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The algebraic Ricci solitons of Lie groups $\mathbb{H}^2 \times \mathbb{R}$ and Sol_3

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Abstract. In this article, we study the algebraic Ricci solitons of three-dimensional Lie group $\mathbb{H}^2 \times \mathbb{R}$, endowed with a left-invariant Riemannian metric. Also, we examine the existence of sol-solitons on the three-dimensional Lie group Sol_3, endowed with a left-invariant Riemannian metric.

Keywords: Algebraic Ricci soliton, Lie group, Riemannian metric.

1. Introduction

Suppose G be a three-dimensional Lie group. If ∇ be the Levi-Civita connection, the corresponding curvature tensor is given by the identity

$$\mathcal{R}(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Now, the Ricci tensor ρ and the Ricci operator Ric are given by

$$\rho(X, Y) = \operatorname{tr}\Big\{Z \mapsto \mathcal{R}(Z, X)Y\Big\} \quad \text{and} \quad \rho(X, Y) = g\Big(Ric(X), Y\Big),$$

respectively.

The problem of constructing a metric with a distinguish feature is a important subject in differential geometry. For example, Hamilton [5] and Friedan [4]

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introduced Ricci flow, which we recall here. Suppose $g(t)$ be a one-parameter family of metrics into a Riemannian manifold M, then $g(t)$ is a solution to the Ricci flow if it satisfies the equation

$$\frac{\partial}{\partial t}g(t) = -2\rho\big(g(t)\big). \tag{1.1}$$

On the other side, we also have the so called Ricci solitons, which is also introduced by Hamilton [6]. A Riemannian manifold (M, g) is a Ricci soliton if there exist a smooth vector field X such that

$$\rho = \kappa g + \mathcal{L}_X g, \tag{1.2}$$

where $\mathcal{L}_X$ is the Lie derivative of the metric in the direction of X, and κ is a real number. Furthermore, if $\kappa < 0$, $\kappa = 0$, or $\kappa > 0$, then, (M, g) is a expanding, steady, or shrinking Ricci soliton, respectively. Note that, if $X = 0$, then, it will be satisfies in Einstein condition. Therefore, Ricci solitons are a generalization of Einstein manifolds. Ricci solitons have a significant work in conception the uniqueness of the Ricci flow, because they are the self-similar solutions. After Hamilton's work [6], the perusal of the Ricci solitons has been one of main question in differential geometry. For example in [11], the well-know mathematician Perelman was able to prove Thurston's geometrization conjecture by using the notion of Ricci solitons.

In 2001, Lauret first introduced the notion of algebraic Ricci soliton in the Riemannian setting [9].

Definition 1.1. *Suppose G be a Lie group with Lie algebra $\mathfrak{g}$. A left-invariant Riemannian metric g on Lie group G is called an algebraic Ricci soliton if*

$$Ric = \kappa\ Id + \mathcal{D}, \tag{1.3}$$

where $\kappa \in \mathbb{R}$ and $\mathcal{D} \in Der(\mathfrak{g})$ is derivation of the corresponding Lie algebra $\mathfrak{g}$, i.e.,

$$\mathcal{D}[X, Y] = [\mathcal{D}X, Y] + [X, \mathcal{D}Y], \qquad \forall\ X, Y \in \mathfrak{g}. \tag{1.4}$$

In the special case, if G is a solvable Lie group (a nilpotent Lie group), then an algebraic Ricci soliton is called a sol-soliton (a nil-soliton).

It is clear that any Einstein metric on a Lie group is an algebraic Ricci soliton. Also, any algebraic Ricci soliton is a Ricci soliton, whereas the converse is an open problem (see[7, 9]).

In [9], Lauret considered the relationship among sol-solitons and Ricci solitons on Riemannian setting. Actually, he denote that any sol-soliton is a Ricci soliton. It is worth noting that all these examples are either expanding or steady. Note that problems related to Ricci solitons are second-order differential equations. But, problems related to algebraic Ricci solitons are algebraic equations. Thus, algebraic Ricci solitons permit us to make Ricci solitons in an algebraic method, i.e., the study of Ricci solitons on homogeneous spaces is algebraic Ricci soliton by using algebraic Ricci soliton theory.

The classification of algebraic Ricci solitons on Lie groups is very significant, because any Ricci soliton on homogeneous spaces is an algebraic Ricci soliton [8]. Therefore, the algebraic Ricci solitons have been perused in many articles. Furthermore, the three-dimensional case has been investigated as an attractive original of instances. For example, In [1], Batat and Onda have perused algebraic Ricci solitons of three-dimensional Lorentzian Lie groups. Also, Onda [10] offered examples of algebraic Ricci solitons in the Pseudo-Riemannian setting.

In [12], Salimi Moghaddam completely classifies algebraic Ricci solitons on three-dimensional Lie groups. He obtains a scale for a Lie group endowed with a left-invariant Riemannian metric to be a Ricci soliton based on structural constants of the corresponding Lie algebra. First, we recall the following Lemma.

Lemma 1.2. [12] *Suppose G be a n-dimensional Lie group with a left invariant Riemannian metric g. Suppose that $\{u_1, \cdots, u_n\}$ is an orthonormal basis for the Lie algebra $\mathfrak{g}$ of G, with respect to the Riemannian metric g. If α_{ijk} denote the structural constants defined by*

$$[u_i, u_j] = \sum_{k=1}^{n} \alpha_{ijk} u_k,$$

then the Ricci operator of g can be computed as follows,

$$Ric(u_i) = \frac{1}{4} \sum_{l=1}^{n} \sum_{j=1}^{n} \sum_{r=1}^{n} 2\alpha_{rjj}(\alpha_{irl} + \alpha_{lir} - \alpha_{rli}) - 2\alpha_{ijr}(\alpha_{rjl} + \alpha_{lrj} - \alpha_{jlr}) \\ - (\alpha_{ijr} + \alpha_{rij} - \alpha_{jri})(\alpha_{jrl} + \alpha_{ljr} - \alpha_{rlj})u_i.$$

Considering that a left-invariant Rirmannian metric on a Lie group is a Ricci soliton if and only if it is an algebraic Ricci soliton [8]. He proved the following.

Theorem 1.3. [12] *Suppose that G is a n-dimensional Lie group, endowed with a left invariant Riemannian metric g. Assume that $\{u_1, \cdots, u_n\}$ is an orthonormal basis for $\mathfrak{g}$, the Lie algebra of G, and α_{ijk} are the structural constants with respect to this basis. The Riemannian metric g is a Ricci soliton if and only if there exists a constant $\kappa \in \mathbb{R}$ such that, for any $t, p, q = 1, \cdots, n$ we have*

$$\kappa\alpha_{qpt} + \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{r=1}^{n} 2\alpha_{rjj}\Big(\alpha_{iqt}(\alpha_{pri} + \alpha_{ipr} - \alpha_{rip}) - \alpha_{ipt}(\alpha_{qri} + \alpha_{iqr} - \alpha_{riq})\Big) \\ + 2(\alpha_{rji} + \alpha_{irj} - \alpha_{jir})(\alpha_{ipt}\alpha_{qjr} - \alpha_{iqt}\alpha_{pjr})$$

$$+ (\alpha_{jri} + \alpha_{ijr} - \alpha_{rij})\Big(\alpha_{ipt}(\alpha_{qjr} + \alpha_{rqj} - \alpha_{jrq}) - \alpha_{iqt}(\alpha_{pjr} + \alpha_{rpj} - \alpha_{jrp})\Big)$$
$$- 2\alpha_{pqi}\alpha_{rjj}(\alpha_{irt} + \alpha_{tir} - \alpha_{rti}) + 2\alpha_{pqi}\alpha_{ijr}(\alpha_{rjt} + \alpha_{trj} - \alpha_{jtr})$$
$$+ \alpha_{pqi}(\alpha_{ijr} + \alpha_{rij} - \alpha_{jri})(\alpha_{jrt}\alpha_{tjr} - \alpha_{tjr}\alpha_{rtj}) = 0.$$

By using this theorem, he classified an algebraic Ricci soliton on simply-connected three-dimensional Lie groups.

In this article, we focus specifically on the two Lie groups $\mathbb{H}^2 \times \mathbb{R}$ and Sol_3, endowed with a left-invariant Riemannian metric and work based on their corresponding structures of Lie algebraic. Actually, we consider the algebraic Ricci solitons of the Lie group $\mathbb{H}^2 \times \mathbb{R}$, endowed with a left-invariant Riemannian metric. Also, we check the existence of sol-solitons on the three-dimensional Lie group Sol_3, endowed with left-invariant Riemannian metric.

It is worth noting that Belarbi [2], has studied the Ricci solitons on the Lie group $\mathbb{H}^2 \times \mathbb{R}$, endowed with a left-invariant Riemannian metric. Furthermore he has considered the Ricci solitons of the Lie group Sol_3, endowed with a left-invariant Riemannian metric [3].

We organize this article as follows. In section 2, we examine the left-invariant Riemannian metrics accepted by the three-dimensional Lie group $\mathbb{H}^2 \times \mathbb{R}$ and show the existence algebraic Ricci solitons on this group. In section 3, we examine left-invariant Riemannian metrics accepted by the three-dimensional solvable Lie group Sol_3 and consider sol-solitons of this group.

2. The lie group $\mathbb{H}^2 \times \mathbb{R}$

2.1. **Curvature of the Lie group $\mathbb{H}^2 \times \mathbb{R}$.** Suppose $\mathbb{H}^2$ be expressed by the upper half-plane model

$$\mathbb{H}^2 := \Big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\Big\},$$

endowed with the metric

$$g_{\mathbb{H}^2} = \frac{1}{x_2^2}\Big(dx_1^2 + dx_2^2\Big).$$

The space $\mathbb{H}^2$, with the group structure derivative by the combination of suitable affine map, is a Lie group and the metric $g_{\mathbb{H}^2}$ is left-invariant. It follows that the Riemannian product space $\mathbb{H}^2 \times \mathbb{R}$ is a Lie group with respected to the action

$$(x_1, x_2, x_3) \star (x_1^{'}, x_2^{'}, x_3^{'}) = (x_1^{'}x_2 + x_1, x_2x_2^{'}, x_3 + x_3^{'}),$$

and the left-invariant product metric

$$g = \frac{1}{x_2^2}\Big(dx_1^2 + dx_2^2\Big) + dx_3^2. \tag{2.1}$$

A left-invariant orthonormal basis $\{u_1, u_2, u_3\}$ in the Riemannian Lie group $\mathbb{H}^2 \times \mathbb{R}$ is

$$\begin{aligned} u_1 &= x_2\partial_1, \\ u_2 &= x_2\partial_2, \\ u_3 &= \partial_3, \end{aligned} \tag{2.2}$$

where we put

$$\partial_i := \frac{\partial}{\partial x_i}, \quad \forall i = 1, 2, 3.$$

So, the non-zero Lie brackets are

$$\begin{aligned} [u_1, u_2] &= -u_1, \\ [u_2, u_3] &= 0, \\ [u_3, u_1] &= 0. \end{aligned} \tag{2.3}$$

In the whole article, we consider the Lie group $\mathbb{H}^2 \times \mathbb{R}$ with a left-invariant Riemannian metric g.

The components of the Levi-Civita connection ∇ of the Lie group $\mathbb{H}^2 \times \mathbb{R}$ are given by

$$\begin{aligned} \nabla_{u_1} u_1 &= u_2, \\ \nabla_{u_1} u_2 &= -u_1, \\ \nabla_{u_1} u_3 &= 0, \\ \nabla_{u_2} u_1 &= 0, \\ \nabla_{u_2} u_2 &= 0, \\ \nabla_{u_2} u_3 &= 0, \\ \nabla_{u_3} u_1 &= 0, \\ \nabla_{u_3} u_2 &= 0, \\ \nabla_{u_3} u_3 &= 0. \end{aligned}$$

The non-zero components of the Riemann curvature tensor $\mathcal{R}$ are computed as follows

$$\begin{aligned} \mathcal{R}(u_1, u_2)u_1 &= u_2, \\ \mathcal{R}(u_1, u_2)u_2 &= -u_1, \end{aligned}$$

and the Ricci operator is given by

$$Ric = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

2.2. **The algebraic Ricci solitons of the Lie group $\mathbb{H}^2 \times \mathbb{R}$.** Now, we check the existence of algebraic Ricci solitons on the Lie group $(\mathbb{H}^2 \times \mathbb{R}, g)$, endowed with the left-invariant Riemannian metric. Suppose $\mathcal{D} \in Der(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of the Lie group $\mathbb{H}^2 \times \mathbb{R}$. For any $i = 1, 2, 3$, we put

$$\mathcal{D}u_i = \lambda_i^1 u_1 + \lambda_i^2 u_2 + \lambda_i^3 u_3.$$

Equation (1.4), using the above equation can be written as follows.

$$\begin{aligned}
&\mathcal{D}[u_1, u_2] = [\mathcal{D}u_1, u_2] + [u_1, \mathcal{D}u_2] \\
&-\mathcal{D}u_1 = [\lambda_1^1 u_1 + \lambda_1^2 u_2 + \lambda_1^3 u_3, u_2] + [u_1, \lambda_2^1 u_1 + \lambda_2^2 u_2 + \lambda_2^3 u_3] \\
&-\lambda_1^1 u_1 - \lambda_1^2 u_2 - \lambda_1^3 u_3 = \lambda_1^1 [u_1, u_2] + \lambda_1^2 [u_2, u_2] + \lambda_1^3 [u_3, u_2] \\
&\qquad\qquad + \lambda_2^1 [u_1, u_1] + \lambda_2^2 [u_1, u_2] + \lambda_2^3 [u_1, u_3]
\end{aligned}$$

So, by using (2.3), we obtain

$$\lambda_2^2 = \lambda_1^2 = \lambda_1^3 = 0.$$

By continuing the same process we have

$$\lambda_1^2 = \lambda_1^3 = \lambda_2^2 = \lambda_3^2 = \lambda_3^1 = 0.$$

Thus, we showed the following lemma.

Lemma 2.1. *Suppose $\mathfrak{g}$ is the Lie algebra of the Lie group $\mathbb{H}^2 \times \mathbb{R}$. Then $\mathcal{D} \in Der(\mathfrak{g})$ if and only if*

$$\mathcal{D} = \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & 0 \\ 0 & 0 & 0 \\ 0 & \lambda_2^3 & \lambda_3^3 \end{pmatrix}.$$

By using the above Lemma, we have the following.

Theorem 2.2. *The Lie group $\mathbb{H}^2 \times \mathbb{R}$ equipped with the left-invariant Riemannian metric g is an algebraic Ricci soliton. In particular,*

$$\mathcal{D} = \begin{pmatrix} -(\kappa+1) & -(\kappa+1) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\kappa \end{pmatrix},$$

and κ is an arbitrary real number.

Proof. Now, using Lemma 2.1, we examine that algebraic Ricci soliton condition (1.3) on the Lie group $\mathbb{H}^2 \times \mathbb{R}$. We have

$$\begin{aligned}
&Ric(u_1) = \kappa\, Id(u_1) + \mathcal{D}(u_1) \\
&-u_1 = \kappa u_1 + \lambda_1^1 u_1 \\
&\kappa + \lambda_1^1 = -1.
\end{aligned}$$

By continuing this process, we see that the Lie group $\mathbb{H}^2 \times \mathbb{R}$ is an algebraic Ricci soliton if and only if

$$\begin{aligned} &\lambda_1^1 = \lambda_2^1 = -(\kappa + 1), \\ &\lambda_3^3 = -\kappa, \\ &\lambda_2^3 = 0. \end{aligned}$$

This completes the proof. □

3. The Lie group Sol_3

3.1. **Curvature of the Lie group Sol_3.** It is easy to see that the Riemannian solvable Lie group Sol_3 is a Lie group $\mathbb{R}^3$ endowed with a left-invariant Riemannian metric

$$g = e^{2x_3} dx_1^2 + e^{-2x_3} dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) are the usual coordinates of $\mathbb{R}^3$. Also, the group structure of three-dimensional Lie group Sol_3 is of form

$$(x_1^{'}, x_2^{'}, x_3^{'}) \star (x_1, x_2, x_3) = (e^{-x_3^{'}} x_1 + x_1^{'}, e^{x_3^{'}} x_2 + x_2^{'}, x_3 + x_3^{'}).$$

The isometries under this Riemannian metric are

$$(x_1, x_2, x_3) \mapsto (\pm e^{-c} x_1 + a, \pm e^{-c} x_2 + b, x_3 + c),$$

and

$$(x_1, x_2, x_3) \mapsto (\pm e^{-c} x_2 + a, \pm e^{-c} x_1 + b, -x_3 + c),$$

where a, b, c are arbitrary real numbers.

A left-invariant orthonormal basis $\{u_1, u_2, u_3\}$ in the Riemannian Lie group Sol_3 is

$$\begin{aligned} &u_1 = e^{-x_3} \partial_1, \\ &u_2 = e^{x_3} \partial_2, \\ &u_3 = \partial_3, \end{aligned} \tag{3.1}$$

where we put

$$\partial_i := \frac{\partial}{\partial x_i}, \quad \forall i = 1, 2, 3.$$

So, the non-zero Lie brackets are

$$\begin{aligned} &[u_1, u_2] = 0, \\ &[u_2, u_3] = -u_2, \\ &[u_3, u_1] = -u_1. \end{aligned} \tag{3.2}$$

In the whole article, we consider the Lie group Sol_3 with left-invariant Riemannian metric g.

The components of the Levi-Civita connection ∇ of the Lie group $\mathbb{H}^2 \times \mathbb{R}$ are given by

$$\begin{aligned}
\nabla_{u_1} u_1 &= -u_3, \\
\nabla_{u_1} u_2 &= 0, \\
\nabla_{u_1} u_3 &= u_1, \\
\nabla_{u_2} u_1 &= 0, \\
\nabla_{u_2} u_2 &= u_3, \\
\nabla_{u_2} u_3 &= -u_2, \\
\nabla_{u_3} u_1 &= 0, \\
\nabla_{u_3} u_2 &= 0, \\
\nabla_{u_3} u_3 &= 0.
\end{aligned}$$

The non-zero components of the Riemann curvature tensor $\mathcal{R}$ are computed as follows

$$\begin{aligned}
\mathcal{R}(u_1, u_2)u_1 &= -u_2, \\
\mathcal{R}(u_1, u_2)u_2 &= u_1, \\
\mathcal{R}(u_1, u_3)u_1 &= u_3, \\
\mathcal{R}(u_1, u_3)u_3 &= -u_1, \\
\mathcal{R}(u_2, u_3)u_2 &= u_3, \\
\mathcal{R}(u_2, u_3)u_3 &= -u_2,
\end{aligned}$$

and the Ricci operator is given by

$$Ric = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

3.2. **The algebraic Ricci solitons of the Lie group Sol_3.** Here, we consider the existence of sol-solitons on the Lie group Sol_3, endowed with the left-invariant Riemannian metric.

Suppose $\mathcal{D} \in Der(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of the Lie group Sol_3. For any $i = 1, 2, 3$, we put

$$\mathcal{D}u_i = \lambda_i^1 u_1 + \lambda_i^2 u_2 + \lambda_i^3 u_3.$$

Equation (1.4), using the above equation can be written as follows.

$$\begin{aligned}
&\mathcal{D}[u_1, u_2] = [\mathcal{D}u_1, u_2] + [u_1, \mathcal{D}u_2] \\
&0 = [\lambda_1^1 u_1 + \lambda_1^2 u_2 + \lambda_1^3 u_3, u_2] + [u_1, \lambda_2^1 u_1 + \lambda_2^2 u_2 + \lambda_2^3 u_3] \\
&0 = \lambda_1^1 [u_1, u_2] + \lambda_1^2 [u_2, u_2] + \lambda_1^3 [u_3, u_2] + \lambda_2^1 [u_1, u_1] + \lambda_2^2 [u_1, u_2] + \lambda_2^3 [u_1, u_3]
\end{aligned}$$

So, by using (2.3), we obtain

$$\lambda_1^3 = \lambda_2^3 = 0.$$

By continuing the same process we have

$$\lambda_1^2 = \lambda_1^3 = \lambda_2^1 = \lambda_2^3 = \lambda_3^3 = 0.$$

Thus, we showed the following lemma.

Lemma 3.1. *Suppose $\mathfrak{g}$ is the Lie algebra of the Lie group Sol_3. Then $\mathcal{D} \in Der(\mathfrak{g})$ if and only if*

$$\mathcal{D} = \begin{pmatrix} \lambda_1^1 & 0 & \lambda_3^1 \\ 0 & \lambda_2^2 & \lambda_3^2 \\ 0 & 0 & 0 \end{pmatrix}.$$

By using the above lemma, we have the following theorem.

Theorem 3.2. *The Lie group Sol_3 equipped with the left-invariant Riemannian metric g is a sol-soliton. In particular,*

$$\mathcal{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \kappa = -2.$$

Proof. By using Lemma 3.1, we examine that algebraic Ricci soliton condition (1.3) on the Lie group Sol_3. We have

$$\begin{aligned} &Ric(u_1) = \kappa\, Id(u_1) + \mathcal{D}(u_1) \\ &0 = \kappa u_1 + \lambda_1^1 u_1 \\ &\lambda_1^1 = -\kappa. \end{aligned}$$

By continuing this process, we see that the Lie group Sol_3 is an algebraic Ricci soliton if and only if

$$\begin{aligned} &\lambda_1^1 = \lambda_2^2 = -\kappa = 2, \\ &\lambda_3^1 = \lambda_3^2 = 0. \end{aligned}$$

So, this finish the proof. □

4. CONCLUSION AND REMARKS

In this work, we studied two specified Lie groups. We show that the Lie group $\mathbb{H}^2 \times \mathbb{R}$, endowed with a left-invariant Riemannian metric is algebraic Ricci soliton. Also, we prove that the solvable Lie group Sol_3, endowed with a left-invariant Riemannian metric is sol-soliton.

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