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On projectively related spherically symmetric metrics in Finsler geometry

Hassan Sadeghi[a*] and Morad Bahadori[b]

[a]Faculty of Science, Department of Mathematics
University of Qom, Qom, Iran
[b]Faculty of Science, Department of Mathematics
University of Qom, Qom, Iran

E-mail: sadeghihassan64@gmail.com
E-mail: moradbahadori3@gmail.com

Abstract. Inspired by the notion of projectively related spherically symmetric metrics, we study the class of Finsler metrics whose geodesics have the same shape with a difference in rotation or reflection of their graphs. This class of metrics contains the class of projectively related Finsler metrics. First, we characterize the class of Randers metrics, (α, β)-metrics and spherically symmetric metrics in this class of metrics.

Keywords: Projectively related metrics, (α, β)-metric, Randers metric, spherically symmetric metric.

1. Introduction

The 4-th Problem of Hilbert is finding Finsler metrics on an open subset $U \subseteq \mathbb{R}^n$ whose geodesics are straight lines. The Finsler metrics which satisfy the mentioned condition are called projective Finsler metrics (see [6], [7], [9], [11], [14], [16]). Hamel obtained a system of PDE's that characterizes projective Finsler metrics [4]. He showed that a Finsler metric $F = F(x, y)$ on $U \subseteq \mathbb{R}^n$ is projective if and only if its geodesic coefficients G^i are given by

$$G^i(x, y) = P(x, y)y^i,$$

*Corresponding Author

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where $P : TU \to \mathbb{R}$ is a positively homogeneous function of degree one with respect to y. The function $P = P(x, y)$ is called the projective factor of F.

In Riemannian geometry, two Riemannian metrics α and $\bar{\alpha}$ on a manifold M are projectively related metrics if the following relation holds

$$G^i_\alpha(x, y) = \bar{G}^i_{\bar{\alpha}}(x, y) + \theta(x, y)y^i,$$

where G^i_α and $\bar{G}^i_{\bar{\alpha}}$ are the geodesic coefficients of α and $\bar{\alpha}$, respectively, $\theta = \theta(x)y^k$ is a closed 1-form on M, and (x^i, y^i) denotes the local coordinates in TM.

In Finsler geometry, two Finsler metrics $F = F(x, y)$ and $\bar{F} = \bar{F}(x, y)$ on a manifold M are projectively related if the following holds

$$G^i(x, y) = \bar{G}^i(x, y) + P(x, y)y^i,$$

where G^i and $\bar{G}^i$ are the geodesic coefficients of F and $\bar{F}$, respectively. In this case, F and $\bar{F}$ have the same geodesics as point sets [2][8][10].

Let F be a Finsler metric defined on a convex domain $\Omega \subseteq \mathbb{R}^n$. If the orthogonal transformations of $\mathbb{R}^n$ act as isometry of (Ω, F), then (Ω, F) is called a spherically symmetric space [17]. Let $|.|$ and $<,>$ denote the Euclidean norm and inner product in $\mathbb{R}^n$, respectively. In [17], Zhou showed that (Ω, F) is spherically symmetric if and only if

$$F(x, y) = u\phi(r, s),$$

where

$$r := |x|, \quad u := |y|, \quad s := \frac{< x, y >}{|y|}.$$

The geodesic spray coefficients of $F = u\phi(r, s)$ is given by

$$G^i = uPy^i + u^2 Qx^i, \tag{1.1}$$

where

$$\begin{aligned} P &:= \frac{1}{\phi}\left((s^2 - r^2)\phi_s - s\phi\right) Q + \frac{1}{2r\phi}\left(s\phi_r + r\phi_s\right), \\ Q &:= \frac{-\phi_r + s\phi_{rs} + r\phi_{ss}}{2r[\phi - s\phi_s + (r^2 - s^2)\phi_{ss}]}. \end{aligned}$$

See [13]. Define Randers metrics $F(x, y) = u\phi(r, s)$ and $\bar{F} = u\bar{\phi}(r, s)$ by

$$\phi(r, s) := \frac{1}{1 + r^2} + p(r)s, \tag{1.2}$$

$$\bar{\phi}(r, s) := \frac{1}{1 + \lambda^2 r^2} + q(r)s, \tag{1.3}$$

where $p := p(r)$ and $q := q(r)$ are two functions in $\mathbb{R}$. By (1.1), we get

$$G^i(x, y) = uP(r, s)y^i + \frac{u^2}{1 + r^2}x^i, \tag{1.4}$$

$$\bar{G}^i(x, y) = u\bar{P}(r, s)y^i + \frac{u^2\lambda^2}{1 + \lambda^2 r^2}x^i, \tag{1.5}$$

where G^i and $\bar{G}^i$ are the spray coefficients of F and $\bar{F}$, respectively. (1.4) and (1.5) imply that

$$G^i(x,y) = \lambda \bar{G}^i(\lambda x, y) + \mathcal{P}(x,y)y^i, \tag{1.6}$$

where λ is a real constant and $\mathcal{P} = \mathcal{P}(x,y)$ is a positively homogeneous scalar function of degree one with respect to y (see [12]).

By definition, every projectively related Finsler metrics satisfy (1.6) with $\lambda = 1$. The Randers metrics defined by (1.3) are special Finsler metrics, also. Therefore the following is a natural question:

"Whenever two arbitrary Finsler metrics satisfy (1.6)?"

Since the idea of Finsler metrics satisfying (1.6) arises from two special Randers metrics, we find the necessary and sufficient conditions under which two arbitrary Randers metrics satisfying (1.6).

Theorem 1.1. *Two Randers metrics $F = \alpha + \beta$ and $\bar{F} = \bar{\alpha} + \bar{\beta}$ on $\mathbb{R}^n$ satisfy (1.6) if and only if one of the following holds*

(i) $\alpha(x,y) = c\bar{\alpha}(\lambda x, y)$, and β and $\bar{\beta}$ satisfy

$$s^i{}_j(x) = \lambda/c \ \bar{s}^i{}_j(x),$$

where $c = c(x)$ is a positive scalar function on M.

(ii) α and $\bar{\alpha}$ satisfy (1.6), and β and $\bar{\beta}$ are closed 1-forms.

Then, we find the necessary and sufficient condition under which an (α,β)-metric and a Randers metric satisfy (1.6).

Theorem 1.2. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric and $\bar{F} = \bar{\alpha} + \bar{\beta}$ be a Randers metric on $\mathbb{R}^n$ $(n \geq 3)$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\bar{\alpha} = \sqrt{\bar{a}_{ij}(x)y^iy^j}$ are two Riemannian metrics, $\beta = b_i(x)y^i$ and $\bar{\beta} = \bar{b}_i(x)y^i$ are two nonzero one forms on $\mathbb{R}^n$. Suppose that ϕ satisfies*

$$\phi = ks\int \frac{(p + rs^2)^{\frac{-1}{2r}}}{s^2}\, ds + ts, \tag{1.7}$$

where k, p, r and t are constants. Then F and $\bar{F}$ satisfy (1.6) if and only if the following hold

(i) $\bar{\beta}$ is a closed form;

(ii) $b_{i|j} = 2\tau\Big\{(p + b^2)a_{ij} + (r-1)b_ib_j\Big\}$;

(iii) $G^i_\alpha(x,y) = \lambda G^i_{\bar{\alpha}}(\lambda x, y) - \tau\alpha^2 b^i + \theta y^i$,

where $\theta = a_i(x)y^i$ *is a* 1*-form and* $\tau = \tau(x)$ *is a scalar function on* M.

We remark that (1.7) comes from the following ODE

$$\phi - s\phi' = (p + rs^2)\phi''$$

which have many solutions in the class of (α, β)-metrics. See the Section 4.

Also, the Randers metrics (1.3) which satisfy (1.6) are written in the form of spherically symmetric Finsler metrics. In order to find the non-trivial examples of Finsler metrics satisfying (1.6), we come back to spherically symmetric metrics and prove the following.

Theorem 1.3. *Let* $F = u\phi(r,s)$ *and* $\bar{F} = u\bar{\phi}(r,s)$ *be two spherically symmetric Finsler metrics on* $\mathbb{R}^n$. *Then the spray coefficients of* F *and* $\bar{F}$ *satisfy (1.6) if and only if there exists a scalar function* $g := g(x)$ *on* $\mathbb{R}$ *such that*

$$\phi(r,s) = s\int \frac{g\Big(\lambda\sqrt{r^2-s^2}\,\big[\bar{\phi}(\lambda r, \lambda s) - \lambda s\bar{\phi}_s(\lambda r, \lambda s)\big]\Big)}{s^2\sqrt{r^2-s^2}}ds + c(r)s, \qquad (1.8)$$

where $\phi_s := \partial\phi/\partial s$ *and* $c := c(t)$ *is a scalar function on* $\mathbb{R}$.

2. Preliminaries

For an n-dimensional Finsler manifold (M, F), there is a special vector field $\mathbf{G}$ which is induced by F on $TM_0 := TM\backslash\{0\}$. In a standard coordinates (x^i, y^i) for TM_0, it is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where

$$G^i(x,y) := \frac{g^{il}}{4}\Big\{\frac{\partial^2 F^2}{\partial x^k \partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\Big\}$$

The homogeneous scalar functions G^i are called the geodesic coefficients of F. The vector field $\mathbf{G}$ is called the associated spray to (M, F). The projection of an integral curve of spray $\mathbf{G}$ is called a geodesic in M. A curve $c = c(t)$ is a geodesic of F if and only if its coordinates $(c^i(t))$ satisfy the ODE

$$\ddot{c}^i + 2G^i(\dot{c}) = 0.$$

For more details, see [3].

The Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1 \wedge \cdots \wedge dx^n$ related to F is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\Big\{(y^i) \in \mathbb{R}^n \ \Big| \ F\Big(y^i \frac{\partial}{\partial x^i}|_x\Big) < 1\Big\}},$$

where $\mathbb{B}^n(1)$ denotes the unit ball in $\mathbb{R}^n$.

The distortion $\tau = \tau(x, y)$ on TM associated with the Busemann-Hausdorff volume form on M, i.e., $dV_{BH} = \sigma(x)dx^1 \wedge dx^2 ... \wedge dx^n$, is defined by following

$$\tau(x, y) = \ln \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)}.$$

Then the S-curvature is defined by

$$\mathbf{S}(x, y) = \frac{d}{dt}\Big[\tau\big(c(t), \dot{c}(t)\big)\Big]_{t=0},$$

where $c = c(t)$ is the geodesic with $c(0) = x$ and $\dot{c}(0) = y$. In a local coordinates, the S-curvature is given by

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial(\ln \sigma)}{\partial x^m}.$$

A Finsler metric F on an n-dimensional manifold M is said to be of isotropic S-curvature if

$$\mathbf{S} = (n + 1)\sigma F,$$

where $\sigma = \sigma(x)$ is a scalar function on M.

For $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ by

$$\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}|_x,$$

where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l},$$

$u = u^i \frac{\partial}{\partial x^i}|_x$, $v = v^i \frac{\partial}{\partial x^i}|_x$ and $w = w^i \frac{\partial}{\partial x^i}|_x$. The non-Riemannian quantity $\mathbf{B}$ is called the Berwald curvature. F is called a Berwald metric if $\mathbf{B} = \mathbf{0}$.

For a non-zero vector $y \in T_xM_0$, define $\mathbf{E}_y : T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{E}_y(u, v) := E_{ij}(y)u^i v^j,$$

where

$$E_{ij} := \frac{1}{2} B^m{}_{ijm},$$

Then, $\mathbf{E}$ is called mean Berwald curvature and F is called a weakly Berwald metric if $\mathbf{E} = \mathbf{0}$.

Also, by using the Berwald and mean Berwald curvatures of F, one can define the Douglas curvature $\mathbf{D}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{D}_y(u,v,w) := D^i{}_{jkl}(y)u^iv^jw^k\frac{\partial}{\partial x^i}|_x$, where

$$D^i{}_{jkl} := B^i{}_{jkl} - \frac{2}{n+1}\Big\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk,l}y^i\Big\}.$$

The Finsler metric F satisfies $\mathbf{D} = 0$ is called a Douglas metric.

3. Proof of Theorem 1.1

An (α,β)-metric is defined by

$$F := \alpha\phi(s), \qquad s = \frac{\beta}{\alpha},$$

where $\phi = \phi(s)$ is a scalar function on $(-b_0, b_0)$, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. For an (α,β)-metric $F := \alpha\phi(s)$, $s = \beta/\alpha$, one can define

$$b_{i;j}\theta^j := db_i - b_j\theta_i^j,$$

where $\theta^i := dx^i$ and $\theta_i^j := \Gamma^j_{ik}dx^k$ are the Levi-Civita connection forms of Riemannian metric α. Put

$$\begin{gathered} r_{ij} := \tfrac{1}{2}\big(b_{i;j} + b_{j;i}\big), \quad s_{ij} := \tfrac{1}{2}\big(b_{i;j} - b_{j;i}\big), \\ r_j := b^ir_{ij}, \quad r := b^ib^jr_{ij}, \quad s_j := b^is_{ij}, \quad r_0 := r_jy^j, \\ s_0 := s_jy^j, \quad r_{i0} := r_{ij}y^j, \quad r_{00} := r_{ij}y^iy^j, \quad s_{i0} := s_{ij}y^j, \\ s^i{}_j := a^{im}s_{mj}, \quad r^i{}_j := a^{im}r_{mj}, \end{gathered}$$

where $a^{ij} = (a_{ij})^{-1}$ and $b^i := a^{ij}b_j$.

Let G^i and G^i_α denote the geodesic coefficients of F and α, respectively. By a direct computation, one gets the following formula

$$G^i = G^i_\alpha + By^i + T^i, \tag{3.1}$$

where

$$\begin{aligned} B &:= \Theta\alpha^{-1}(r_{00} - 2Q\alpha s_0), \\ T^i &:= \alpha Qs^i_0 + \psi(r_{00} - 2Q\alpha s_0)b^i, \\ Q &:= \frac{\phi'}{\phi - s\phi'}, \\ \Theta &= \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\big[(\phi - s\phi') + (b^2 - s^2)\phi''\big]}, \\ \psi &:= \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}. \end{aligned}$$

For a Finsler metric F with spray coefficients G^i, let us define the following quantity.

$$\Pi^i_F := G^i - \frac{1}{n+1} G^m{}_m y^i.$$

Then we prove the following.

Lemma 3.1. *Two Finsler metrics F and $\bar{F}$ on a manifold M satisfy (1.6) if and only if the following holds*

$$\Pi^i_F(x,y) = \lambda \Pi^i_{\bar{F}}(\lambda x, y). \tag{3.2}$$

Proof. Let the spray coefficients of F and $\bar{F}$ satisfy

$$G^i(x,y) = \lambda \bar{G}^i(\lambda x, y) + P(x,y) y^i \tag{3.3}$$

Thus

$$P(x,y) = \frac{1}{n+1}[G^m_m(x,y) - \lambda \bar{G}^m_m(\lambda x, y)] \tag{3.4}$$

Substitution (3.4) in (3.3), we get (3.2). □

Next we calculate the quantity Π^i_F for an arbitrary (α,β)-metric.

Lemma 3.2. *For an (α,β)- metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, on an n-dimensional manifold M, the following holds*

$$\begin{aligned}\Pi^i_F = \Pi^i_\alpha + \alpha Q s^i{}_0 \quad + \quad & \Psi \Gamma b^i - \frac{1}{n+1}\Big[\Psi' \alpha^{-1}(b^2 - s^2)\Gamma + Q' s_0 \\ & +2\Psi\big(r_0 - Q'(b^2-s^2)s_0 - Qss_0\big)\Big] y^i,\end{aligned} \tag{3.5}$$

where

$$\Gamma := r_{00} - 2Q\alpha s_0.$$

Proof. By a direct computation, we obtain from (3.1) that

$$\Pi^i_F = \Pi^i_\alpha + T^i - \frac{1}{n+1}\Big[(\alpha Q s^m{}_0)_{y^m} + [\Psi(r_{00} - 2Q\alpha s_0) b^m]_{y^m}\Big] y^i. \tag{3.6}$$

It easy to see that

$$\begin{aligned}(\alpha Q s^m{}_0)_{y^m} &= \alpha^{-1} y_m Q s^m{}_0 + \alpha^{-2} Q'[b_m \alpha^2 - \beta y_m] s^m{}_0 \\ &= Q' s_0,\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}[\Psi(r_{00} - 2Q\alpha s_0) b^m]_{y^m} \quad = \quad & \Psi' \alpha^{-1}(b^2 - s^2)(r_{00} - 2Q\alpha s_0) \\ & +2\Psi(r_0 - Q'(b^2 - s^2)s_0 - Qss_0).\end{aligned} \tag{3.8}$$

Substituting (3.7) and (3.8) in (3.6), we get (3.5). □

Proof of Theorem 1.1: Let $F=\alpha+\beta$ and $\bar{F}=\bar{\alpha}+\bar{\beta}$ be two Randers metrics and G^i and $\bar{G}^i$ be the spray coefficients of F and $\bar{F}$, respectively. Suppose that G^i_α and $\bar{G}^i_\alpha$ are the spray coefficients of α and $\bar{\alpha}$, respectively. By Lemma 3.1, we have

$$\Pi^i_F(x,y)=\lambda\Pi^i_{\bar{F}}(\lambda x,y) \tag{3.9}$$

Substituting $\phi=1+s$ and $\bar{\phi}=1+\bar{s}$ in (3.5), we obtain

$$\Pi^i_F \quad = \quad \Pi^i_\alpha+\alpha s^i{}_0, \tag{3.10}$$

and

$$\Pi^i_{\bar{F}} \quad = \quad \Pi^i_{\bar{\alpha}}+\bar{\alpha}\bar{s}^i{}_0. \tag{3.11}$$

By (3.9), (3.10) and (3.11) we get

$$\Pi^i_\alpha(x,y)+\alpha(x,y)s^i{}_0(x,y)=\lambda\Pi^i_\alpha(\lambda x,y)+\lambda\bar{\alpha}(\lambda x,y)\bar{s}^i{}_0(\lambda x,y). \tag{3.12}$$

It is easy to see that

$$\Pi^i_\alpha(x,-y)=\Pi^i_\alpha(x,y),$$
$$\Pi^i_{\bar{\alpha}}(\lambda x,-y)=\Pi^i_{\bar{\alpha}}(\lambda x,y).$$

Thus by replacing y by $-y$ in (3.12), we obtain

$$\Pi^i_\alpha(x,y)=\lambda\Pi^i_{\bar{\alpha}}(\lambda x,y), \tag{3.13}$$

and

$$\alpha(x,y)s^i_0(x,y)=\lambda\bar{\alpha}(\lambda x,y)\bar{s}^i_0(\lambda x,y). \tag{3.14}$$

It follows from (3.13) that α and $\bar{\alpha}$ satisfy (1.6). If

$$\alpha(x,y)=c(x)\bar{\alpha}(\lambda x,y)$$

then from (3.14) we get

$$s^i{}_j(x)=\frac{\lambda}{c(x)}\bar{s}^i{}_j(\lambda x).$$

Otherwise

$$s^i{}_j(x)=\bar{s}^i{}_j(x)=0.$$

This completes the proof. □

A Finsler metric F is said to be isotropic Berwald metric if its Berwald curvature is in the following form

$$B^i{}_{jkl}=\sigma\Big\{F_{y^jy^k}\delta^i{}_l+F_{y^ky^l}\delta^i{}_j+F_{y^ly^j}\delta^i{}_k+F_{y^jy^ky^l}y^i\Big\}, \tag{3.15}$$

for some scalar function $\sigma=\sigma(x)$ on M. Consider following Finsler metric on the unit ball $\mathbb{B}^n\subset\mathbb{R}^n$,

$$F(y):=\frac{\sqrt{|y|^2-(|x|^2|y|^2-<x,y>^2)}+<x,y>}{1-|x|^2},\quad y\in T_x\mathbb{B}^n=\mathbb{R}^n,$$

where $|.|$ and $<,>$ denote the Euclidean norm and inner product in $\mathbb{R}^n$, respectively. F is called the Funk metric which is a Randers metric on $\mathbb{B}^n$. One can show that F is positively complete on $\mathbb{B}^n$. The Funk metrics are also nontrivial isotropic Berwald metrics $\sigma = \frac{1}{2}$. Shen proved that every Berwald metric satisfies $\mathbf{S} = 0$. Then Tayebi-Rafie Rad generalized his result and proved the following

Lemma 3.3. ([15]) *Every Finsler metric on an n-dimensional manifold M with isotropic Berwald curvature (3.15) has isotropic S-curvature $\mathbf{S} = (n+1)\sigma F$.*

The converse of Lemma 3.3 is not true in general. It is interesting to find some Finsler metrics that these notions of curvatures for those are equal. Here, we study two Randers metrics satisfy (1.6) such that the one has isotropic Berwald curvature. Then, we prove the following.

Theorem 3.4. *Let $F = \alpha + \beta$ and $\bar{F} = \bar{\alpha} + \bar{\beta}$ be two Randers metrics satisfy (1.6). Suppose that F has isotropic Berwald curvature. Then $\bar{F}$ has isotropic Berwald curvature if and only if it has isotropic S-curvature.*

Proof. Suppose that $\bar{F}$ has isotropic S-curvature

$$\bar{\mathbf{S}} = (n+1)\bar{c}\bar{F},$$

where $\bar{c} = \bar{c}(x)$ is a scaler function on $\mathbb{R}^n$. Then

$$\bar{E}_{ij} = \frac{n+1}{2}\bar{c}\bar{F}_{y^iy^j}. \tag{3.16}$$

Since F and $\bar{F}$ satisfy (1.6), then

$$\bar{G}^i(x,y) = \lambda G^i(\lambda x, y) + P(x,y)y^i. \tag{3.17}$$

By assumption, we have

$$B^i{}_{jkl} = c\Big\{F_{y^jy^k}\delta^i_l + F_{y^jy^l}\delta^i_k + F_{y^ky^l}\delta^i_j + F_{y^jy^ky^l}y^i\Big\}$$

which implies that

$$E_{ij} = \frac{n+1}{2}cF_{y^iy^j}. \tag{3.18}$$

By (3.16), (3.17) and (3.18), we get

$$\bar{c}\bar{F}_{y^iy^j} = c\lambda F_{y^iy^j} + P_{y^iy^j}$$

which yields

$$\bar{c}\bar{F}_{y^iy^jy^k} = c\lambda F_{y^iy^jy^k} + P_{y^iy^jy^k}.$$

Thus

$$\begin{aligned}\bar{B}^i_{jkl} = \frac{\partial^3 \bar{G}^i}{\partial y^j \partial y^k \partial y^l} &= \lambda\Big\{B^i_{jkl} + P_{y^jy^k}\delta^i_l + P_{y^jy^l}\delta^i_k + P_{y^ky^{kl}}\delta^i_j + P_{y^jy^ky^ly^i}\Big\}\\ &= \bar{c}\Big\{\bar{F}_{y^ly^k}\delta^i_j + \bar{F}_{y^jy^l}\delta^i_k + \bar{F}_{y^jy^k}\delta^i_l + \bar{F}_{y^jy^ky^l}y^i\Big\}.\end{aligned}$$

This means that $\bar{F}$ has isotropic Berwald curvature. By Lemma 3.3, the converse is trivial. $\Box$

4. Proof of Theorem 1.2

In [1], Bácsó-Cheng-Shen studied the ODE (1.7). For certain values of p and r, they found some solutions of (1.7) which can be expressed in terms of elementary functions as follows:

(i) For $r = -1$ and $p = \pm 1$, we get

$$\phi = \begin{cases} \sqrt[2]{1-s^2} + s\arctan(\frac{s}{1-s^2}) + \varepsilon s & \text{if } \ p = 0, \\ \sqrt[2]{1+s^2} - \ln(s + \sqrt[2]{1+s^2})^s + \varepsilon s & \text{if } \ p = -1. \end{cases} \tag{4.1}$$

(ii) For $r = 1$ and $p = \pm 1$, we have

$$\phi = \begin{cases} \sqrt[2]{1+s^2} + \varepsilon s & \text{if } \ p = 1, \\ \sqrt[2]{1-s^2} + \varepsilon s & \text{if } \ p = -1. \end{cases} \tag{4.2}$$

(iii) For $r = 1$ and $p = \pm\frac{1}{3}$, we get

$$\phi = \begin{cases} \sqrt{1+s^2} + \frac{s^2}{\sqrt{1+s^2}} + \varepsilon s, & \text{if } \ p = \frac{1}{3}, \\ \sqrt{1-s^2} - \frac{s^2}{\sqrt{1-s^2}} + \varepsilon s, & \text{if } \ p = \frac{-1}{3} \end{cases} \tag{4.3}$$

Here, we consider the class of (α, β)-metrics $F{=}\alpha\phi(s)$, where $\phi{=}\phi(s)$ satisfies the following equation

$$\phi - s\phi' = (p + rs^2)\phi'', \tag{4.4}$$

where r and p are constants. (4.4) is equal to

$$\phi = ks \int \frac{(p + rs^2)^{\frac{-1}{2r}}}{s^2} \mathrm{d}s + ts, \tag{4.5}$$

where k and t are constants.

In [5], Li-Shen-Shen considered the class of Douglas (α, β)-metrics and proved the following.

Theorem 4.1. (Li-Shen [5]) *Let $F = \phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$ $(n \geq 3)$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i \neq 0$. Let $b := ||\beta_x||_\alpha$.*

Suppose that the following conditions hold: (a) β is not parallel with respect to α, (b) F is not of Randers type, and (c) $db = 0$ everywhere or $b =$ constant on $\mathcal{U}$. Then F is a Douglas metric on $\mathcal{U}$ if and only if the function $\phi = \phi(s)$ satisfies the following ODE:

$$\Big\{1 + (k_1 + k_2 s^2)s^2 + k_3 s^2\Big\}\phi''(s) = (k_1 + k_2 s^2)\Big\{\phi(s) - s\phi'(s)\Big\} \tag{4.6}$$

and β satisfies

$$b_{i|j} = 2\tau\Big\{(1+k_1b^2)a_{ij} + (k_2b^2+k_3)b_ib_j\Big\}, \tag{4.7}$$

where $\tau = \tau(x)$ is a scalar function on $\mathcal{U}$ and k_1, k_2 and k_3 are constants with $(k_2, k_3) \neq (0,0)$.

Let us consider the (α,β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, where $\phi = \phi(s)$ satisfying (4.4). If β satisfies

$$b_{i|j} = 2\tau\Big\{(p+b^2)a_{ij} + (r-1)b_ib_j\Big\}, \tag{4.8}$$

where $\tau = \tau(x)$ is a scaler function, then β is a closed 1-form. In this case, F is a Douglas metric. See page 22 in [1].

Proof of Theorem 1.2: Suppose that F and $\bar{F}$ satisfy (1.6). By Lemma 3.1 the following holds

$$\Pi^i_F(x,y) = \lambda\ \Pi^i_{\bar{F}}(\lambda x, y). \tag{4.9}$$

Substituting

$$\phi = ks\int \frac{(p+rs^2)^{\frac{-1}{2r}}}{s^2}\mathrm{d}s + ts, \quad \text{and} \quad \bar{\phi} = 1+\bar{s}$$

into (3.5) yields

$$\begin{aligned}\Pi^i_F \;=\;& \Pi^i_\alpha + \alpha A s^i{}_0 + C(r_{00} - 2A\alpha s_0)b^i \\ & -\frac{1}{n+1}\Big\{D\alpha^{-1}(b^2-s^2)(r_{00}-2A\alpha s_0) + Bs_0 \\ & \qquad +2C(r_0 - B(b^2-s^2)s_0 - Ass_0)\Big\}y^i,\end{aligned} \tag{4.10}$$

$$\Pi^i_{\bar{F}} \;=\; \Pi^i_{\bar{\alpha}} + \bar{\alpha}\bar{s}^i_0. \tag{4.11}$$

where

$$\begin{aligned}
A &= -(p+rs^2)\int \frac{(p+rs^2)^{\frac{-1}{2r}}}{s^2}\mathrm{d}s - tk^{-1}(p+rs^2)^{\frac{1}{2r}} - s^{-1},\\
B &= -s(p+rs^2)^{\frac{1-2r}{2r}}\int \frac{(p+rs^2)^{\frac{-1}{2r}}}{s^2}\mathrm{d}s - tk^{-1}s(p+rs^2)^{\frac{1-2r}{2r}},\\
C &= \frac{1}{2(p+rs^2+(b^2-s^2))},\\
D &= \frac{(1-r)s}{(p+rs^2+(b^2-s^2))^2}.
\end{aligned}$$

It follows from (4.9), (4.10) and (4.11) that

$$\Pi^i_\alpha + \alpha A s^i{}_0 + C(r_{00} - 2A\alpha s_0)b^i - \frac{1}{n+1}\Big\{ D\alpha^{-1}(b^2 - s^2)(r_{00} - 2A\alpha s_0)$$
$$+ Bs_0 + 2C\big[r_0 - B(b^2 - s^2)s_0 - Ass_0\big]\Big\} y^i$$
$$= \lambda\ \Pi^i_{\bar\alpha}(\lambda x, y) + \lambda\ \bar\alpha(\lambda x, y)\bar s^i_0(\lambda x, y). \tag{4.12}$$

Replacing y^i with $-y^i$ in (4.12), one can see that

$$tk^{-1}\Big\{2Cs_0 b^i + \frac{2}{n+1} m s_0 y^i - s^i{}_0\Big\} = \lambda\bar\alpha(\lambda x, y)\alpha^{-1} W^{1-\frac{1}{2r}}\ \bar s^i{}_0(\lambda x, y), \tag{4.13}$$

where

$$W := p + rs^2 \tag{4.14}$$
$$m := \frac{(p + rs^2)(r-1)(b-s)(b+s)\beta}{(p + rs^2 + b^2 - s^2)^2\alpha^2}. \tag{4.15}$$

The left side of (4.13) is rational in y while its right side is irrational. So, it follows that

$$s^i{}_0 = \bar s^i{}_0 = 0.$$

Substituting $\bar s^i{}_0 = 0$ into (4.12) implies that

$$\Pi^i_F(x, y) = \lambda\ \Pi^i_{\bar\alpha}(\lambda x, y). \tag{4.16}$$

By (4.16), it follows that F is a Douglas metric. Then, by Theorem 4.1 we have

$$b_{i|j} = 2\tau\Big\{(p + b^2)a_{ij} + (r-1)b_i b_j)\Big\}. \tag{4.17}$$

Substituting

$$\bar s^i{}_0 = 0 \quad \text{and} \quad b_{i|j} = 2\tau\Big\{(p + b^2)a_{ij} + (r-1)b_i b_j\Big\}$$

into (4.12), we have

$$\Pi^i_F(x, y) = \lambda\Pi^i_{\bar\alpha}(\lambda x, y) - \tau\alpha^2 b^i + \frac{2}{n+1}\tau\beta y^i. \tag{4.18}$$

Then by (4.18), one can conclude that

$$G^i_\alpha(x, y) = \lambda G^i_{\bar\alpha}(\lambda x, y) - \tau\alpha^2 b^i + \theta y^i, \tag{4.19}$$

where $\theta = a_i(x)y^i$ is a 1-form on M. $\Box$

5. Proof of Theorem 1.3

In this section, we are going to prove Theorem 1.3.

Proof of Theorem 1.3: Let G^i and $\bar{G}^i$ denote the spray coefficients of F and $\bar{F}$, respectively. Then by (1.1), we have

$$G^i = uPy^i + u^2Qx^i, \tag{5.1}$$

$$\bar{G}^i = u\bar{P}y^i + u^2\bar{Q}x^i, \tag{5.2}$$

where

$$P = -\frac{1}{\phi}\Big(s\phi + (r^2 - s^2)\phi_s\Big)Q + \frac{1}{2r\phi}\Big(s\phi_r + r\phi_s\Big),$$

$$Q = \frac{1}{2r}\frac{-\phi_r + s\phi_{rs} + r\phi_{ss}}{\phi - s\phi_s + (r^2 - s^2)\phi_{ss}},$$

$$\bar{P} = -\frac{1}{\bar{\phi}}\Big(s\bar{\phi} + (r^2 - s^2)\bar{\phi}_s\Big)\bar{Q} + \frac{1}{2r\bar{\phi}}\Big(s\bar{\phi}_r + r\bar{\phi}_s\Big),$$

$$\bar{Q} = \frac{1}{2r}\frac{-\bar{\phi}_r + s\bar{\phi}_{rs} + r\bar{\phi}_{ss}}{\bar{\phi} - s\bar{\phi}_s + (r^2 - s^2)\bar{\phi}_{ss}}.$$

Since the spray coefficients of F and $\bar{F}$ satisfy (1.6), then there exists a scalar function $\mathbb{P} = \mathbb{P}(x, y)$ defined on $\mathbb{R}^n_0$ such that the following holds

$$G^i(x, y) = \lambda\bar{G}^i(\lambda x, y) + \mathbb{P}(x, y)y^i. \tag{5.3}$$

By (5.1), (5.2) and (5.3) we get

$$\Big(\lambda u\bar{P}(\lambda r, \lambda s) - uP(r, s) + \mathbb{P}(x, y)\Big)y^i + u^2\Big(\lambda^2\bar{Q}(\lambda r, \lambda s) - Q(r, s)\Big)x^i = 0. \tag{5.4}$$

It follows from (5.4) that

$$Q(r, s) = \lambda^2\bar{Q}(\lambda r, \lambda s), \tag{5.5}$$

(5.5) is equal to following

$$\frac{-\lambda\bar{\phi}_r(\lambda r, \lambda s) + \lambda^2 s\bar{\phi}_{rs}(\lambda r, \lambda s) + \lambda^2 r\bar{\phi}_{ss}(\lambda r, \lambda s)}{\bar{\phi}(\lambda r, \lambda s) - \lambda s\bar{\phi}_s(\lambda r, \lambda s) + \lambda^2(r^2 - s^2)\bar{\phi}_{ss}(\lambda r, \lambda s)} = \frac{-\phi_r(r, s) + s\phi_{rs}(r, s) + r\phi_{ss}(r, s)}{\phi(r, s) - s\phi_s(r, s) + (r^2 - s^2)\phi_{ss}(r, s)}. \tag{5.6}$$

Put

$$A(r, s) := \sqrt{r^2 - s^2}\big[\phi(r, s) - s\phi_s(r, s)\big],$$

$$B(r, s) := \lambda\sqrt{r^2 - s^2}\big[\bar{\phi}(\lambda r, \lambda s) - \lambda s\bar{\phi}_s(\lambda r, \lambda s)\big].$$

A direct computation yields

$$\frac{sA_r(r, s) + rA_s(r, s)}{(r^2 - s^2)A_s(r, s)} = \frac{\Pi}{\Omega}, \tag{5.7}$$

$$\frac{sB_r(r, s) + rB_s(r, s)}{(r^2 - s^2)B_s(r, s)} = \frac{\bar{\Pi}}{\bar{\Omega}}, \tag{5.8}$$

where

$$\begin{aligned}
\Pi &:= -\phi_r(r,s) + s\phi_{rs}(r,s) + r\phi_{ss}(r,s),\\
\bar{\Pi} &:= \lambda^2 s\bar{\phi}_{rs}(\lambda r,\lambda s) - \lambda\bar{\phi}_r(\lambda r,\lambda s) + \lambda^2 r\bar{\phi}_{ss}(\lambda r,\lambda s),\\
\Omega &:= \phi(r,s) - s\phi_s(r,s) + (r^2-s^2)\phi_{ss}(r,s),\\
\bar{\Omega} &:= \bar{\phi}(\lambda r,\lambda s) - \lambda s\bar{\phi}_s(\lambda r,\lambda s) + \lambda^2(r^2-s^2)\bar{\phi}_{ss}(\lambda r,\lambda s).
\end{aligned}$$

By (5.6), (5.7) and (5.8), one can see that

$$\frac{A_r(r,s)}{A_s(r,s)} = \frac{B_r(r,s)}{B_s(r,s)}. \tag{5.9}$$

So there exist a function $g := g(x)$ such that

$$A(r,s) = g(B(r,s)). \tag{5.10}$$

It is easy to see that (5.10) is equal to (1.8). □

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