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On semi-P-reducible Finsler manifolds with relatively isotropic Landsberg curvature

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Abstract. The class of semi-P-reducible Finsler metrics is a rich and basic class of Finsler metrics that contains the class of L-reducible metrics, C-reducible metrics and Landsberg metrics. In this paper, we prove that every semi-P-reducible manifolds with isotropic Landsberg curvature reduce to semi-C-reducible manifolds. Also, we prove that a semi-P-reducible Finsler metric of relatively isotropic mean Landsberg curvature has relatively isotropic Landsberg curvature if and only if it is a semi-C-reducible Finsler metric.

Keywords: Finsler metric, semi-C-reducible metric, C-reducible metric.

1. Introduction

Let (M, F) be an n-dimensional Finsler manifold. The second derivatives of F_x^2 at $y \in T_xM_0$ is an inner product $\mathbf{g}_y$ on T_xM. The third order derivatives of F_x^2 at $y \in T_xM_0$ is a symmetric trilinear forms $\mathbf{C}_y$ on T_xM. We call $\mathbf{g}_y$ and $\mathbf{C}_y$ the fundamental form and the Cartan torsion, respectively. A Finsler metric reduces to a Riemannian metric if and only if $\mathbf{C} = 0$. There is a weaker notion of Cartan torsion, namely mean Cartan torsion. Set

$$\mathbf{I}_y := \sum_{i=1}^{n} \mathbf{C}_y(e_i, e_i, \cdot),$$

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where $\{e_i\}$ is an orthonormal basis for $(T_xM, \mathbf{g}_y)$. $\mathbf{I}_y$ is called the mean Cartan torsion of F. By Diecke Theorem, a positive-definite metric F is Riemannian if and only if $\mathbf{I}_y = 0$. In [4], by using the Cartan and mean Cartan torsions of a Finsler metric, Matsumoto introduced the notion of C-reducible Finsler metrics. Indeed, F is called C-reducible if its Cartan torsion is give by

$$C_{ijk} = \frac{1}{n+1}\Big\{I_i h_{jk} + I_j h_{ik} + I_k h_{ij}\Big\}, \tag{1.1}$$

where h_{ij} is the angular metric. Matsumoto proved that any Randers metric is C-reducible [4]. Later on, Matsumoto-Hōjō proves that the converse is true too [8]. A Randers metric $F = \alpha+\beta$ is just a Riemannian metric α perturbated by a one form β. Randers metrics have important applications both in mathematics and physics [13].

By considering the form of Cartan torsion of a non-Riemannian (α, β)-metric on a manifold M of dimension $n \geq 3$, Matsumoto-Shibata introduced the notion of semi-C-reducibility [6]. A Finsler metric F is called semi-C-reducible if its Cartan tensor is given by

$$C_{ijk} = \frac{P}{n+1}\Big\{h_{ij}I_k + h_{jk}I_i + h_{ki}J_j\Big\} + \frac{Q}{||\mathbf{I}||^2} I_i I_j I_k,$$

where $P = P(x, y)$ and $Q = Q(x, y)$ are scalar function on TM and $||\mathbf{I}||^2 = g^{ij} I_i I_j$. The function P is called characteristic scalar of F. We have two special cases as follows:

(i) If $Q = 0$, then F reduces to a C-reducible metric;

(ii) If $P = 0$, then F reduces to a C2-like metric.

It is remarkable that, an (α, β)-metric is a Finsler metric on M defined by $F := \alpha\phi(s)$, $s = \beta/\alpha$, where $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, α is a Riemannian metric and β is a 1-form on M.

The rate of change of $\mathbf{C}_y$ along geodesics is the Landsberg curvature $\mathbf{L}_y$ on T_xM for any $y \in T_xM_0$. F is said to be Landsbergian if $\mathbf{L} = 0$. There is a weaker notion of metrics- weakly Landsberg metrics. Set

$$\mathbf{J}_y := \sum_{i=1}^{n} \mathbf{L}_y(e_i, e_i, \cdot).$$

Then $\mathbf{J}_y$ is called the mean Landsberg curvature. A Finsler metric F is said to be weakly Landsbergian if $\mathbf{J} = 0$.

As a generalization of C-reducible metrics, Matsumoto-Shimada introduced the notion of L-reducible metrics [7,15]. For this aim, they used the Landsberg and mean Landsberg curvatures of a Finsler metric. Indeed, F is said to be

L-reducible if its Landsberg curvature is give by

$$L_{ijk} = \frac{1}{n+1}\Big\{J_i h_{jk} + J_j h_{ik} + J_k h_{ij}\Big\}. \tag{1.2}$$

In [21], Tayebi-Sadeghi considered a new class of Finsler metrics, namely generalized P-reducible metrics. This class of Finsle metrics contains the class of L-reducible Finsler metrics. They studied generalized P-reducible (α, β)-metrics with vanishing S-curvature and showed that such metrics reduce to Berwald metrics or Randers metric. It follows that there is not any non-trivial L-reducible (α, β)-metric with vanishing S-curvature. In [18], Tayebi-Bahadori-Sadeghi studied the Cartan torsion and Landsberg curvature of spherically symmetric Finsler metrics. They investigated C-reducibility and L-reducibility of spherically symmetric Finsler metrics and proved the following.

Theorem 1.1. ([18]) Let $F = u\phi(r, s)$ be a spherically symmetric Finsler metric on a domain $\Omega \subseteq \mathbb{R}^n$. Then the following hold:

(i) F is a semi-C-reducible Finsler metric;

(ii) F is a L-reducible metric if and only if satisfies the following PDE

$$(\phi - s\phi_s)\, L_1 - 3\phi_{ss} L_2 = 0. \tag{1.3}$$

It is easy to see that, if $L_1, L_2 \neq 0$ then one can get a L-reducible spherically symmetric Finsler metric which is not C-reducible. It is remarkable that Matsumoto was proved that every (α, β)-metric on a manifold M of dimension $n \geq 3$ is semi-C-reducible [7]. There are many spherically symmetric Finsler metrics which are not (α, β)-metrics, namely the Bryant metrics. Thus the mentioned Matsumoto's theorem was not proved for the class of spherically symmetric Finsler metrics. The part (i) of the above Theorem is an answer to this unsolved problem.

In [11], B. N. Prasad introduced a new class of Finsler spaces which contains the notion of P-reducible spaces. A Finsler metric F is called generalized P-reducible if its Landsberg curvature is given by following

$$L_{ijk} = \lambda C_{ijk} + a_i h_{jk} + a_j h_{ki} + a_k h_{ij},$$

where $\lambda = \lambda(x, y)$ is a scalar function on TM, $a_i = a_i(x)$ is scalar function on M and $h_{ij} = g_{ij} - F^{-2} y_i y_j$ is the angular metric. λ and a_i are homogeneous function of degree 1 and degree 0 with respect to y, respectively. By definition, we have $a_i y^i = 0$. We have two special cases as follows:

(i) If $a_i = 0$, then F is reduce to general relatively isotropic Landsberg metric;

(ii) If $\lambda = 0$ then F is a L-reducible metric.

In [14], Rastogi introduced a new class of Finsler spaces named by semi-P-reducible spaces which contains the notion of C-reducible and L-reducible metrics, as a special case. A Finsler metric F is called semi-P-reducible if its Landsberg tensor is given by

$$L_{ijk} = \lambda\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\} + 3\mu J_i J_j J_k, \tag{1.4}$$

where $\lambda = \lambda(x, y)$ and $\mu = \mu(x, y)$ are scalar functions on TM. We have some special cases as follows:

(i) If $\mu = 0$, then F is a L-reducible metric;

(ii) If $\lambda = 0$, then F is a $L2$-like metric;

(iii) If $\mu = \lambda = 0$, then F is a Landsberg metric.

Indeed, complex and interesting special forms of Cartan, mean Cartan, Landsberg and mean Landsberg tensors have been obtained by some Finslerians (see [2, 3, 5, 6, 9–12, 19] and [20]). Since the class of semi-P-reducible metrics contain the class of C-reducible metrics as a special case, therefore the study of this class of Finsler spaces will enhance our understanding of the geometric meaning of Randers metrics.

The quotient $\mathbf{L}/\mathbf{C}$ is regarded as the relative rate of change of $\mathbf{C}$ along Finslerian geodesics. Then F is said to be isotropic Landsberg metric if $\mathbf{L} = cF\mathbf{C}$, where $c = c(x)$ is a scalar function on M. In this paper, we consider semi-P-reducible manifolds with isotropic Landsberg curvature and prove the following.

Theorem 1.2. *Let (M, F) be a semi-P-reducible Finsler manifold of dimension $n \geq 3$. Suppose that F has relatively isotropic Landsberg curvature*

$$\mathbf{L} = cF\mathbf{C}, \tag{1.5}$$

where $c = c(x)$ is a scalar function on M. Then one of the following holds

(i) F is a Landsberg metric;

(ii) F is a semi-C-reducible metric.

There are many connections in Finsler geometry [17]. In this paper, we use the Berwald connection on Finsler manifolds. Also, the h- and v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively.

2. Preliminaries

Let M be a n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:
(i) F is C^∞ on $TM_0 := TM \setminus \{0\}$;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;
(iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\Big[F^2(y + su + tv)\Big]|_{s,t=0}, \quad u, v \in T_xM.$$

Let $x \in M$. To measure the non-Euclidean feature of $F_x := F|_{T_xM}$, one can define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]|_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that **C=0** if and only if F is Riemannian.

For $y \in T_xM_0$, define mean Cartan torsion $\mathbf{I}_y$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where

$$I_i := g^{jk} C_{ijk}$$

and $u = u^i \partial/\partial x^i|_x$. By Diecke Theorem, a positive-definite Finsler metric F is Riemannian if and only if $\mathbf{I}_y = 0$ [16].

For $y \in T_xM_0$, define the Matsumoto torsion $\mathbf{M}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by $\mathbf{M}_y(u, v, w) := M_{ijk}(y)u^iv^jw^k$ where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\Big\{I_i h_{jk} + I_j h_{ik} + I_k h_{ij}\Big\},$$

and $h_{ij} := FF_{y^iy^j} = g_{ij} - F^{-2} g_{ip}y^p g_{jq}y^q$ is the angular metric. A Finsler metric F is said to be C-reducible if $\mathbf{M}_y = 0$. This quantity is introduced by Matsumoto [4]. Matsumoto proved that every Randers metric satisfies that $\mathbf{M}_y = 0$. Later on, Matsumoto-Hōjō proves that the converse is true too.

Lemma 2.1. ([8]) A Finsler metric F on a manifold of dimension $n \geq 3$ is a Randers metric if and only if $\mathbf{M}_y = 0$, $\forall y \in TM_0$.

The horizontal covariant derivatives of **C** along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ which is defined by

$$\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^iv^jw^k,$$

where

$$L_{ijk} := C_{ijk|s}y^s,$$

$u = u^i\partial/\partial x^i|_x$, $v = v^i\partial/\partial x^i|_x$ and $w = w^i\partial/\partial x^i|_x$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y\in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L}$=$\mathbf{0}$.

The quotient $\mathbf{L}/\mathbf{C}$ is regarded as the relative rate of change of $\mathbf{C}$ along geodesics. A Finsler metric F on a manifold M is said to be isotropic Landsberg metric if

$$\mathbf{L} = cF\mathbf{C},$$

where $c = c(x)$ is a scalar function on M [1].

The horizontal covariant derivatives of $\mathbf{I}$ along geodesics give rise to the mean Landsberg curvature $\mathbf{J}_y(u) := J_i(y)u^i$, where

$$J_i := g^{jk}L_{ijk}.$$

A Finsler metric is said to be weakly Landsberg if $\mathbf{J} = 0$.

The quantity $\mathbf{J}/\mathbf{I}$ is regarded as the relative rate of change of $\mathbf{I}$ along geodesics. Then F is said to be isotropic mean Landsberg metric if

$$\mathbf{J} = cF\mathbf{I},$$

where $c = c(x)$ is a scalar function on M [1].

Define $\bar{\mathbf{M}}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\bar{\mathbf{M}}_y(u, v, w) := \bar{M}_{ijk}(y)u^iv^jw^k$ where

$$\bar{M}_{ijk} := L_{ijk} - \frac{1}{n+1}\Big\{J_ih_{jk} + J_jh_{ik} + J_kh_{ij}\Big\}.$$

A Finsler metric F is said to be P-reducible if $\bar{\mathbf{M}}_y = 0$. The notion of P-reducibility was given by Matsumoto-Shimada [9].

A curve $c = c(t)$ is called a geodesic if it satisfies

$$\frac{d^2c^i}{dt^2} + 2G^i(\dot{c}(t)) = 0, \tag{2.1}$$

where $G^i(x, y)$ are local functions on TM given by

$$G^i := \frac{1}{4}g^{il}\Big\{\frac{\partial^2[F^2]}{\partial x^k\partial y^l}y^k - \frac{\partial[F^2]}{\partial x^l}\Big\}, \qquad y \in T_xM. \tag{2.2}$$

F is called a Berwald metric if $G^i(y)$ are quadratic in $y \in T_xM$ for all $x \in M$.

Let $U(t)$ be a vector field along a curve $c(t)$. The canonical covariant derivative $\mathrm{D}_{\dot{c}}U(t)$ is defined by

$$\mathrm{D}_{\dot{c}}U(t) := \Big\{\frac{dU^i}{dt}(t) + U^j(t)\frac{\partial G^i}{\partial y^j}(\dot{c}(t))\Big\}\frac{\partial}{\partial x^i}|_{c(t)}. \tag{2.3}$$

$U(t)$ is said to be *parallel* along c if $\mathrm{D}_{\dot{c}(t)}U(t) = 0$.

Let F be a Finsler metric on an n-dimensional manifold M. Let (x^i, y^i) be a standard coordinate system in TM_0 and $G^i(y)$ denote the geodesic coefficients of F in (2.2). Put

$$\omega_j{}^i := \frac{\partial^2 G^i}{\partial y^j \partial y^k}(y) dx^k. \tag{2.4}$$

$\{\omega_j{}^i\}$ are called the Berwald connection forms. Put

$$g_{ij}(y) := \mathbf{g}_y(e_i, e_j), \tag{2.5}$$

$$C_{ijk}(y) := \mathbf{C}_y(e_i, e_j, e_k), \quad L_{ijk}(y) := \mathbf{L}_y(e_i, e_j, e_k). \tag{2.6}$$

where $\{e_i = \frac{\partial}{\partial x^i}|_{\pi(y)}\}$ is a natural local frame on M. They are local functions on TM_0. With the Berwald connection forms, we define $C_{ijk;l}$ and $C_{ijk\cdot l}$ by

$$dC_{ijk} - C_{pjk}\omega_i{}^p - C_{ipk}\omega_j{}^p - C_{ijp}\omega_k{}^p = C_{ijk;l}\omega^l + C_{ijk\cdot l}\omega^{n+l}. \tag{2.7}$$

The definition of Landsberg curvature $\mathbf{L}$ is equivalent to the following

$$L_{ijk}(y) := C_{ijk;l}(y) y^l. \tag{2.8}$$

In literatures, $C_{ijk;l}(y)y^l$ are also denoted by $C_{ijk|0}(y)$. Thus $L_{ijk} = C_{ijk|0}$.

Let $\omega^i := dx^i$ and $\omega^{n+i} := dy^i + y^j\omega_j{}^i$. $\{\omega^i, \omega^{n+i}\}_{i=1}^n$ is a natural coframe for $T^*(TM_0)$. They satisfy the following structure equations

$$d\omega^i = \omega^j \wedge \omega_j{}^i, \tag{2.9}$$

$$dg_{ij} - g_{kj}\omega_i{}^k - g_{ik}\omega_j{}^k = -2L_{ijk}\omega^k + 2C_{ijk}\omega^{n+k}. \tag{2.10}$$

3. Proof of Theorem 1.2

In this section, we study the relation between the class of semi-C-reducible metrics and the class of semi-P-reducible metrics and prove the Theorem 1.2.

Proof of Theorem 1.2: First, we remark that every Finsler surface is C-reducible. Then we assume that $n \geq 3$. Now, by assumption we have

$$L_{ijk} = \lambda\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\} + 3\mu J_i J_j J_k, \tag{3.1}$$

where $\lambda = \lambda(x, y)$ and $\mu = \mu(x, y)$ are scalar functions on TM. Multiplying (3.1) with g^{ij} implies that

$$J_k = (n+1)\lambda J_k + 3\mu||\mathbf{J}||^2 J_k, \tag{3.2}$$

where $||\mathbf{J}||^2 := J^m J_m$. (3.2) is equal to

$$\Big\{1 - (n+1)\lambda - 3\mu\mathbf{J}^2\Big\} J_k = 0. \tag{3.3}$$

By (3.3), we have two main cases as follows:

If $J_k = 0$, then by (3.1) we get $L_{ijk} = 0$. This means that F is a Landsberg metric.

If $J_k \neq 0$, then (3.3) implies that

$$(n+1)\lambda + 3\mu||\mathbf{J}||^2 = 1,$$

which yields

$$\mu = \frac{1-(n+1)\lambda}{3||\mathbf{J}||^2}. \tag{3.4}$$

By assumption $\mathbf{J} = cF\mathbf{I}$ and then

$$||\mathbf{J}||^2 = c^2F^2||\mathbf{I}||^2, \tag{3.5}$$

where $||\mathbf{I}||^2 := I_m I^m$. Putting (3.5) in (3.4) result that

$$\mu = \frac{1-(n+1)\lambda}{3c^2F^2||\mathbf{I}||^2}. \tag{3.6}$$

On the other hand, by putting

$$\mathbf{L} = cF\mathbf{C}, \quad \text{and} \quad \mathbf{J} = cF\mathbf{I}$$

in (3.1) we get

$$cFC_{ijk} = \lambda cF\Big\{I_ih_{jk} + I_jh_{ki} + I_kh_{ij}\Big\} + 3\mu c^3F^3I_iI_jI_k. \tag{3.7}$$

Since $c \neq 0$, then (3.7) implies that

$$C_{ijk} = \lambda\Big\{I_ih_{jk} + I_jh_{ki} + I_kh_{ij}\Big\} + 3\mu c^2F^2I_iI_jI_k. \tag{3.8}$$

By (3.6) and (3.8) we have

$$C_{ijk} = \lambda\Big\{I_ih_{jk} + I_jh_{ki} + I_kh_{ij}\Big\} + \frac{1}{||\mathbf{I}||^2}\Big(1-(n+1)\lambda\Big)I_iI_jI_k. \tag{3.9}$$

Then F is a semi-C-reducible with

$$P = (n+1)\lambda, \qquad Q = 1-(n+1)\lambda.$$

This completes the proof. □

Remark 3.1. In [22], Tayebi-Sadeghi considered semi-C-reducible Finsler metrics of dimension $n \geq 3$ and showed that the norm of Cartan and mean Cartan torsion of F satisfy in following relation

$$\|\mathbf{C}\| = \sqrt{\frac{3p^2 + 6p\,q + (n+1)q^2}{n+1}}\;\|\mathbf{I}\|, \tag{3.10}$$

where $p = p(x, y)$ and $q = q(x, y)$ are scalar function on TM satisfying

$$p + q = 1.$$

For Finsler surface, (3.10) reduces to following

$$||\mathbf{C}|| = ||\mathbf{I}||.$$

For a semi-P-reducible metric, we have

$$L_{ijk} = \lambda\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\} + 3\mu J_i J_j J_k. \tag{3.11}$$

Contracting (3.11) with $g^{mi}g^{pj}g^{qk}$ yields

$$L^{ijk} = \lambda\Big\{J^i h^{jk} + J^j h^{ki} + J^k h^{ij}\Big\} + 3\mu J^i J^j J^k. \tag{3.12}$$

(3.11)×(3.12) implies that

$$\begin{aligned} ||\mathbf{L}||^2 = \lambda^2\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\}\Big\{J^i h^{jk} + J^j h^{ki} + J^k h^{ij}\Big\} \\ +3\mu\lambda J_i J_j J_k\Big\{J^i h^{jk} + J^j h^{ki} + J^k h^{ij}\Big\} + 9||\mathbf{J}||^6 \\ +3\mu\lambda\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\}J^i J^j J^k \end{aligned}$$

which yields

$$||\mathbf{L}|| = \sqrt{9||\mathbf{J}||^4 + 18\mu\lambda||\mathbf{J}||^2 + 3(n+1)\lambda^2}\ ||\mathbf{J}||. \tag{3.13}$$

The equation (3.13) denotes the relation between the norm of Landsberg curvature and mean Landsberg curvature of a semi-P-reducible Finsler metric. Also, it follows that for a semi-P-reducible Finsler metric $\mathbf{L} = 0$ if and only if $\mathbf{J} = 0$.

Now, we study the converse of Theorem 1.2 and prove the following.

Theorem 3.2. *Let (M, F) be a semi-P-reducible Finsler manifold. Suppose that F has relatively isotropic mean Landsberg curvature*

$$\mathbf{J} = cF\mathbf{I},$$

where $c = c(x)$ is a non-zero scalar function on M. Then the following are equivalent

(1) *F has relatively isotropic Landsberg curvature*

$$\mathbf{L} = cF\mathbf{C},$$

where $c = c(x)$ is a non-zero scalar function on M.

(2) *F is a semi-C-reducible metric with*

$$P = (n+1)\lambda, \qquad Q = 2\mu c^2||\mathbf{I}||^2F^2.$$

Proof. Let F be a semi-P-reducible metric

$$L_{ijk} = \lambda\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\} + 3\mu J_i J_j J_k. \tag{3.14}$$

Plugging $\mathbf{J} = cF\mathbf{I}$ in (3.14) yields

$$L_{ijk} = cF\Bigg\{\lambda\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} + 3\mu c^2F^2 I_i I_j I_k\Bigg\}. \tag{3.15}$$

By (3.15), F is a isotropic Landsberg metric if and only if the following holds

$$C_{ijk} = \lambda\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} + 3\mu c^2 F^2 I_i I_j I_k, \tag{3.16}$$

which means that F must be a semi-C-reducible metric with

$$P = (n+1)\lambda, \qquad Q = 3\mu c^2 ||\mathbf{I}||^2 F^2.$$

This completes the proof. □

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