 1$. Note that $1 + (\alpha-1)\eta \ge 0$ for any $\alpha > 1$. Solving $\mathcal{F}$ from the quadratic inequality (3.17) of $\mathcal{F}$ and by inequality

$\sqrt{a^2+b^2} \le a+b$ for any $a, b \ge 0$, we have

$$
\begin{aligned}
0 < \mathcal{F}(x_0,t_0) &\le \frac{1}{2A}\left(B+\sqrt{B^2+4AC}\right)\\
&\le \frac{1}{A}\left(B+\sqrt{AC}\right)\\
&= \frac{N\alpha^2 t_0}{2(1-\varepsilon)\left[1+(\alpha-1)\eta\right]^2}\Big\{\Big[2(\alpha-1)\sqrt{C_1}+2K_3+\big|1-\frac{2\mu}{N-n}\big|\varepsilon\Big]\eta\\
&\quad+\frac{1}{t_0}+\left[1+(\alpha-1)\eta\right]\sqrt{\frac{2(1-\varepsilon)}{N\varepsilon}\left(nC_1+2K_4^2\right)}\Big\}. \qquad (3.18)
\end{aligned}
$$

By inequality $a+b \ge 2\sqrt{ab}$ for any $a, b \ge 0$, we have

$$
\frac{\eta}{(1+(\alpha-1)\eta)^2} \le \frac{1}{4(\alpha-1)}.
$$

Note that

$$
\frac{1}{(1+(\alpha-1)\eta)^2} \le \frac{1}{1+(\alpha-1)\eta} \le 1.
$$

Substituting above inequalities into (3.18) yields

$$
\begin{aligned}
\mathcal{F}(x_0,t_0) \quad &\le \quad \frac{N\alpha^2}{2(1-\varepsilon)}+\frac{N\alpha^2 t_0}{2(1-\varepsilon)}\Big\{\frac{2(\alpha-1)\sqrt{C_1}+2K_3+\big|1-\frac{2\mu}{N-n}\big|\varepsilon}{4(\alpha-1)}\\
&\quad+\sqrt{\frac{2(1-\varepsilon)}{N\varepsilon}\left(nC_1+2K_4^2\right)}\Big\}. \qquad (3.19)
\end{aligned}
$$

Then, because of $t \ge t_0$, we get the following

$$
\begin{aligned}
\mathcal{F}(x,t) \quad &\le \quad \mathcal{F}(x_0,t_0)\\
&\le \quad \frac{N\alpha^2}{2(1-\varepsilon)}+\frac{N\alpha^2 t}{2(1-\varepsilon)}\Big\{\frac{2(\alpha-1)\sqrt{C_1}+2K_3+\big|1-\frac{2\mu}{N-n}\big|\varepsilon}{4(\alpha-1)}\\
&\quad+\sqrt{\frac{2(1-\varepsilon)}{N\varepsilon}\left(nC_1+2K_4^2\right)}\Big\}.
\end{aligned}
$$

Finally, we conclude that the following inequality holds for any $(x,t) \in M \times (0,T]$ and $\alpha > 1, 0 < \varepsilon < 1$:

$$
\begin{aligned}
&F^2\left(\nabla(\log u)(x,t)\right)-\alpha\partial_t(\log u)(x,t)\\
&\le \frac{N\alpha^2}{2(1-\varepsilon)t}+\frac{N\alpha^2}{2(1-\varepsilon)}\Big\{\frac{2(\alpha-1)\sqrt{C_1}+2K_3+\big|1-\frac{2\mu}{N-n}\big|\varepsilon}{4(\alpha-1)}\\
&\quad+\sqrt{\frac{2(1-\varepsilon)}{N\varepsilon}\left(nC_1+2K_4^2\right)}\Big\}. \qquad (3.20)
\end{aligned}
$$

This completes the proof of Theorem 1.1. □

Integrating the gradient estimate in space-time as in [8], we can prove the Harnack type inequality given in Corollary 1.2 from Theorem 1.1.

Proof of Corollary 1.2. Take any curve γ satisfying the assumption and set

$$l(s) := \log u(\gamma(s), \tau(s)) = f(\gamma(s), \tau(s)), \quad \tau(s) := (1-s)t_2 + st_1.$$

Then $l(0) = f(x_2, t_2)$, $l(1) = f(x_1, t_1)$. By a direct computation, we have

$$\begin{aligned}
l(1) - l(0) = f(x_1, t_1) - f(x_2, t_2) &= \int_0^1 \frac{d}{ds}\left(f(\gamma(s), \tau(s))\right) ds \\
&= \int_0^1 \left\{ \frac{\partial f}{\partial x^m}(\gamma(s), \tau(s))\dot{\gamma}^m(s) + \partial_t f(\gamma(s), \tau(s))(t_1 - t_2) \right\} ds \\
&= \int_0^1 (t_2 - t_1)\left(\frac{df(\dot{\gamma}(s))}{t_2 - t_1} - \partial_t f(\gamma(s), \tau(s)) \right) ds \\
&\le \int_0^1 (t_2 - t_1)\left\{ \frac{F(\dot{\gamma}(s))F(\nabla f)}{t_2 - t_1} - \partial_t f(\gamma(s), \tau(s)) \right\} ds.
\end{aligned}$$

By (1.4), we have

$$\begin{aligned}
-\partial_t f(\gamma(s), \tau(s)) \le \frac{N\alpha}{2(1-\varepsilon)\tau(s)} + \frac{N\alpha}{2(1-\varepsilon)} &\left\{ \frac{2(\alpha-1)\sqrt{C_1} + 2K_3 + \left|1 - \frac{2\mu}{N-n}\right|\varepsilon}{4(\alpha-1)} \right. \\
&\left. + \sqrt{\frac{2(1-\varepsilon)}{N\varepsilon}\left(nC_1 + 2K_4^2\right)} \right\} - \frac{1}{\alpha}F^2\left(\nabla f\right).
\end{aligned}$$

Then we have the following

$$\begin{aligned}
f(x_1, t_1) - f(x_2, t_2) &\le \int_0^1 (t_2 - t_1)\left\{ \frac{F(\dot{\gamma}(s))F(\nabla f)}{t_2 - t_1} - \frac{F^2(\nabla f)}{\alpha} + \frac{N\alpha}{2(1-\varepsilon)\tau(s)} \right\} ds \\
&\quad + \frac{(t_2 - t_1)N\alpha}{2(1-\varepsilon)}\left\{ \frac{2(\alpha-1)\sqrt{C_1} + 2K_3 + \left|1 - \frac{2\mu}{N-n}\right|\varepsilon}{4(\alpha-1)} \right. \\
&\quad \left. + \sqrt{\frac{2(1-\varepsilon)}{N\varepsilon}\left(nC_1 + 2K_4^2\right)} \right\} \\
&\le \int_0^1 \left\{ \frac{\alpha}{4}\frac{F^2\left(\dot{\gamma}(s)\right)|_{\tau(s)}}{t_2 - t_1} + \frac{(t_2 - t_1)N\alpha}{2(1-\varepsilon)\tau(s)} \right\} ds \\
&\quad + \frac{(t_2 - t_1)N\alpha}{2(1-\varepsilon)}\left\{ \frac{2(\alpha-1)\sqrt{C_1} + 2K_3 + \left|1 - \frac{2\mu}{N-n}\right|\varepsilon}{4(\alpha-1)} \right. \\
&\quad \left. + \sqrt{\frac{2(1-\varepsilon)}{N\varepsilon}\left(nC_1 + 2K_4^2\right)} \right\},
\end{aligned}$$

where we have used the following inequality

$$\frac{F(\dot{\gamma}(s))}{t_2 - t_1}F(\nabla f) - \frac{1}{\alpha}F^2(\nabla f) \le \frac{\alpha}{4}\frac{F^2(\dot{\gamma}(s))}{(t_2 - t_1)^2}.$$

Furthermore, we have the following

$$\begin{aligned}\log\left(\frac{u(x_1,t_1)}{u(x_2,t_2)}\right) \leq & \frac{\alpha}{4}\int_0^1 \frac{F^2\left(\dot{\gamma}(s)\right)|_{\tau(s)}}{t_2-t_1}ds + \frac{N\alpha}{2(1-\varepsilon)}\log\left(\frac{t_2}{t_1}\right)\\ & + \frac{(t_2-t_1)N\alpha}{2(1-\varepsilon)}\Bigg\{\frac{2(\alpha-1)\sqrt{C_1}+2K_3+\left|1-\frac{2\mu}{N-n}\right|\varepsilon}{4(\alpha-1)}\\ & + \sqrt{\frac{2(1-\varepsilon)}{N\varepsilon}\left(nC_1+2K_4^2\right)}\Bigg\}.\end{aligned}$$

Thus, we can get the following

$$\begin{aligned}u(x_1,t_1) \leq u(x_2,t_2)\left(\frac{t_2}{t_1}\right)^{\frac{N\alpha}{2(1-\varepsilon)}} & \exp\Bigg\{\frac{\alpha}{4}\int_0^1 \frac{F^2\left(\dot{\gamma}(s)\right)|_{\tau(s)}}{t_2-t_1}ds\\ & + \frac{(t_2-t_1)N\alpha}{2(1-\varepsilon)}\Big[\frac{2(\alpha-1)\sqrt{C_1}+2K_3+\left|1-\frac{2\mu}{N-n}\right|\varepsilon}{4(\alpha-1)}\\ & + \sqrt{\frac{2(1-\varepsilon)}{N\varepsilon}\left(nC_1+2K_4^2\right)}\Big]\Bigg\},\end{aligned}$$

which is just (1.5). This completes the proof of Corollary 1.2. □

Remark 3.4. In [5], the first author established first order gradient estimates and the corresponding Harnack inequality for positive solutions of the heat equation under closed Finsler-Ricci flow with the condition that the weighted Ricci curvature Ric^∞ has a non-positive lower bound. However, the main results in [5] can not be obtained from the corresponding results in the present paper by letting $N \to \infty$.

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