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On compact L-reducible Finsler manifolds

Asma Ghasemi[a*]

[a]Department of Mathematics, University of Hormozgan
Bandar-Abbas, Iran

E-mail: ghasemi.asmaa@gmail.com

Abstract. The class of L-reducible Finsler metric was introduced by Matsumoto as a generalization of Randers metrics. One of the open problems in Finsler Geometry is to find a L-reducible metric which is not of Randers-type. In this paper, we are going to study 3-dimensional L-reducible metrics. Let (M, F) be a compact 3-dimensional L-reducible metric. Suppose that F has constant relatively isotropic mean Landsberg curvature. Then we show that F reduces to a Randers metric.

Keywords: L-reducible metric, Randers metric, Landsberg metric.

1. Introduction

Randers metrics are natural non-Riemannian Finsler metrics which were introduced by Norwegian Physician Gunnar Randers in order to study of general relativity in 4-dimensional manifolds [20]. His metric is in the form

$$F = \alpha + \beta,$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is gravitation field and $\beta = b_i(x)y^i$ is the electromagnetic field. Randers regarded these metrics not as Finsler metrics but as "affinely connected Riemannian metrics", which is a rather confusing notion. This metric was first recognized as a kind of Finsler metric in 1957 by the Polish Physician, Roman Stanisław Ingarden, who first named them Randers

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*Corresponding Author
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metrics [6]. Randers metrics have been widely applied in many areas of natural science, including Seismic Ray Theory, Biology, Physics, and etc.

An interesting reality about the Randers metrics is related to their Cartan torsions. First, we introduce some notions and then explain the mentioned property. Let (M, F) be a Finsler manifold. The second derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ is an inner product $\mathbf{g}_y$ on T_xM. The third order derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ is a symmetric trilinear forms $\mathbf{C}_y$ on T_xM. We call $\mathbf{g}_y$ and $\mathbf{C}_y$ the fundamental form and the Cartan torsion, respectively. Taking a trace of $\mathbf{C}$ gives the mean Cartan torsion $\mathbf{I}$. A Finsler metric F on an n-dimensional manifold M is C-reducible if its Cartan torsion is give by

$$C_{ijk} = \frac{1}{n+1}\Big\{I_i h_{jk} + I_j h_{ik} + I_k h_{ij}\Big\}. \tag{1.1}$$

In [9], Matsumoto introduced the notion of Matsumoto torsion and proved that any Randers metric has vanishing Matsumoto torsion. Every Finsler metric with vanishing Matsumoto torsion is called C-reducible. Thus by Matsumoto's result, Randers metrics are C-reducible. Later on, Matsumoto-Hōjō proved that the converse is true too [13]. In [16], Mo-Shen proved that every Finsler metric of negative scalar flag curvature on a compact manifold of dimension $n \geq 3$ is a Randers metric. By using the main scalar and its derivation in Finsler plans, Mo-Huang found a quantity that characterized Randers plans among the Minkowski plans [15]. They pointed out that the Matsumoto torsion is just the cubic form of the indicatrix with its Blaschke structure. Hence the Matsumoto-Hōjō's Theorem is a corollary of the Maschke-Pick-Berwald Theorem (see page 53 in [18]). In [3], Bao-Robles-Shen showed that a Finsler metric is of Randers type if and only if it is a solution of the navigation problem on a Riemannian manifold. Then Javaloyes-Vitório define the Matsumoto torsion of a conic pseudo-Finsler metric and proved that a conic pseudo-Finsler manifold of dimension at least 3 is of pseudo-Randers-Kropina type if and only if its Matsumoto tensor vanishes identically [7]. Recently, Yan give some new characterizations of Randers norms by proving a maximum property of Randers norms and some integral inequalities on the indicatrix [25]. In [13], Matsumoto-Hōjō proved that a Finsler metric F is C-reducible if and only if it is a Randers metric or an almost regular Finsler metric, namely Kropina metric.

The rate of change of $\mathbf{C}_y$ along geodesics is the Landsberg curvature $\mathbf{L}_y$ on T_xM for any $y \in T_xM_0$. F is said to be Landsbergian if $\mathbf{L} = 0$. Taking a trace of $\mathbf{L}$ give us mean Landsberg curvature $\mathbf{J}$. A Finsler metric F on an n-dimensional manifold M is L-reducible if its Landsberg curvature is give by

$$L_{ijk} = \frac{1}{n+1}\Big\{J_i h_{jk} + J_j h_{ik} + J_k h_{ij}\Big\}. \tag{1.2}$$

By taking a horizontal derivation from (1.1), one can get (1.2). Thus every C-reducible metric is L-reducible. But the converse may not true in general.

There are some other generalization of C-reducible metrics, namely generalized P-reducible metrics. A Finsler metric F is called generalized P-reducible if its Landsberg curvature is given by following

$$L_{ijk} = \lambda C_{ijk} + a_i h_{jk} + a_j h_{ki} + a_k h_{ij},$$

where $\lambda = \lambda(x, y)$ is a scalar function on TM, $a_i = a_i(x)$ is scalar function on M and $h_{ij} = g_{ij} - F^{-2} y_i y_j$ is the angular metric. λ and a_i are homogeneous function of degree 1 and degree 0 with respect to y, respectively. The class of generalized P-reducible metrics was introduced by Prasad in [19]. In [24], Tayebi-Sadeghi characterized generalized P-reducible (α, β)-metrics with vanishing S-curvature and proved the following.

Theorem A. ([24]) Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on a manifold M. Suppose that F is a generalized P-reducible metric with vanishing S-curvature. Then F is a Berwald metric or C-reducible metric.

By Theorem A, it follows that there is no concrete P-reducible (α, β)-metric with vanishing S-curvature. For more information about the class of (α, β)-metrics, see [5], [12], [21] and [23].

In [17], Moór constructed an intrinsic orthonormal frame field on three dimensional Finsler manifolds which was a generalization of the Berwald frame of two-dimensional Finsler manifolds. Then, Matsumoto gave a systematic description of a general theory of 3-dimensional Finsler spaces based on Moór's frame, that is, on a frame whose first vector is the normalized supporting element and the second one is taken as the normalized torsion vector [10][11]. In addition to three main scalars and nine scalars representing the curvature tensor, he introduces two important vector fields, called h-connection and v-connection vectors. He proved that a non-Riemannian Berwald 3-space is characterized by the fact that the h-connection vector h_i vanishes and the main scalars $\mathcal{H}$, $\mathcal{I}$, $\mathcal{J}$ are h-covariant constant.

In [4], Beizavi studied the class of L-reducible metrics with relatively isotropic mean Landsberg curvature and proved the following.

Theorem B. ([4]) Let (M, F) be a 3-dimensional L-reducible Finsler manifold such that $b_i = b_i(x, y)$ is constant along Finslerian geodesics. Suppose that F has relatively isotropic mean Landsberg curvature

$$\mathbf{J} = cF\mathbf{I}, \tag{1.3}$$

where $c = c(x)$ is a scalar function on M. Then one of the following holds

(1) F is a Randers metric;
(2) F is a Landsberg metric;

In this paper, we are going to find a condition under which a L-reducible Finsler metric reduces to a C-reducible metric, or equivalently a Randers metric by Matsumoto-Hōjō Theorem. Then, we prove the following.

Theorem 1.1. *Let (M, F) be a compact 3-dimensional L-reducible manifold. Suppose that F has non-zero constant relatively isotropic mean Landsberg curvature*

$$\mathbf{J} = cF\mathbf{I},$$

where c is a real constant. Then F is a Randers metric.

It is easy to see that every L-reducible Finsler metric with vanishing mean Landsberg curvature ($c = 0$) reduces to a Landsberg metric.

2. Preliminaries

A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:
(i) F is C^∞ on $TM_0 := TM \setminus \{0\}$;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;
(iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\Big[F^2(y + su + tv)\Big]|_{s,t=0}, \quad u, v \in T_xM.$$

Let $x \in M$. To measure the non-Euclidean feature of $F_x := F|_{T_xM}$, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\left[\mathbf{g}_{y+tw}(u, v)\right]|_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C}{=}\mathbf{0}$ if and only if F is Riemannian.

For $y \in T_xM_0$, define mean Cartan torsion $\mathbf{I}_y$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where

$$I_i := g^{jk}C_{ijk}$$

and $u = u^i \frac{\partial}{\partial x^i}|_x$. By Diecke Theorem, a positive-definite Finsler metric F is Riemannian if and only if $\mathbf{I}_y = 0$ (see [8]).

For a non-zero vector $y \in T_xM_0$, one can define the Matsumoto torsion $\mathbf{M}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{M}_y(u, v, w) := M_{ijk}(y)u^iv^jw^k$ where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\Big\{I_ih_{jk} + I_jh_{ik} + I_kh_{ij}\Big\},$$

and

$$h_{ij} := FF_{y^iy^j} = g_{ij} - \frac{1}{F^2} g_{ip}y^p g_{jq}y^q$$

is the angular metric. A Finsler metric F is said to be C-reducible if $\mathbf{M}_y = 0$. This quantity is introduced by Matsumoto [9]. Matsumoto proves that every Randers metric satisfies that $\mathbf{M}_y = 0$. A Randers metric $F = \alpha + \beta$ on a manifold M is just a Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ perturbated by a one form $\beta = b_i(x)y^i$ on M such that $\|\beta\|_\alpha < 1$. Later on, Matsumoto-Hōjō proves that the converse is true too.

Lemma 2.1. ([13]) A Finsler metric F on a manifold of dimension $n \geq 3$ is a Randers metric if and only if $\mathbf{M}_y = 0$, $\forall y \in TM_0$.

The horizontal covariant derivatives of $\mathbf{C}$ along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ defined by $\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^iv^jw^k$, where

$$L_{ijk} := C_{ijk|s}y^s,$$

$u = u^i \frac{\partial}{\partial x^i}|_x$, $v = v^i \frac{\partial}{\partial x^i}|_x$ and $w = w^i \frac{\partial}{\partial x^i}|_x$ (see [2]). The family $\mathbf{L} := \{\mathbf{L}_y\}_{y \in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$. The quantity $\mathbf{L}/\mathbf{C}$ is regarded as the relative rate of change of $\mathbf{C}$ along geodesics. Then F is said to be relatively isotropic Landsberg metric if

$$\mathbf{L} = cF\mathbf{C},$$

for some scalar function $c = c(x)$ on M.

The horizontal covariant derivatives of $\mathbf{I}$ along geodesics give rise to the mean Landsberg curvature $\mathbf{J}_y(u) := J_i(y)u^i$, where

$$J_i := g^{jk}L_{ijk}.$$

A Finsler metric is called a weakly Landsberg metric if $\mathbf{J} = 0$. The quantity $\mathbf{J}/\mathbf{I}$ is regarded as the relative rate of change of $\mathbf{I}$ along geodesics. Then F is said to be relatively isotropic mean Landsberg metric if

$$\mathbf{J} = cF\mathbf{I},$$

for some scalar function $c = c(x)$ on M.

Let us consider the following Randers metric on $\mathbb{R}^2$

$$F = \frac{\sqrt{(1-\epsilon^2)(xu+yv)^2 + \epsilon(u^2+v^2)(1+\epsilon(x^2+y^2))}}{1+\epsilon(x^2+y^2)} + \frac{\sqrt{1-\epsilon^2}(xu+yv)}{1+\epsilon(x^2+y^2)},$$

where $0 < \epsilon \leq 1$ is a real number. By calculation, we get $\mathbf{J} + cF\,\mathbf{I} = 0$, where

$$c = \frac{\sqrt{1-\epsilon^2}}{2(\epsilon + x^2 + y^2)}.$$

3. L-Reducible Finsler Metrics

In [17], Moór introduced a special orthonormal frame field (ℓ^i, m^i, n^i) in the three dimensional Finsler space. The first vector of the frame is the normalized supporting element, the second is the normalized mean Cartan torsion vector, and third is the unit vector orthogonal to them. Let (M, F) be a 3-dimensional Finsler manifold. Suppose that $\ell_i := F_{y^i}$ is the unit vector along the element of support, m_i is the unit vector along mean Cartan torsion I_i, i.e.,

$$m_i := \frac{1}{||\mathbf{I}||} I_i,$$

where $||\mathbf{I}|| := \sqrt{I_i I^i}$ and n_i is a unit vector orthogonal to the vectors ℓ_i and m_i. Then the triple (ℓ_i, m_i, n_i) is called the Moór frame.

For 3-dimensional Finsler manifolds, we have

$$g_{ij} = \ell_i \ell_j + m_i m_j + n_i n_j.$$

Thus

$$g^{ij} = \ell^i \ell^j + m^i m^j + n^i n^j. \tag{3.1}$$

Then the Cartan torsion of F is written as follows

$$FC_{ijk} = \mathcal{H} m_i m_j m_k - \mathcal{J}\Big\{ m_i m_j n_k + m_j m_k n_i + m_k m_i n_j - n_i n_j n_k \Big\} + \mathcal{I}\Big\{ n_i n_j m_k + n_j n_k m_i + n_i n_k m_j \Big\}, \tag{3.2}$$

where $\mathcal{H}$, $\mathcal{I}$ and $\mathcal{J}$ are called the main scalars of F. Thus multiplying (3.2) with (3.1) implies that

$$FI_k = (\mathcal{H} + \mathcal{I}) m_k. \tag{3.3}$$

In [22], Tayebi-Najafi obtained the following.

Lemma 3.1. ([22]) Let (M, F) be a 3-dimensional non-Riemannian Finsler manifold. Then the Cartan torsion of F is given by following

$$C_{ijk} = \Big\{ a_i h_{jk} + a_j h_{ki} + a_k h_{ij} \Big\} + \Big\{ b_i I_j I_k + I_i b_j I_k + I_i I_j b_k \Big\}, \tag{3.4}$$

where $a_i = a_i(x, y)$ and $b_i = b_i(x, y)$ are scalar functions on tangent bundle TM and given by

$$a_i := \frac{1}{3F}\Big[3\mathcal{I} m_i + \mathcal{J} n_i \Big], \qquad b_i := \frac{F}{3(\mathcal{H} + \mathcal{I})^2}\Big[(\mathcal{H} - 3\mathcal{I}) m_i - 4\mathcal{J} n_i \Big]. \tag{3.5}$$

By (3.5), one can see that

$$a_i y^i = b_i y^i = 0.$$

Throughout this paper, we assume that $\mathcal{H} + \mathcal{I} \neq 0$. By (3.3), we assume that F is not Riemannian in this paper.

By taking a horizontal derivation of (3.4), one can get the Landsberg curvature of 3-dimensional Finsler manifolds, as follows.

Lemma 3.2. *Let (M, F) be a 3-dimensional Finsler manifold. Then the Landsberg curvature of F is given by following*

$$\begin{aligned} L_{ijk} = & -\frac{1}{2}\Big\{J^m b_m + b'_m I^m\Big\}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} \\ & +\frac{1}{4}\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\} + \Big\{b'_i I_j I_k + b'_j I_i I_k + b'_k I_i I_j\Big\} \\ & -\frac{1}{4}\Big\{I_m J^m + J_m I^m\Big\}\Big\{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\Big\} \\ & -\frac{b_m I^m}{2}\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\} - \frac{||\mathbf{I}||^2}{4}\Big\{b'_i h_{jk} + b'_j h_{ki} + b'_k h_{ij}\Big\} \\ & +\Big\{b_i(J_j I_k + I_j J_k) + b_j(J_i I_k + I_i J_k) + b_k(J_i I_j + I_i J_j)\Big\}, \qquad (3.6) \end{aligned}$$

where $b'_i := b_{i|s} y^s$.

In [22], Tayebi-Najafi characterized 3-dimensional non-Riemannian almost regular Landsberg (α, β)-metrics as follows.

Theorem C. ([22]) Every 3-dimensional non-Riemannian almost regular Landsberg (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, belongs to the one of the following three classes of Finsler metrics:

(1) F is a Berwald metric. In this case, F is a Randers metric or a Kropina metric;
(2) ϕ is given by the ODE

$$\phi^{4-4c}(\phi - s\phi')^{4-c}\Big[\phi - s\phi' + (b^2 - s^2)\phi''\Big]^{-c} = e^{k_0}, \qquad (3.7)$$

where c is a nonzero real constant, k_0 is a real number and $b := ||\beta||_\alpha$. In this case, F is a Berwald metric (regular case) or an almost regular unicorn.

In [1], Amini study the weakly Landsberg 3-dimensional Finsler metrics and prove the following.

Theorem C. ([1]) Let (M, F) be a non-Riemannian 3-dimensional weakly Landsberg manifold. Then F is a Landsberg metric if and only if the quantity $b_i = b_i(x, y)$ is horizontally constant along Finsler geodesics.

As a generalization of C-reducible metrics, Matsumoto-Shimada introduced the notion of L-reducible (P-reducible) metrics [14]. This class of Finsler metrics has some interesting physical and mathematical means and contains Randers metrics as a special case [24]. Here, we consider 3-dimensional L-reducible Finsler metrics and prove the following.

Lemma 3.3. *Let (M, F) be a 3-dimensional Finsler manifold. Suppose that F is L-reducible. Then F satisfies following*

$$2b_m J^m I_k - 2b_m I^m J_k - 2J_m I^m b_k - ||\mathbf{I}||^2 b'_k = 0. \tag{3.8}$$

Proof. Let F be a L-reducible metric

$$L_{ijk} = \frac{1}{4}\Big\{h_{ij}J_k + h_{jk}J_i + h_{ki}J_j\Big\}. \tag{3.9}$$

Then (3.6) reduces to following

$$\begin{aligned}
&||\mathbf{I}||^2\Big\{b'_i h_{jk} + b'_j h_{ki} + b'_k h_{ij}\Big\} - 4\Big\{b_i(J_j I_k + I_j J_k) + b_j(J_i I_k + I_i J_k)\\
&+b_k(J_i I_j + I_i J_j)\Big\} + 2\Big(b_m J^m + b'_m I^m\Big)\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\}\\
&+2I_m J^m\Big\{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\Big\} - 4\Big\{b'_i I_j I_k + b'_j I_i I_k + b'_k I_i I_j\Big\}\\
&+2b_m I^m\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\} = 0.
\end{aligned} \tag{3.10}$$

Multiplying (3.10) with I^i yields

$$\begin{aligned}
&||\mathbf{I}||^2\Big\{b'_p I^p h_{jk} + b'_j I_k + b'_k I_j\Big\} - 4\Big\{b_p I^p(J_j I_k + I_j J_k) + b_j(J_p I^p I_k + ||\mathbf{I}||^2 J_k)\\
&+b_k(J_p I^p I_j + ||\mathbf{I}||^2 J_j)\Big\} + 2\Big(b_m J^m + b'_m I^m\Big)\Big\{||\mathbf{I}||^2 h_{jk} + 2I_j I_k\Big\}\\
&+2I_m J^m\Big\{b_p I^p h_{jk} + b_j I_k + b_k I_j\Big\} - 4\Big\{b'_p I^p I_j I_k + ||\mathbf{I}||^2 b'_j I_k + ||\mathbf{I}||^2 b'_k I_j\Big\}\\
&+2b_m I^m\Big\{J_p I^p h_{jk} + J_j I_k + J_k I_j\Big\} = 0.
\end{aligned} \tag{3.11}$$

Contracting (3.11) with I^j implies (3.8). □

Remark 3.4. *The horizontal derivation of Moór frame are giving by following*

$$\ell_{i|j} = 0, \quad m_{i|j} = h_j n_i, \quad n_{i|j} = -h_j m_i,$$

where h_i are called the h-connection vectors. Thus

$$m'_i := m_{i|j}y^j = h_0 n_i, \quad n'_i := n_{i|j}y^j = -h_0 m_i,$$

where $h_0 := h_i y^i$.

Now, by taking a horizontal derivation of (3.3), we get

$$FJ_k = (\mathcal{H}' + \mathcal{I}')m_k + (\mathcal{H} + \mathcal{I})h_0 n_k. \tag{3.12}$$

Let us put

$$B_1 := \frac{1}{3F||\mathbf{I}||^2}\Big(\mathcal{H} - 3\mathcal{I}\Big),$$
$$B_2 := \frac{-4}{3F||\mathbf{I}||^2}\mathcal{J},$$

Then, (3.5) can be written as follows

$$b_i = B_1 m_i + B_2 n_i.$$

Let us put

$$P_1 := \frac{1}{3F||\mathbf{I}||^4}\Big[(\mathcal{H}' - 3\mathcal{I}')||\mathbf{I}||^2 + 4\mathcal{J}h_0||\mathbf{I}||^2 - 2I_m J^m(\mathcal{H} - 3\mathcal{I})\Big],$$
$$P_2 := \frac{1}{3F||\mathbf{I}||^4}\Big[(\mathcal{H} - 3\mathcal{I})||\mathbf{I}||^2 h_0 - 4||\mathbf{I}||^2\mathcal{J}' + 8I_m J^m \mathcal{J}\Big].$$

Then

$$b_i' = P_1 m_i + P_2 n_i.$$

By (3.12), we get

$$b_s J^s = \frac{1}{F}\Big[(\mathcal{H}' + \mathcal{I}')B_1 + (\mathcal{H} + \mathcal{I})h_0 B_2\Big],$$
$$b_s I^s = \frac{1}{F}B_1\Big(\mathcal{H} + \mathcal{I}\Big),$$
$$J_m I^m = \frac{1}{F^2}\Big(\mathcal{H}' + \mathcal{I}'\Big)\Big(\mathcal{H} + \mathcal{I}\Big),$$
$$I_m I^m = \frac{1}{F^2}\Big(\mathcal{H} + \mathcal{I}\Big)^2,$$

Then

$$P_1 = \frac{1}{3F^3||\mathbf{I}||^4}\Bigg[(\mathcal{H}' - 3\mathcal{I}')(\mathcal{H} + \mathcal{I})^2 + 4\mathcal{J}(\mathcal{H} + \mathcal{I})^2 h_0 - 2(\mathcal{H}' + \mathcal{I}')(\mathcal{H} + \mathcal{I})(\mathcal{H} - 3\mathcal{I})\Bigg],$$

$$P_2 = \frac{1}{3F^3||\mathbf{I}||^4}\Bigg[(\mathcal{H} - 3\mathcal{I})(\mathcal{H} + \mathcal{I})^2 h_0 - 4(\mathcal{H} + \mathcal{I})^2\mathcal{J}' + 8(\mathcal{H}' + \mathcal{I}')(\mathcal{H} + \mathcal{I})\mathcal{J}\Bigg].$$

By putting the above relations in (3.8), we get

$$\frac{2}{F^2}\Big[(\mathcal{H}'+\mathcal{I}')B_1+(\mathcal{H}+\mathcal{I})h_0B_2\Big](\mathcal{H}+\mathcal{I})m_k$$
$$-\frac{1}{F^2}(\mathcal{H}+\mathcal{I})^2(P_1m_k+P_2n_k)$$
$$-\frac{2}{F^2}B_1(\mathcal{H}+\mathcal{I})\Big[(\mathcal{H}'+\mathcal{I}')m_k+(\mathcal{H}+\mathcal{I})h_0n_k\Big]$$
$$-\frac{2}{F^2}(\mathcal{H}'+\mathcal{I}')(\mathcal{H}+\mathcal{I})(B_1m_k+B_2n_k)=0. \tag{3.13}$$

Since $(\mathcal{H}+\mathcal{I})\neq 0$, then by contracting (3.13) with m^k and n^k, we get the following

$$2(\mathcal{H}'+\mathcal{I}')B_1-2(\mathcal{H}+\mathcal{I})h_0B_2+(\mathcal{H}+\mathcal{I})P_1=0, \tag{3.14}$$

and

$$2(\mathcal{H}+\mathcal{I})h_0B_1+2(\mathcal{H}'+\mathcal{I}')B_2+(\mathcal{H}+\mathcal{I})P_2=0. \tag{3.15}$$

Then we conclude the following.

Proposition 3.5. *Let (M,F) be a 3-dimensional L-reducible Finsler manifold. Then F satisfies* (3.8) *if and only if it satisfies* (3.14) *and* (3.15)*.*

Here, we prove an extension of Theorem 1.1. More precisely, we prove the following.

Theorem 3.6. *Let (M,F) be a complete 3-dimensional L-reducible manifold with bounded main scalars. Suppose that F has constant relatively isotropic mean Landsberg curvature*

$$\mathbf{J}=cF\mathbf{I},$$

where c is a non-zero real constant. Then F is a Randers metric.

Proof. Now, let F has constant relatively isotropic mean Landsberg curvature $\mathbf{J}=cF\mathbf{I}$, where c is a real number. Then (3.8) reduces to following

$$b'_k+2cFb_k=0. \tag{3.16}$$

On Finslerian geodesics, (3.16) is written as follows

$$\mathbf{b}'+2c\mathbf{b}=0. \tag{3.17}$$

The solution of (3.17) is

$$\mathbf{b}(t)=e^{-2ct}\mathbf{b}(0). \tag{3.18}$$

Since the main scalars are bounded then $||\mathbf{b}||<\infty$. Thus letting $t\to\infty$ implies that $\mathbf{b}=0$. In this case, (3.4) reduces to following

$$C_{ijk}=\Big\{a_ih_{jk}+a_jh_{ki}+a_kh_{ij}\Big\}. \tag{3.19}$$

Contracting (3.19) with g^{ij} yields

$$a_k = \frac{1}{n+1} I_k. \tag{3.20}$$

Putting (3.20) in (3.19) implies that F is C-reducible. By Matsumoto-Hōjō's Lemma, F is a Randers metric. □

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