Journal of Finsler Geometry and its Applications
Vol. 2, No. 1 (2021), pp 40-50
DOI: 10.22098/jfga.2021.1262

A class of Finsler metrics with bounded non-Riemannian curvatures

S. Hedayatian[a*], N. Izadian[b] and M. Yar Ahmadi[c]

[a,b,c]Department of Mathematics, Faculty of Mathematical Sciences and Computer, Shahid Chamran University of Ahvaz, Ahvaz, Iran.

E-mail: hedayatian@scu.ac.ir
E-mail: n-izadian@stu.scu.ac.ir
E-mail: m.yarahmadi@scu.ac.ir

Abstract. In this paper, we find a necessary and sufficient condition for a class of Finsler metrics defined by a Riemannian metric and a 1-form on a manifold which has bounded mean Cartan torsion. Moreover, we obtain a necessary and sufficient condition under which the above mentioned class of Finsler metrics has bounded mean Landsberg curvature. Next, we investigate these metrics with bounded mean Cartan torsion and mean Landsberg curvature. Furthermore, we give explicit examples of this type of metrics.

Keywords: (α, β)-metric, Cartan torsion, mean Cartan torsion, Landsberg curvature, mean Landsberg curvature.

1. Introduction

One of the challenging problems in Finsler geometry is whether or not each Finsler manifold can be isometrically immersed into a Minkowski space. The answer is affirmative for Riemannian manifolds. In [14], J. Nash proved that any n-dimensional Riemannian manifold can be isometrically imbedded into a higher dimensional Euclidean space. In [11], R.S. Ingarden proved that every n-dimensional Finsler manifold can be locally isometrically imbedded into a $2n$-dimensional "Weak" Minkowski space, i.e., the indicatrix is not necessarily

---

*Corresponding Author
AMS 2020 Mathematics Subject Classification: 53B40, 53C30

strongly convex. Burago-Ivanov showed that any compact C^r manifold ($r \geq 3$) with a C^2 Finsler metric admits a C^r imbedding into a finite-dimensional Banach spaces [4]. However for general Finsler manifolds, the answer is negative as Shen in [17] proved that a Finsler manifold with unbounded Cartan torsion cannot isometrically imbedded to any Minkowski space. The Cartan torsion is one of the most fundamental non-Riemannian quantities in Finsler geometry. It was first introduced by P. Finsler in [10] and emphasized by E. Cartan in [5]. The Cartan torsion $\mathbf{C}$ characterized Riemannian manifolds among the class of Finsler manifolds, i.e., a Finsler metric is Riemannian if and only if it has vanishing Cartan torsion. Thus the norm of Cartan torsion plays an important role for studying of immersion theory in Finsler geometry.
There are many Finsler metrics with bounded Cartan torsion. In [18], Z. Shen proved that the Cartan torsion of the Randers metrics $F = \alpha + \beta$ is uniformly bounded by $3/\sqrt{2}$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on a manifold M. Mo extended his result to a more general Randers metrics, i.e., $F = (\alpha + \beta)^s/\alpha^{s-1}$, where $s \in [1, 2]$ is a constant [13]. Then, Tayebi-Sadeghi found a relation between the norm of Cartan and mean Cartan torsions of Finsler metrics defined by a Riemannian metric and a 1-form on a manifold, so called (α, β)-metrics [20]. They find a subclass of (α, β)-metrics which have bounded Cartan torsion. Moreover the (generalized) Kropina metrics $F = \alpha^2/\beta$ have bounded Cartan torsion. It is remarkable that the class of (α, β)-metrics forms a rich class of computable Finsler metrics. Many (α, β)-metrics with special curvature properties have been found and discussed [1], [7] and [23].
The mean Cartan torsion $\mathbf{I}$ is another important quantity in Finsler geometry that characterizes Riemannian metrics. Z. Shen proved that every complete Finsler surface with non-positive Gaussian curvature, constant $\mathbf{S}$-curvature and bounded mean Cartan torsion is either Riemannian or locally Minkowskian. In the corollary 3.1, we characterize the (α, β)-metrics with isotropic $\mathbf{S}$-curvature and bounded mean Cartan torsion. In the following two theorems, we find some conditions under which an (α, β)- metrics have bounded mean Cartan torsion and bounded mean Landsberg torsion.

Theorem 1.1. *Let $F = \alpha\phi(s)$, where $s = \beta/\alpha$, be an (α, β)-metric on an n-dimensional manifold M. Then F has bounded mean Cartan torsion if and only if the function*

$$B(s) := \frac{(b^2 - s^2)\phi}{\phi - s\phi' + (b^2 - s^2)\phi''}\Upsilon_n(s), \tag{1.1}$$

is bounded for all $|s| < b$, where

$$\Upsilon_n(s) := \Big((n+1)\frac{\phi'}{\phi} - (n-2)\frac{s\phi''}{\phi - s\phi'} + \frac{(b^2 - s^2)\phi''' - 3s\phi''}{\phi - s\phi' + (b^2 - s^2)\phi''}\Big)^2.$$

Recall that the mean Landsberg curvature $\mathbf{J}$ is the rate of change of the mean Cartan torsion $\mathbf{I}$ along geodesics for any $y \in T_xM_0$. It has been shown that on a weakly Landsberg manifold, the volume function $\mathrm{Vol}(x)$ is constant [3]. In [19], Z. Shen shows that if $\mathbf{J} = 0$, then all the slit tangent spaces T_xM_0 are minimal in TM_0.

Theorem 1.2. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an n-dimensional manifold M, Suppose that α and β satisfy*

$$b_{i|j} = c(b^2 a_{ij} - b_i b_j), \tag{1.2}$$

where $c := c(x)$ is a scalar function on M. Then F has bounded mean Landsberg torsion if and only if the function

$$H(s) := \frac{(b^2 - s^2)\phi}{\phi - s\phi' + (b^2 - s^2)\phi''} \left(\frac{c\Psi_1}{\Delta}\right)^2, \tag{1.3}$$

is a bounded function for $|s| < b$ where Δ and Ψ_1 are defined by

$$\Delta := 1 + sQ + (b^2 - s^2)Q',$$
$$\Psi_1 := \sqrt{b^2 - s^2}\,\Delta^{\frac{1}{2}} \left[\frac{\sqrt{b^2 - s^2}}{\Delta^{\frac{3}{2}}}\right]',$$

where

$$Q := \frac{\phi'}{\phi - s\phi'}.$$

Finally in Section 5, we give some examples of (α, β)-metrics with bounded Cartan torsion and bounded mean Cartan torsion.

2. Preliminaries

A Finsler metric on a manifold M is a non-negative function F on TM having the following properties: (i) F is C^∞ on $TM \setminus \{0\}$; (ii) $F(\lambda y) = \lambda F(y)$, $\forall \lambda > 0$, $y \in TM$; and (iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\Big[F^2(y + su + tv)\Big]\Big|_{s,t=0}, \qquad u, v \in T_xM. \tag{2.1}$$

At each point $x \in M$, $F_x := F|_{T_xM}$ is an Euclidean norm if and only if $\mathbf{g}_y$ is independent of $y \in T_xM \setminus \{0\}$.
To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]\Big|_{t=0}, \qquad u, v, w \in T_xM. \tag{2.2}$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM\setminus\{0\}}$ is called the *Cartan torsion*. Obviously, F is Riemannian metric if and only if $\mathbf{C}_y = 0$. The norm of Cartan torsion $\mathbf{C}$ at point $x \in M$ is defined by

$$\|\mathbf{C}\|_x := \sup_{y,v\in T_xM\setminus\{0\}} \frac{F(x,y)|\mathbf{C}_y(v,v,v)|}{[\mathbf{g}_y(v,v)]^{\frac{3}{2}}}.$$

and the norm of Cartan torsion on M is defined by

$$\|\mathbf{C}\| := \sup_{x\in M} \|\mathbf{C}\|_x.$$

For more details, see [15].

Taking a trace of Cartan torsion yields the mean Cartan torsion $\mathbf{I}_y$. It is defined by

$$\mathbf{I}_y(u) := g^{ij}(y)\ \mathbf{C}_y\Big(u, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big).$$

The norm of mean Cartan torsion $\mathbf{I}$ at point $x \in M$ is defined by

$$\|\mathbf{I}\|_x := \sup_{y,v\in T_xM\setminus\{0\}} \frac{F(x,y)|\mathbf{I}_y(v)|}{[\mathbf{g}_y(v,v)]^{\frac{1}{2}}}, \tag{2.3}$$

and the norm of mean Cartan torsion on M is defined by $\|\mathbf{I}\| := \sup_{x\in M} \|\mathbf{I}\|_x$.

For an n-dimensional Finsler manifold (M, F), there is a special vector field $\mathbf{G}$ which is induced by F on $TM_0 := TM\setminus\{0\}$. In a standard coordinates (x^i, y^i) for TM_0, it is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where

$$G^i := \frac{g^{il}}{4}\Big\{\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l}\Big\}.$$

The homogeneous scalar functions G^i are called the geodesic coefficients of F. The vector field $\mathbf{G}$ is called the associated spray to (M, F).

For $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{B}_y(u,v,w) := B^i{}_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}|_x$ where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

The quantity $\mathbf{B}$ is called the Berwald curvature of the Finsler metric F.

For $y \in T_xM$, define the Landsberg curvature $\mathbf{L}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{L}_y(u,v,w) := -\frac{1}{2}\mathbf{g}_y\big(\mathbf{B}_y(u,v,w), y\big).$$

F is called a Landsberg metric if $\mathbf{L}_y = 0$. By definition, every Berwald metric is a Landsberg metric. We remark that the horizontal covariant derivatives of the Cartan torsion $\mathbf{C}$ along geodesics give rise to the Landsberg curvature $\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^i v^j w^k$, where

$$L_{ijk} := C_{ijk|s} y^s.$$

The horizontal covariant derivatives of the mean Cartan torsion $\mathbf{I}$ along geodesics give rise to the mean Landsberg curvature $\mathbf{J}_y : T_xM \to \mathbb{R}$ which are defined by $\mathbf{J}_y(u) := J_i(y)u^i$, where

$$J_i := I_{i|s} y^s.$$

The family $\mathbf{J} := \{\mathbf{J}_y\}_{y \in TM_0}$ is called the mean Landsberg curvature.
The norm of mean Landsberg $\mathbf{J}$ at point $x \in M$ is defined by

$$\|\mathbf{J}\|_x := \sup_{y,v \in T_xM \setminus \{0\}} \frac{F(x, y)|\mathbf{J}_y(v)|}{[\mathbf{g}_y(v, v)]^{\frac{1}{2}}}. \tag{2.4}$$

and the norm of mean Landsberg torsion on M is defined by

$$\|\mathbf{J}\| := \sup_{x \in M} \|\mathbf{J}\|_x.$$

The Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1 \wedge \cdots \wedge dx^n$ related to F is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\Big\{(y^i) \in \mathbb{R}^n \;\Big|\; F\Big(y^i \frac{\partial}{\partial x^i}|_x\Big) < 1\Big\}},$$

where $\mathbb{B}^n(1)$ denotes the unit ball in $\mathbb{R}^n$.
The distortion $\tau = \tau(x, y)$ on TM associated with the Busemann-Hausdorff volume form on M, i.e., $dV_{BH} = \sigma(x)dx^1 \wedge dx^2 ... \wedge dx^n$, is defined by following

$$\tau(x, y) = \ln \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)}.$$

Then the S-curvature is defined by

$$\mathbf{S}(x, y) = \frac{d}{dt}\Big[\tau\big(c(t), \dot{c}(t)\big)\Big]_{t=0},$$

where $c = c(t)$ is the geodesic with $c(0) = x$ and $\dot{c}(0) = y$. In a local coordinates, the S-curvature is given by

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial(\ln \sigma)}{\partial x^m}.$$

A Finsler metric F on an n-dimensional manifold M is said to be of isotropic S-curvature if

$$\mathbf{S} = (n + 1)\sigma F,$$

where $\sigma = \sigma(x)$ is a scalar function on M.

Let $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ be a Riemannian metric and $\beta = b_i(x)y^i$ be a 1-form on an n-dimensional manifold M. Using α and β one can define a function on $\mathcal{TM}$ as follows

$$F = \alpha\phi(s), \qquad s := \frac{\beta}{\alpha}.$$

where $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$. The norm $\|\beta_x\|_\alpha$ of β with respect to α is defined by

$$\|\beta_x\|_\alpha := \sqrt{a^{ij}(x)b_i(x)b_j(x)}.$$

In order to define F, β must satisfy the condition $\|\beta_x\|_\alpha < b_0$ for all $x \in M$. For an (α, β)-metric, let us define $b_{i|j}$ by $b_{i|j}\theta^j := db_i - b_j\theta_i^j$, where $\theta^i := dx^i$ and $\theta_i^j := \Gamma^j_{ik}dx^k$ denote the Levi-Civita connection form of α. Let

$$\begin{gathered}
r_{ij} := \tfrac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \tfrac{1}{2}(b_{i|j} - b_{j|i}), \\
r_{i0} := r_{ij}y^j, \quad r_{00} := r_{ij}y^i y^j, \quad r_j := b^i r_{ij}, \\
s_{i0} := s_{ij}y^j, \quad s_j := b^i s_{ij}, \quad r_0 := r_j y^j, \quad s_0 := s_j y^j.
\end{gathered}$$

By a direct computation, one gets the following formula for matrix $(g_{ij} := \frac{1}{2}\frac{\partial F^2}{\partial y^i \partial y^j})$ and inverse matrix $(g^{ij}) := (g_{ij})^{-1}$[9].

$$g_{ij} = \rho\delta_{ij} + \rho_0 b_i b_j + \rho_1(b_i\frac{y_j}{\alpha} + b_j\frac{y_i}{\alpha}) - s\rho_1\frac{y_i}{\alpha}\frac{y_j}{\alpha}, \tag{2.5}$$
$$g^{ij} = \rho^{-1}\Big(\delta^{ij} - \frac{\rho_1 + s\rho_0}{s\rho + b^2(\rho_1 + s\rho_0)}b^i b^j + \frac{s\rho_1}{\rho + (b^2 - s^2)\lambda\rho_1}Y^i Y^j\Big), \tag{2.6}$$

where $y_i := a_{ij}y^j$ and

$$\begin{aligned}
\rho &:= \phi(\phi - s\phi'), \\
\rho_0 &:= \phi'^2 + \phi\phi'', \\
\rho_1 &:= (\phi - s\phi')\phi' - s\phi\phi'', \\
\lambda &:= -\frac{\rho + s(s\rho_0 + \rho_1)}{s\rho + b^2(s\rho_0 + \rho_1)}, \\
Y^i &:= \frac{1}{\alpha}y^i + \lambda b^i.
\end{aligned}$$

The mean Cartan curvature and mean Landsberg curvature of (α,β)- metrics are given by following

$$I_i = \frac{1}{2\alpha}\left[(n+1)\frac{\phi_s}{\phi} - (n-2)\frac{s\phi_{ss}}{\phi - s\phi_s} + \frac{(b^2-s^2)\phi_{sss} - 3s\phi_{ss}}{\phi - s\phi_s + (b^2-s^2)\phi_{ss}}\right]h_i, \tag{2.7}$$

$$\begin{aligned} J_i = -\frac{1}{2\alpha^4\Delta}\Bigg(&\frac{2\alpha^3}{b^2-s^2}\Big[\frac{\Phi}{\Delta} + (n+1)(Q - sQ')\Big](r_0+s_0)h_i \\ &+\frac{\alpha^2}{b^2-s^2}\Big[\Psi_1 + s\frac{\Phi}{\Delta}\Big](r_{00} - 2\alpha Q s_0)h_i + \alpha\Big[-\alpha^2 Q' s_0 h_i + \alpha Q(\alpha^2 s_i - y_i s_0) \\ &+\alpha^2\Delta s_{i0} + \alpha^2(r_{i0} - 2\alpha Q s_0) - (r_{00} - 2\alpha Q s_0)y_i\Big]\frac{\Phi}{\Delta}\Bigg), \end{aligned} \tag{2.8}$$

where

$$h_i := b_i - \alpha^{-1} s y_i,$$

$$Q := \frac{\phi'}{\phi - s\phi'},$$

$$\Delta := 1 + sQ + (b^2 - s^2)Q', \tag{2.9}$$

$$\Psi_1 := \sqrt{b^2-s^2}\,\Delta^{\frac{1}{2}}\left[\frac{\sqrt{b^2-s^2}}{\Delta^{\frac{3}{2}}}\right]', \tag{2.10}$$

$$\Phi := -(Q - sQ')(n\Delta + 1 + sQ) - (b^2-s^2)(1+sQ)Q''. \tag{2.11}$$

For more details, see [8] and [12].

In [6], Cheng-Shen characterize (α,β)-metrics with isotropic S-curvature.

Theorem 2.1. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Riemannian (α,β)-metric on a manifold and $b := \|\beta_x\|_\alpha$. Suppose that F is not a Finsler metric of Randers type. Then F is of isotropic S-curvature, $\boldsymbol{S} = (n+1)cF$, if and only if one of the following holds*

a): *β satisfies*

$$r_j + s_j = 0, \tag{2.12}$$

and $\phi = \phi(s)$ satisfies

$$\Phi = 0. \tag{2.13}$$

In this case, $\mathbf{S} = 0$.

b): *β satisfies*

$$r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j), \quad s_j = 0, \tag{2.14}$$

where $\varepsilon = \varepsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2-s^2}, \tag{2.15}$$

where k is a constant. In this case, $\boldsymbol{S} = (n+1)cF$ with $c = k\varepsilon$.

c): *β satisfies*

$$r_{ij} = 0, \qquad s_j = 0. \tag{2.16}$$

In this case, $\boldsymbol{S} = 0$, regardless of choices of a particular ϕ.

3. Proof of Theorem 1.1

Proof. By the definition of the mean Cartan torsiongiven in (2.3), we have

$$\|\mathbf{I}\| = \sup_{x\in M,\ y\in T_xM} F\sqrt{g^{ij}I_iI_j}. \tag{3.1}$$

It follows from (2.6) and (2.7) that

$$g^{ij}I_iI_j \quad = \quad \frac{(b^2-s^2)\Gamma^2}{4\alpha^2\phi(\phi - s\phi' + (b^2-s^2)\phi'')}, \tag{3.2}$$

where

$$\Gamma(s) := (n+1)\frac{\phi'}{\phi} - (n-2)\frac{s\phi''}{\phi - s\phi'} + \frac{(b^2-s^2)\phi''' - 3s\phi''}{\phi - s\phi' + (b^2-s^2)\phi''}.$$

Substituting (3.2) in (3.1) yields

$$\begin{aligned}
\|\mathbf{I}\| &= \sup_{|s|<b} F\sqrt{\frac{(b^2-s^2)\Gamma^2}{4a^2\phi(\phi - s\phi' + (b^2-s^2)\phi'')}}\\
&= \sup_{|s|<b} \frac{F}{2a\phi}\sqrt{\frac{(b^2-s^2)\phi}{\phi - s\phi' + (b^2-s^2)\phi''}}\ \Gamma\\
&= \frac{1}{2}\sup_{|s|<b}\sqrt{\frac{(b^2-s^2)\phi}{\phi - s\phi' + (b^2-s^2)\phi''}}\ \Gamma
\end{aligned}$$

Thus, we conclude that $\|\mathbf{I}\|$ is bounded if and only if the function defined in (1.1) is bounded for $|s| < b$. □

Corollary 3.1. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Randers type (α,β)-metric on a manifold M and $b := \|\beta_x\|_\alpha$. Suppose that F has isotropic S-curvature, $\boldsymbol{S} = (n+1)cF$, where $c := c(x)$ is a nonzero function on M. Then F has bounded mean Cartan torsion if and only if the function*

$$\chi(s) := \frac{\phi^3(\phi - s\phi' + (b^2-s^2)\phi'')}{(b^2-s^2)(\phi - s\phi')^2}, \tag{3.3}$$

is bounded for $|s| < b$.

Proof. It is easy to see that

$$(b^2-s^2)\Big(\frac{\Phi}{\Delta^2}\Big)^2 = \frac{(\phi - s\phi')^2}{\phi(\phi - s\phi' + (b^2-s^2)\phi'')}B(s), \tag{3.4}$$

where Φ, Δ and $B(s)$ are defined in (1.1), (2.9) and (2.11), respectively. Since F has isotropic S-curvature with a non-zero function $c(x)$ on M, by part b): of Theorem 2.1, from (2.15) we conclude

$$\frac{(b^2-s^2)\Phi}{\phi\Delta^2} = -2(n+1)k, \tag{3.5}$$

where k is a constant. It follows from (3.4) and (3.5) that

$$B(s) = \frac{4(n+1)^2k^2\phi^3(\phi - s\phi' + (b^2-s^2)\phi'')}{(b^2-s^2)(\phi - s\phi')^2}. \tag{3.6}$$

By Theorem 1.1, we conclude that F has mean bounded Cartan torsion if and only if the function (3.3) is bounded. □

4. Proof of Theorem 1.2

By the definition of mean Landsberg torsion given in (2.4) we have

$$\|\mathbf{J}\| = \sup_{x\in M, y\in T_xM} F\sqrt{g^{ij}J_iJ_j}. \tag{4.1}$$

Substituting (1.2) into (2.8) yields

$$J_i = \frac{c\Psi_1}{2\Delta}h_i. \tag{4.2}$$

It follows from (2.6) and (4.2) that

$$g^{ij}J_iJ_j = \frac{(b^2-s^2)}{4\alpha^2\phi\big(\phi - s\phi' + (b^2-s^2)\phi''\big)}\left(\frac{c\Psi_1}{\Delta}\right)^2. \tag{4.3}$$

Substituting (4.3) in (4.1) yields

$$\|\mathbf{J}\| = \frac{1}{2}\sup_{|s|<b}\sqrt{\frac{(b^2-s^2)\phi}{\phi - s\phi' + (b^2-s^2)\phi''}}\left(\frac{c\Psi_1}{\Delta}\right).$$

Thus, we conclude that $\|\mathbf{J}\|$ is bounded if and only if the function defined in (1.3) is bounded for $|s| < b$.

5. Finsler metrics with bounded Cartan torsions

In this section we give explicit examples of Finsler metrics with bounded Cartan torsion.

Example 5.1. *In* [16], *Z. Shen finds a Family of Landsberg* (α,β)*- metrics which are not Berwaldian. They are in the form*

$$\phi(s) := \exp\Big[\int_0^s \frac{kt + q\sqrt{b^2-t^2}}{1 + t(kt + q\sqrt{b^2-t^2})}dt\Big], \tag{5.1}$$

where k and q are constants. Recently, H. Sadeghi proved that an (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, on a manifold M has bounded Cartan torsion if and only if the function

$$A(s) := \frac{3s(\phi'^2 + \phi\phi'') - 3\phi\phi' - (b^2 - s^2)(\phi\phi''' + 3\phi'\phi'')}{2\phi^{\frac{1}{2}}(b^2 - s^2)^{-\frac{1}{2}}(\phi - s\phi' + (b^2 - s^2)\phi'')^{\frac{3}{2}}}, \tag{5.2}$$

is a bounded function for $|s| < b$ where $b := \|\beta_x\|_\alpha$. Substituting (5.1) in (5.2) and (1.1) yields

$$A(s) = \frac{2b^2 q}{\sqrt{1 + kb^2}}, \quad B(s) = \frac{n^2 b^4 q^2}{1 + kb^2}.$$

Thus by theorem 1.1, these metrics have bounded Cartan torsion and bounded mean Cartan torsion.

Example 5.2. *In* [21] *G. Yang study a family of (α, β)- metrics in form*

$$\phi(s) = s^m (b^2 - s^2)^{\frac{1-m}{2}}, \tag{5.3}$$

where m is a constant. This type of metrics have special curvature properties [22]. *Substituting (5.3) in (5.2) and (1.1) yields*

$$A(s) = \frac{2m - 1}{\sqrt{m(m-1)}}, \quad B(s) = 4\frac{(nm-1)^2}{m(m-1)}. \tag{5.4}$$

Since $B(s)$ is bounded (constant), thus by Theorem 1.1 these metrics have bounded Cartan torsion and bounded mean Cartan torsion.

Acknowledgment: We are grateful to the Research Council of Shahid Chamran University of Ahvaz (Ahvaz-Iran) for financial support (Grant Number : SCU.MM 99.650).

References

1. S. Bácsó, X. Cheng, and Z. Shen, *Curvature properties of (α, β)-metrics*, Adv. Stud. Pure. Math. Soc. Japan. **48**(2007), 73-110.
2. D. Bao and S. S. Chern, *A note on the Gauss-Bonnet theorem for Finsler spaces*, Ann. Math. **143**(1996), 233252.
3. D. Bao and Z. Shen, *Finsler metrics of constant positive curvature on the Lie group S^3*, J. London. Math. Soc. **66**(2002), 453-467.
4. D. Burago and S. Ivanov, *Isometric embedding of Finsler manifolds*, Algebra i Analiz, **5**(1993), 179-192.
5. E. Cartan, *Les spaces de Finsler.* Actualites Scientifiques et industrilles, no. 79, Hermann, Paris, 1934.
6. X. Cheng and Z. Shen, *A class of Finsler metrics with isotropic S-curvature*, Israel J. Math. **169**(2009), 317-340.
7. X. Cheng, Z. Shen and Y. Tian, *A class of Einstein (α, β)-metrics*, Israel J. Math, **192**(2012), 221-249.
8. X. Cheng, H. Wang and M. Wang, *(α, β)-metrics with relatively isotropic mean Landsberg curvature*, Publ. Math. Debrecen, **72**(2008), 475-485.
9. S.S. Chern and Z. Shen, *Riemann-Finsler Geometry*, World Scientific, Singapore, 2005.

10. P. Finsler, *Über Kurven und Flächen in allgemeinen Räumen.* Verlag Birkhäuser, Basel, 1951.
11. R. S. Ingarden, *Über die Einbetting eines Finslerschen Rammes in einan Minkowskischen Raum*, Bull. Acad. Polon. Sci. III, **2**(1954), 305-308.
12. B. Li and Z. Shen, *On a class of weak Landsberg metrics*, Science in China Series A. **50**(2007),75-85.
13. X. Mo and L. Zhou, *A class of Finsler metrics with bounded Cartan torsion*, Canad. Math. Bull. **53**(2010), 122-132.
14. J. Nash, *The imbedding problem for Riemannian manifolds*, Ann. Math. **73**(1957), 20-37.
15. T. Rajabi, *On the norm of Cartan torsion of two classes of* (α, β)*-metrics*, Journal of Finsler Geometry and its Applications, **1**(1) (2020), 66-72.
16. Z. Shen, *On a class of Landsberg metrics in Finsler geometry*, Canadian. J. Math. **61**(2009), 1357-1374.
17. Z. Shen, *On Finsler geometry of submanifolds*, Math. Ann. **311**(1998), 549-576.
18. Z. Shen, *On R-quadratic Finsler spaces*, Publ. Math. Debrecen, **58**(2001), 263-274.
19. Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, 2001.
20. A. Tayebi and H. Sadeghi, *On Cartan torsion of Finsler metrics*, Publ. Math. Debrecen, **82**(2013), 461-471.
21. G. Yang, *On a class of singular Douglas and projectively flat Finsler metrics*, Differ. Geom. Appl. **32**(2014), 113-129.
22. G. Yang, *On a class of singular projectively flat Finsler metrics with constant flag curvature*, Int. J. Math. **24**(2013), 16 pages, 1350087.
23. L. Zhou, *A local classification of a class of* (α, β)*-metrics with constant flag curvature.* Differ. Geom. Appl. **28**(2010), 170-193

Received: 27.01.2021
Accepted: 01.06.2021