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On New Classes of Stretch Finsler Metrics

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Abstract. In this paper, we introduce two classes of stretch Finsler metrics. A Finsler metric with vanishing stretch $\widetilde{\mathbf{B}}$-curvature (stretch $\mathbf{H}$-curvature) is called $\widetilde{\mathbf{B}}$-stretch ($\mathbf{H}$-stretch) metric (respectively). The class of $\widetilde{\mathbf{B}}$-stretch ($\mathbf{H}$-stretch) metric contain the class of Berwald (weakly Berwald) metric (respectively). First, we show that every complete $\widetilde{\mathbf{B}}$-stretch metric ($\mathbf{H}$-stretch metric) is a $\widetilde{\mathbf{B}}$-metric ($\mathbf{H}$-metric). Then we prove that every compact Finsler manifold with non-negative (non-positive) relatively isotropic stretch $\widetilde{\mathbf{B}}$-curvature (stretch $\mathbf{H}$-curvature) is $\widetilde{\mathbf{B}}$-metric ($\mathbf{H}$-metric).

Keywords: stretch curvature, complete stretch metric, Berwald curvature, **H**-curvature, relatively isotropic stretch curvature.

1. **Introduction**

Riemann hinted in a remark at generalized case of Riemannian metrics, which later labeled Finsler metric and denoted by F [11]. Along the time lots mathematicians aimed to adjust mathematical tools which were effective in Riemannian geometry such as the theory of connections, Jacobi vector fields, sectional curvature to a more general one. In 1918, Finsler devoted his Ph.D. dissertation to clear the way to start that approach in the field of Finsler

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Geometry [6]. To describe the nature of Finsler geometry can be done by investigating several quantities: the Cartan torsion $\mathbf{C}$, the Berwald curvature $\mathbf{B}$, and the Landsberg curvature $\mathbf{L}$, etc. These are said to be non-Riemannian quantities because all of them vanish for the Riemannian case. (See [9], [13], [14]).

Let (M,F) be a Finsler manifold. The second and third order derivatives of $\left[\frac{1}{2}F^2\right]$ at $y \in \mathcal{T}M$ are called the fundamental tensor and the Cartan torsion, respectively. The rate of change of the Cartan torsion along Finslerian geodesics gives the Landsberg curvature $\mathbf{L}$ of F.

In 1924, Berwald defined the notion of stretch curvature as a generalization of Landsberg curvature and denoted it by $\mathbf{T}$ [3]. In 1925, he published it in the first of his main papers [4]. He showed that $\mathbf{T} = 0$ if and only if the length of a vector remains unchanged under the parallel displacement along infinitesimal parallelograms. In 1928, he formulated a number of Finsler metric classes, including Landsberg metrics and stretch metrics [5]. Then, this curvature investigated was by Shibata in [12] and Matsumoto in [8]. Matsumoto denoted this curvature by $\mathbf{\Sigma}_y$. Najafi and Tayebi in 2017 introduced a new non-Riemannian quantity named as mean stretch curvature by taking trace with respect to $\mathbf{g}_{\mathrm{y}}$ in first and second variables of $\mathbf{\Sigma}_y$ [10]. A Finsler metric has vanishing mean stretch curvature called a weakly stretch metric. Recently many interesting results have been obtained in this direction. (See [15], [16], [17], [18]).

Z. Shen introduced a non-Riemannian quantity $\widetilde{\mathbf{B}}$ which is obtained from the Berwald curvature $\mathbf{B}$ by the covariant horizontal differentiation along Finslerian geodesics. For a vector $y \in \mathcal{T}_pM$, define $\widetilde{\mathbf{B}}_y : T_pM \times T_pM \times T_pM \to T_pM$ by $\widetilde{\mathbf{B}}_y(u,v,w) := \widetilde{B}^i_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}\mid_x$, where

$$\widetilde{B}^i_{jkl} := B^i_{jkl|m}y^m.$$

The Finsler metric F is called $\widetilde{\boldsymbol{B}}$*-metric* if and only if $\widetilde{\mathbf{B}} = 0$. $\widetilde{\mathbf{B}}_y$ is symmetric in $u,v,w \in T_pM$. (See [13], page 139).

In this paper, we use the Berwald curvature instead of the Cartan torsion, and investigate the relationships among the classes obtained analogously to the Landsberg and the stretch curvatures. This will enhance the understanding of the role of the relevant tensors in characterizing the new classes of Finsler metrics.

For a vector $y \in \mathcal{T}_pM$, we define $\boldsymbol{\mathcal{K}}_y : T_pM \times T_pM \times T_pM \times T_pM \to T_pM$ by

$$\boldsymbol{\mathcal{K}}_y(u,v,w,z) := \mathcal{K}^i_{jklm}(y)u^jv^kw^lz^m\frac{\partial}{\partial x^i}\mid_x,$$

where

$$\mathcal{K}^i_{jklm} := 2\left(\widetilde{B}^i_{jkl|m} - \widetilde{B}^i_{jkm|l}\right),$$

and " $|$ " is the horizontal derivation with respect to the Berwald connection D of F. The family $\boldsymbol{\mathcal{K}} := \{\boldsymbol{\mathcal{K}}_y : y \in \mathcal{T}_pM\}$ is called the stretch $\widetilde{\mathbf{B}}$-curvature. A Finsler metric F is said to be *$\widetilde{\boldsymbol{B}}$-stretch metric* if and only if $\boldsymbol{\mathcal{K}} = 0$. Especially, every $\widetilde{\mathbf{B}}$-metric is a $\widetilde{\mathbf{B}}$-stretch metric. Therefore, on the contrary, it is interesting to find some topological condition on the manifold M such that every $\widetilde{\mathbf{B}}$-stretch metric on M reduces to a $\widetilde{\mathbf{B}}$-metric.

Let us introduce a non-trivial example, where $|.|$ and $\langle , \rangle$ denote the Euclidean norm and the inner product in $\mathbb{R}^n$, respectively.

Example 1.1. *The Finsler function F*

$$F(x,y) = \frac{\left(\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)} + \varepsilon\langle x,y\rangle\right)^2}{(1-|x|^2)^2\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}},$$

on the unit ball $\mathbb{B}^n$ is a $\widetilde{\boldsymbol{B}}$-stretch metric when $n = 2$ and $n = 3$. This can be shown using the Finsler package and Maple program [20]. *We guess it should work in the general dimension but the calculation is very tedious and a bit complicated.*

Example 1.2. *The Finsler metric F with the property that the Berwald curvature satisfies $B^i_{jkl|m} = B^i_{jkm|l}$ is $\widetilde{\boldsymbol{B}}$-stretch metric. Namely, in this case we have $\widetilde{B}^i_{jkl|m} = \left[B^i_{jkl|s}y^s\right]_{|m} = \left[B^i_{jks|l}y^s\right]_{|m} = 0$, so we have $\boldsymbol{\mathcal{K}} = 0$.*

We have the following inclusions:

$$\{\text{Berwald metric}\} \subset \left\{\widetilde{\mathbf{B}}\text{-metric}\right\} \subset \left\{\widetilde{\mathbf{B}}\text{-stretch metric}\right\}.$$

The Finslerian quantity $\mathbf{H}$ was introduced by H. Akbar-Zadeh to characterization of Finsler metrics of constant flag curvature which is obtained from the mean Berwald curvature $\mathbf{E}$ by the covariant horizontal differentiation along geodesics. For a vector $y \in \mathcal{T}_pM$, $\mathbf{H}_y : T_pM \times T_pM \to \mathbb{R}$ is given by $\mathbf{H}_y(u,v) := H_{jk}(y)u^jv^k$, where

$$H_{jk} := E_{jk|l}y^l.$$

The Finsler metric F is called *$\boldsymbol{H}$-metric* if and only if $\mathbf{H} = 0$ [1].

A non-Riemannian quantity is considered, namely *stretch $\boldsymbol{H}$-curvature* which is $\boldsymbol{\kappa} := \{\boldsymbol{\kappa}_y : y \in \mathcal{T}_pM\}$, where $\boldsymbol{\kappa}_y : T_pM \times T_pM \times T_pM \to \mathbb{R}$, by

$$\boldsymbol{\kappa}_y(u,v,w) := \kappa_{jkl}(y)u^jv^kw^l,$$

where

$$\kappa_{jkl} := 2\left(H_{jk|l} - H_{jl|k}\right).$$

The Finsler metric F is called *$\boldsymbol{H}$-stretch metric* if and only if $\boldsymbol{\kappa} = 0$. We have the following inclusion relations

$$\{\text{weakly Berwald metric}\} \subset \{\mathbf{H}\text{-metric}\} \subset \{\mathbf{H}\text{-stretch metric}\}\,.$$

In this paper, we prove the following theorems.

Theorem 1.3. *Suppose that F is a positively complete $\widetilde{\boldsymbol{B}}$-stretch metric with bounded Berwald torsion. Then F must be a $\widetilde{\boldsymbol{B}}$-metric and the Berwald torsion is constant along any geodesic.*

Theorem 1.4. *Every complete $\boldsymbol{H}$-stretch metric with bounded mean Berwald torsion is $\boldsymbol{H}$-metric.*

Let (M, F) be a Finsler manifold. Then F is called a *relatively isotropic stretch $\widetilde{\boldsymbol{B}}$-curvature* if its stretch $\widetilde{\mathbf{B}}$-curvature is given by

$$\boldsymbol{\mathcal{K}} := \lambda F \left(B^i_{jkl|m} - B^i_{jkm|l} \right),$$

where $\lambda := \lambda(x, y)$ is scalar function on TM. In this case, (M, F) is called a *relatively isotropic $\widetilde{\boldsymbol{B}}$-stretch manifold.* If $\lambda \geq 0\,(\lambda \leq 0, \lambda = \text{constant})$, then F is said to be *non-negative (non-positive or constant) relatively isotropic stretch $\widetilde{\boldsymbol{B}}$-curvature* (respectively).

If the stretch $\mathbf{H}$-curvature is given by

$$\boldsymbol{\kappa} := \lambda F \left(E_{jk|l} - E_{jl|k} \right).$$

Then F is said to be *non-negative (non-positive, constant) relatively isotropic stretch $\boldsymbol{H}$-curvature* if we have $\lambda \geq 0\,(\lambda \leq 0, \lambda = \text{constant})$ (respectively).

By Theorem 1.3 every complete $\widetilde{\mathbf{B}}$-stretch Finsler manifold with bounded Berwald torsion is a $\widetilde{\mathbf{B}}$-manifold. Thus, a compact $\widetilde{\mathbf{B}}$-stretch Finsler manifold reduces to a $\widetilde{\mathbf{B}}$-manifold. We generalize this result as follows.

Theorem 1.5. *A compact Finsler manifold with non-negative (non-positive) relatively isotropic stretch $\widetilde{\boldsymbol{B}}$-curvature is $\widetilde{\boldsymbol{B}}$-Finsler manifold. More precisely, a complete Finsler manifold with constant relatively isotropic stretch $\widetilde{\boldsymbol{B}}$-curvature and bounded $\widetilde{\boldsymbol{B}}$-curvature is $\widetilde{\boldsymbol{B}}$-metric.*

By Theorem 1.4 every complete $\mathbf{H}$-stretch Finsler manifold with bounded mean Berwald torsion is a $\mathbf{H}$-manifold. Thus, a compact $\mathbf{H}$-stretch Finsler manifold reduces to a $\mathbf{H}$-manifold. We generalize this result as follows.

Theorem 1.6. *Every compact Finsler manifold with non-positive (non-negative) relatively isotropic stretch $\boldsymbol{H}$-curvature is $\boldsymbol{H}$-Finsler manifold. More precisely, a complete Finsler metric with constant relatively isotropic stretch $\boldsymbol{H}$-curvature and bounded $\boldsymbol{H}$-curvature is $\boldsymbol{H}$-metric.*

2. **Preliminaries**

In this section, we will give a concise description of some quantities in Finsler geometry.

Let M be an n-dimensional C^∞ manifold, by T_pM we denote the tangent space at $p \in M$ and by $TM := \bigcup_{p\in M} T_pM$ we denote the tangent bundle of M. Every element of TM is a pair (p, y) where $p \in M$ and $y \in T_pM$. Denoted the slit tangent manifold by $\mathcal{T}M = TM \setminus \{\mathbf{o}\}$, where $\mathbf{o}$ denotes the zero section of the tangent bundle. The natural projection $\pi : TM \to M$ is given by $\pi(p, y) := p$. The pull-back tangent bundle π^*TM is a vector bundle over $\mathcal{T}M$ whose fiber $\pi^*_zTM = \pi^{-1}(z)$ is isomorphic to $T_{\pi(z)}M$, $\pi(z) := p$, where $\mathcal{T}TM := \bigcup_{z\in\mathcal{T}M} \left\{\pi^{-1}(z) \mid_z \right\}$, and

$$\pi^{-1}(z) = \Big\{(p, y, z) \mid y \in \mathcal{T}M, z \in T_pM\Big\}.$$

A *Finsler metric* on a manifold M is a function $F : TM \to [0, \infty)$ which has the following properties: (i) F is C^∞ mapping over $\mathcal{T}M$, (ii) $F(p, y)$ is positively 1-homogeneous y and (iii) the Hessian matrix $g_{ij}(p, y) := \mathcal{L}_{y^iy^j}$ is positive-definite at each element of $\mathcal{T}M$ where $\mathcal{L} = \frac{1}{2}[F^2(p, y)]$. Given a manifold M and a Finsler structure F on M, the pair (M, F) is called a *Finsler manifold.* The following quadratic form $\mathbf{g}_y : T_pM \times T_pM \to \mathbb{R}$ is called *the fundamental tensor* given by

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2\,(y + su + tv)\Big]_{s=t=0}.$$

Let $p \in M$ and $F_p := F \mid_{T_pM}$. To measure the non-Riemannian feature, one can define a $(0, 3)$-tensor field on π^*TM denoted by $\mathbf{C}$, where $\mathbf{C}_y : T_pM \times T_pM \times T_pM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}[\mathbf{g}_{y+tw}(u, v)]_{t=0}.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in\mathcal{T}M}$ is called *Cartan torsion.* The Finsler metric F is Riemannian if and only if $\mathbf{C} = 0$.

For a vector $y \in \mathcal{T}M$, $\mathbf{I}_y : T_pM \to \mathbb{R}$ is defined by

$$\mathbf{I}_y(u) := \sum_{i=1}^{n} \mathbf{C}_y(e_i, e_j, u)g^{ij}(y),$$

where $\{e_i\}_{i=1}^n$ is basis vectors for T_pM at $p \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in\mathcal{T}M}$ is called the *mean Cartan torsion.*

For a given n-dimensional Finsler manifold (M, F) *a spray* $\mathbf{G}$ is a smooth vector field induced by F on $\mathcal{T}M$, which it a map $\mathbf{G} : \mathcal{T}M \to T(\mathcal{T}M)$ and it is a section of $(T\mathcal{T}M, \nu_\pi, \mathcal{T}M)$, i.e. $\nu_\pi \circ \mathbf{G} = id_{\mathcal{T}M}$.

In a standard local coordinate system (x^i, y^i) for $\mathcal{T}M$ is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i},$$

where $G^i(\lambda y) = \lambda^2 G^i(y)$ for all $\lambda > 0$, and G^i are smooth at $(0 \neq y) \in \mathcal{T}M$, it called *spray coefficients* of $\mathbf{G}$, shown as below

$$G^i(y) = \frac{1}{4}g^{im}(y)\Big\{y^k[F^2]_{x^k y^m} - [F^2]_{x^m}\Big\}.$$

Assume the following conventions:

$$G^i_{\ j} := \frac{\partial G^i}{\partial y^j}, \quad G^i_{\ jk} := \frac{\partial G^i_{\ j}}{\partial y^k}.$$

The local functions $G^i_{\ j}$ are coefficients of a connection in the pullback tangent bundle π^*TM which is called the Berwald connection denoted by D. The derivatives of a vector field V and a 2-covariant tensor $T = T_{ij}dx^i \otimes dx^j$ is given by:

$$V^i_{|m} = \frac{\delta V^i}{\delta x^m} + V^s G^i_{sm}, \quad V_{i|m} = \frac{\delta V^i}{\delta x^m} - V_s G^s_{im},$$

$$T_{ij|m} = \frac{\delta T_{ij}}{\delta x^m} - T_{sj}G^s_{im} - T_{is}G^s_{mj},$$

where $\frac{\delta}{\delta x^m} := \frac{\partial}{\partial x^m} - G^i_{\ m}\frac{\partial}{\partial y^i}$.

A curve $\gamma = \gamma(t)$ is *a geodesic* if and only if its coordinates $(\gamma^i(t))$ satisfy

$$\ddot{\gamma}^i + 2G^i \circ \dot{\gamma} = 0,$$

where $\dot{\gamma} = \dot{\gamma}^i \frac{\partial}{\partial x^i}$.

For a non-zero vector $y \in \mathcal{T}_pM$, let us define $\mathbf{B}_y : T_pM \times T_pM \times T_pM \to T_pM$ by $\mathbf{B}_y(u, v, w) := B^i_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}\,|_x$ where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

We have a $(0,2)$-tensor, which is $\mathbf{E}_y : T_pM \times T_pM \to \mathbb{R}$ by $\mathbf{E}_y(u, v) := E_{jk}(y)u^j v^k$, where

$$E_{jk} := \frac{1}{2}B^m_{jkm}.$$

The quantities $\mathbf{B}$ and $\mathbf{E}$ are non-Riemannian quantities called the *Berwald curvature* and *mean Berwald curvature* [4], respectively. A Finsler metric F is said to be *Berwald metric* if $\mathbf{B} = 0$, while if $\mathbf{E} = 0$, it called *weakly Berwald metric.*

There are some important classes of Finsler metrics containing the class of Berwald metrics. For $y \in \mathcal{T}_pM$, define the *Landsberg curvature* $\mathbf{L}_y : T_pM \times T_pM \times T_pM \to \mathbb{R}$ and *mean Landsberg curvature* $\mathbf{J}_y : T_pM \to \mathbb{R}$ by

$$\mathbf{L}_y(u, v, w) := -\frac{1}{2}\mathbf{g}_y(\mathbf{B}_y(u, v, w), y), \quad \mathbf{J}_y(u) := \sum_{i,j=1}^{n} \mathbf{L}_y(e_i, e_j, u)g^{ij}(y).$$

In the local coordinates (x^i, y^i)

$$\mathbf{L}_y(u,v,w) := L_{ijk}u^iv^jw^k, \quad \mathbf{J}_y(u) := J_i(y)u^i,$$

where

$$L_{ijk} := -\frac{1}{2}y^m g_{ml}(y)B^l_{ijk}, \quad J_i := g^{jk}C_{ijk}.$$

Note that $\mathbf{L}_y(u,v,w)$ is symmetric in u, v and w and $\mathbf{L}_y(y,v,w) = 0$. A Finsler metric F is called a *Landsberg metric* (*weakly Landsberg metric*) if $\mathbf{L}_y = 0$ ($\mathbf{J}_y = 0$). respectively.

It is easy that every Berwald metric is a Landsberg metric.

For $y \in \mathcal{T}_pM$, the *stretch curvature* $\mathbf{\Sigma}_y : T_pM \times T_pM \times T_pM \times T_pM \to \mathbb{R}$ is given by $\mathbf{\Sigma}_y(u,v,w,z) := \Sigma_{ijkl}u^iv^jw^kz^l$, where

$$\Sigma_{ijkl} := 2\left(L_{ijk|l} - L_{ijl|k}\right).$$

A Finsler metric is said to be a *stretch metric* if and only if $\mathbf{\Sigma} = 0$. Obviously, every $\widetilde{\mathbf{B}}$-stretch metric is a stretch metric.

We have the following relation

$$\{\text{Berwald metrics}\} \subseteq \{\text{Landsberg metrics}\} \subseteq \{\text{Stretch metrics}\}.$$

3. Proof of Theorem 1.3

In this section, we are going to prove Theorem 1.3. We need the following

Proposition 3.1. *Let (M,F) be a Finsler manifold. Suppose that F is $\widetilde{\boldsymbol{B}}$-stretch metric and $\gamma = \gamma(t)$ is a geodesic. Put $\boldsymbol{B}(t) := \boldsymbol{B}_{\dot\gamma}(U(t),V(t),W(t))$, where $U(t), V(t)$ and $W(t)$ are the parallel vector fields along γ. Then, the following equation holds:*

$$\boldsymbol{B}(t) = \widetilde{\boldsymbol{B}}(0)t + \boldsymbol{B}(0). \tag{3.1}$$

Proof. Let p be an arbitrary point of M, $y, u, v, w \in T_pM$ and $\gamma : (-\infty, \infty) \to M$ be the unit speed geodesic passing from p and $\frac{d\gamma}{dt}(0) = y$. For $U(t), V(t)$ and $W(t)$ are the parallel vector fields along γ with $U(0) = u, V(0) = v, W(0) = w$ we put

$$\widetilde{\mathbf{B}}(t) := \widetilde{\mathbf{B}}_{\dot\gamma}(U(t),V(t),W(t)).$$

By definition of $\widetilde{\mathbf{B}}$-curvature, we have

$$\widetilde{\mathbf{B}}(t) = \mathbf{B}'(t). \tag{3.2}$$

Let

$$\widetilde{\mathbf{B}}'(t) := \widetilde{\mathbf{B}}'_{\dot\gamma}(U(t),V(t),W(t)).$$

Since F is $\widetilde{\mathbf{B}}$-stretch metric, then we have

$$\widetilde{\mathbf{B}}'(t) = 0.$$

Which implies that $\widetilde{\mathbf{B}}(t) = \widetilde{\mathbf{B}}(0)$. By (3.2), the proof is complete. $\square$

Now let us begin to prove Theorem 1.3.

Proof of Theorem 1.3: For an arbitrary unit vector $y \in T_pM$ and an arbitrary vector $v \in T_pM$, let $\gamma = \gamma(t)$ be the geodesic with $\dot{\gamma}(0) = y$, and $V(t)$ be the parallel vector field along γ with $V(0) = v$.

Then by Proposition 3.1 we get

$$\mathbf{B}(t) = \widetilde{\mathbf{B}}(0)t + \mathbf{B}(0).$$

Suppose that Berwald torsion is bounded at $x \in M$, i.e.

$$\|\mathbf{B}\|_x := \sup_{y \in \mathcal{T}M} \left[\sup_{v \in T_pM} \frac{\mathbf{B}_y(v)}{\left[\mathbf{g}_y(v,v)\right]^{\frac{3}{2}}} \right] < \infty.$$

By Lemma 7.3.2 in [13] we get that

$$W := \mathbf{g}_{\dot{\gamma}(t)}(V(t), V(t))$$

is a positive constant. Thus

$$|\mathbf{B}(t)| \le W^{\frac{3}{2}} \|\mathbf{B}\| < \infty,$$

Let us put $t \to +\infty$. Then, we get

$$\widetilde{\mathbf{B}}_y(v) = \widetilde{\mathbf{B}}(0) = 0.$$

Therefore $\widetilde{\mathbf{B}} = 0$. This completes the proof. □

It is clear that every Finsler metric with vanishing $\widetilde{\mathbf{B}}$-curvature has vanishing $\mathbf{H}$-curvature, that means, every $\widetilde{\mathbf{B}}$-metric is a $\mathbf{H}$-metric. By Theorem 1.3 a $\widetilde{\mathbf{B}}$-stretch Finsler metric reduces to a $\widetilde{\mathbf{B}}$-metric. Then, we get the following

Corollary 3.2. *Let (M,F) be a Finsler manifold. Then, every $\widetilde{\boldsymbol{B}}$-stretch metric is a $\boldsymbol{H}$-metric.*

Proposition 3.3. *Let (M,F) be a Landsberg space with vanishing Riemannian curvature, then it is a $\widetilde{\boldsymbol{B}}$-stretch space.*

Proof. Since (M,F) is Landsberg space then the horizontal covariant derivatives of Berwald and Cartan connections coincide, i.e.

$$G^h_{ij} = \Gamma^h_{ij}.$$

We have

$$g_{jk|h} = 0. \tag{3.3}$$

Differentiating (3.3) with respect to y^l, we get

$$2C_{jkl|h} - \left(B^r_{hkl}g_{rj} + B^r_{hjl}g_{rk} + B^r_{jhk}g_{rl}\right) + B_{hjkl},$$

where $B^h_{ijk} = \frac{\partial G^h_{ij}}{\partial y^k}$.

We have the property that the tensor $B^r_{hjl} g_{rk} = B_{khjl}$ is totally symmetric. Then we get

$$C_{jkl|h} = B_{jkhl}.$$

Equivalently

$$C^h_{jk|l} = B^h_{jkl}.$$

Now, we have

$$\tilde{B}^h_{jkl} = B^h_{jkl|m} y^m = C^h_{jk|l|m} y^m.$$

Since $R = 0$, $C^h_{jk|l|m} = C^h_{jk|m|l}$ holds, so

$$\tilde{B}^h_{jkl} = C^h_{jk|m|l} y^m = 0,$$

taking into account $C^h_{jk|m} y^m = 0$. Therefore, we obtain $\boldsymbol{\mathcal{K}} = 0$. □

4. **Proof of Theorem 1.4**

To prove Theorem 1.4, we need the following

Proposition 4.1. *Let (M, F) be a Finsler manifold. Suppose that F is $\boldsymbol{H}$-stretch metric. Then, for any geodesic $\gamma = \gamma(t)$ and any parallel vector field $V = V(t)$ along γ, the function $\boldsymbol{E}(t) := \boldsymbol{E}_{\dot\gamma}(V(t))$ must be in the following form:*

$$\boldsymbol{E}(t) = \boldsymbol{H}(0)t + \boldsymbol{E}(0).$$

Proof. Let $\gamma : [0, +\infty] \to M$ be the geodesic parameterized by the arc length on M with the start point $\gamma(0) = p$ and the tangent vector $\dot\gamma(0) = y$. Suppose that $U = U(t), V = V(t)$ are two parallel vector fields along $\gamma = \gamma(t)$ with $U(0) = u, V(0) = v$.

Since F is $\mathbf{H}$-stretch metric, then we get $\boldsymbol{\kappa} = 0$, that means

$$H_{jk|l} = H_{jl|k}. \tag{4.1}$$

Contracting (4.1) with y^l, we have

$$H_{jk|l} y^l = 0.$$

Let

$$\mathbf{H}(t) := \mathbf{H}_{\dot\gamma}(U(t), V(t)) = H_{jk}(\gamma(t), \dot\gamma(t)) U^j(t) V^k(t). \tag{4.2}$$

We have $\mathbf{H}(t) = \mathbf{E}'(t)$, by (4.2)

$$\mathbf{E}''(t) = \mathbf{H}'(t) = H_{jk|l} \dot\gamma^l(t)(\gamma(t), \dot\gamma(t)) U^j(t) V^k(t) = 0.$$

Thus yields $\mathbf{E}(t) = \mathbf{H}(0)t + \mathbf{E}(0)$. □

Let us start to prove Theorem 1.4.

Proof of Theorem 1.4: Let (M, F) be complete Finsler manifold. Suppose that F is $\mathbf{H}$-stretch metric. Take an arbitrary unit vector $y \in T_pM$ and an arbitrary vector $v \in T_pM$. Let $\gamma = \gamma(t)$ be the geodesic with $\gamma(0) = p$ and

$\dot{\gamma}(0) = y$, and $W(t)$ be the parallel vector field along γ with $W(0) = w$. Then by Proposition 4.1, we get

$$\mathbf{E}(t) = \mathbf{H}(0)t + \mathbf{E}(0). \tag{4.3}$$

Suppose that $\mathbf{E}_y$ is bounded, i.e. there is a constant $A < \infty$ such that

$$\|\mathbf{E}\|_x := \sup_{y \in TpM} \left[\sup_{v \in TpM} \frac{\mathbf{E}_y(w)}{\left[\mathbf{g}_y(w,w)\right]^{\frac{3}{2}}} \right] \leq A.$$

By Lemma 7.3.2 in [13], we have

$$|\mathbf{E}(t)| \leq AQ^{\frac{3}{2}} < \infty,$$

for some constant Q. Therefore $\mathbf{E}(t)$ is a bounded function on $(-\infty, \infty)$. Letting $t \to \infty$ in (4.3), it implies that $\mathbf{H}_y(v) = \mathbf{H}(0) = 0$. □

By Theorem 1.4, a **H**-stretch metric reduces to a **H**-metric. Tayebi et al. in [16] proved that any **H**-metric is a $\widetilde{\mathbf{B}}$-metric for a Finsler surface (M, F). Then, we get the following corollary.

Corollary 4.2. *Let (M, F) be a Finsler surface. Then F is a **H**-stretch metric if and only if it is $\widetilde{\boldsymbol{B}}$-metric.*

5. Proof of Theorem 1.5

In this section, we will prove Theorem 1.5. We need the following results:

Theorem 5.1. [7] *Suppose M is a compact, oriented manifold with a volume element ω . Then for every vector field X over M, we have $\int_M (divX)_\omega = 0$.*

Theorem 5.2. [7] *Suppose M is an oriented manifold with the volume form ω and ∇ is a torsion-free connection where $\nabla_\omega = 0$. Then for every vector field X over M, $y \in T_pM$ with $x \in M$ we have $(divX)_x = -trace(Y \to \nabla_Y X) = \nabla_i X^i$.*

Let us begin to prove Theorem 1.5.

Proof of Theorem 1.5: Let $p \in M$, and $y, u, v, w \in T_pM$, and $\gamma : (-\infty, \infty) \to M$ is the geodesic with $\gamma(0) = p$ and $\frac{d\gamma}{dt}(0) = y$ and $U(t), V(t)$ and $W(t)$ are parallel vector fields along γ such that $U(0) = u, V(0) = v, W(0) = w$.

We put

$$\widetilde{\mathbf{B}}(t) = \widetilde{\mathbf{B}}_{\dot{\gamma}}((U(t), V(t), W(t)).$$
$$\widetilde{\mathbf{B}}'(t) = \widetilde{\mathbf{B}}'_{\dot{\gamma}}((U(t), V(t), W(t)).$$

However, the Finsler manifold (M, F) has non-negative (non-positive, respectively) relatively isotropic stretch $\widetilde{\mathbf{B}}$-curvature or is constant. By the definition and multiplying by y^l, it is simple to get:

$$\widetilde{B}^i_{jkm|l} y^l = \lambda F \widetilde{B}^i_{jkm}, \tag{5.1}$$

where $\lambda := \lambda(x,y)$ is a non-negative (non-positive, respectively) or constant homogeneous function over $\mathcal{T}M$.

First suppose $\lambda := \lambda(x,y)$ is a non-negative (non-positive, respectively) function over $\mathcal{T}M$.

By putting

$$\Gamma(x,y) := \widetilde{B}^r_{szmq}\,\widetilde{B}^{szmq}_r,$$

we have

$$\begin{aligned}\dot{\Gamma}(x,y) &= \Gamma_{|n}y^n \\ &= \lambda F \widetilde{B}^r_{szmq}\widetilde{B}^{szmq}_r + \widetilde{B}^r_{szmq}\lambda\widetilde{B}^{szmq}_r \\ &= 2\lambda F\Gamma. \end{aligned} \tag{5.2}$$

As F and Γ have positive values, if λ is non-negative (non-positive), then $\dot{\Gamma}$ is non-negative (non-positive).

By Theorem 5.2 we get

$$\dot{\Gamma}(x,y) = \Gamma_{|n}y^n = \mu(\Gamma) = \overline{Div}(\Gamma\mu).$$

Note that $\delta = y^i\frac{\delta}{\delta x^i}$ is a geodesic vector field on unit sphere tangent bundle SM and $\overline{Div}(\mu) = 0$ [19].

Because M is compact, SM is also compact. The value form ω_{SM} over SM is obtained from the volume form ω on M [2]. According to Theorem 5.1 we get,

$$\int_{SM}\dot{\Gamma}\omega_{SM} = 0.$$

Since $\dot{\Gamma}$ is homogeneous function and its non-negative (non-positive) sign, then $\dot{\Gamma} = 0$.

By equation (5.2), we get $\Gamma = 0$ or $\lambda = 0$. If $\Gamma = 0$, then $\widetilde{\mathbf{B}} = 0$. If $\lambda = 0$, then $\boldsymbol{\mathcal{K}} = 0$.

In this case

$$\widetilde{\mathbf{B}}' = \widetilde{B}^i_{jkm|l}y^l = 0.$$

Thus

$$\widetilde{\mathbf{B}}(t) = \widetilde{\mathbf{B}}(0).$$

So Berwald torsion is equal to

$$\widetilde{\mathbf{B}}(t) = \widetilde{\mathbf{B}}(0)t + \mathbf{B}(0).$$

Letting $t \to \mp\infty$ and using $\|\mathbf{B}\| < \infty$, we get $\widetilde{\mathbf{B}}(0) = 0$. Thus

$$\widetilde{\mathbf{B}}(t) = 0.$$

Now, if $\lambda =$ constant, the general answer to equation (5.1) is as follows:

$$\widetilde{\mathbf{B}}(t) = e^{t\lambda}\widetilde{\mathbf{B}}(0).$$

Using $\|\mathbf{B}\| < \infty$ and letting $t \to \mp\infty$, this implies that

$$\widetilde{\mathbf{B}}(t) = \widetilde{\mathbf{B}}(0) = 0.$$

Thus, the second part of Theorem 1.5 is also proved. □

Corollary 5.3. *Let (M, F) be Finsler manifold. Then, every non-negative (non-positive) relatively isotropic stretch $\widetilde{\boldsymbol{B}}$-curvature is a $\boldsymbol{H}$-metric.*

6. **Proof of Theorem 1.6**

In this section, we are going to prove Theorem 1.6.

Proof of Theorem 1.6: Let $p \in M$ and $y, u, v \in T_pM$. Let $\gamma : (-\infty, \infty) \to M$ be the unit speed geodesic such that $\gamma(0) = p$, $\dot{\gamma}(0) = y$. Suppose $U = U(t), V = V(t)$ are the parallel vector fields along γ with $U(0) = u, V(0) = v$. Put

$$\mathbf{H}(t) = \mathbf{H}(U(t), V(t)),$$
$$\mathbf{H}'(t) = \mathbf{H}'(U(t), V(t)).$$

By assumption, F has non-positive (non-negative) relatively isotropic stretch $\mathbf{H}$-curvature. Then

$$H_{jk|l} - H_{jl|k} = \lambda F \left(E_{jk|l} - E_{jl|k} \right), \tag{6.1}$$

where $\lambda := \lambda(x, y)$ is a non-positive (non-negative) or constant function on TM.

Contraction (6.1) with y^k implies that

$$H_{jl|k} y^k = \lambda F E_{jl|k} y^k,$$

since $H_{jl} = E_{jl|k} y^k$. Thus we get

$$H_{jl|k} y^k = \lambda F H_{jl}.$$

First let $\lambda := \lambda(x, y)$ be a non-negative scalar function on TM. Put

$$\phi := H^{zn} H_{zn}.$$

Then we have

$$\phi' = 2\lambda F \phi.$$

By definition, F and ϕ have positive value. If λ is non-negative (non-positive), then ϕ' is non-negative (non-positive). By Theorem 5.2, we get

$$\phi' = \phi_{|m} y^m = \zeta(\phi) = \overline{Div}(\phi\zeta),$$

where $\zeta = y^i \frac{\delta}{\delta x^i}$ is a geodesic vector field on the unit sphere tangent bundle SM and $\overline{Div}(\zeta) = 0$. By Theorem 5.1, we get

$$\int_{SM} \phi' \omega_{SM} = 0.$$

Thus the volume form ω_{SM} on SM is obtained from volume form ω on M. Since ϕ' is homogeneous function, and its sign is negative (positive), then $\phi' = 0$, and we have $\phi = 0$ or $\lambda = 0$. If $\phi = 0$, then $H = 0$. If $\lambda = 0$, then $\boldsymbol{\kappa} = 0$. In this case

$$\mathbf{H}' = E_{jl|k}y^k = 0.$$

Thus $\mathbf{H}(t) = \mathbf{H}(0)$, which implies that

$$\mathbf{H}(t) = t\mathbf{H}(t) + \mathbf{E}(0).$$

Letting $t \to \mp\infty$, and using $\|\mathbf{E}\| < \infty$, we get $\mathbf{H}(0) = 0$. Thus

$$\mathbf{H}(t) = 0.$$

Now, suppose $\lambda =$ constant, then the general answer of (6.1) is as follows

$$\mathbf{H}(t) = \mathbf{H}(0)\exp(t\lambda).$$

Using $\|\mathbf{E}\| < \infty$ and letting $t \to \pm\infty$ this implies that $\mathbf{H}(0) = 0$, thus

$$\mathbf{H}(t) = 0.$$

This completes the proof. □

Corollary 6.1. *Let (M, F) be Finsler surface. Then, every non-negative (non-positive) relatively isotropic stretch $\boldsymbol{H}$-curvature is a $\widetilde{\boldsymbol{B}}$-metric.*

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