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Invariant vector field on a homogeneous Finsler space with special (α, β)-metric

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Abstract. In a Finsler spaces, we consider a special (α, β)-metric L satisfying $L^2(\alpha, \beta) = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2$, where c_i are constant. In this paper, the existence of invariant vector fields on a special homogeneous (α, β)-space with L metric is proved. Then we study geodesic vectors and investigate the set of all homogeneous geodesics of invariant (α, β)-metric L on homogeneous spaces and simply connected.

Keywords: Homogeneous Finsler space, L-metric, invariant vector field, hypercomplex manifold, Geodesic vector.

1. INTRODUCTION

The geometry of invariant Finsler structures on homogeneous manifolds is an interesting topic in Finsler geometry, which has been studied by some Finsler geometers in recent years (for example, see [1, 14, 15, 19]. An important family of Finsler metrics is the family of (α, β)-metrics. These metrics are introduced by Matsumoto (see [16]). They are considered not only by Finsler geometers because of their simple and interesting structure but also by physicists because of their applications in physics. In fact, the first type of (α, β)-metrics, Randers metrics, introduced by Randers in 1941 for its application in general relativity (see [18]). On the other hand, physicists are also interested in these metrics.

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Invariant (α,β)-metrics have been studied by some Finsler geometers, during recent years.

In this paper, we deal with special (α,β)-metric satisfying

$$L^2(\alpha,\beta) = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2,$$

where c_i, $(i = 1, 2, 3)$ are constants. We study the existence of invariant vector fields on homogeneous Finsler spaces with a special (α,β)-metric L. Homogeneous geodesics have important applications to mechanics. For example, the equation of motion of many systems of classical mechanics reduces to the geodesic equation in an appropriate Riemannian manifold M. Then we study geodesic vectors and investigate the set of all homogeneous geodesics of invariant (α,β)-metric L on homogeneous spaces and simply connected 4-dimensional real Lie groups admitting invariant hypercomplex structure. For more recent papers, see [9] and [11].

2. **Preliminaries**

In this section, we recall briefly some known facts about Finsler spaces (for details, see [2, 20]). Let M be a n-dimensional C^∞ manifold and $TM = \cup_{x\in M} T_xM$ the tangent bundle. A Finsler metric on a manifold M is a non-negative function $F : TM \to \mathbb{R}$ with the following properties:

1) F is smooth on the slit tangent bundle $TM_0 := TM\backslash 0$,
2) $F(x,\lambda Y) = \lambda F(x,Y)$ for any $x \in M$, $Y \in T_xM$ and $\lambda > 0$,
3) The $n \times n$ Hessian matrix

$$[g_{ij}] = \frac{1}{2}\left[\frac{\partial^2 F^2}{\partial y^i \partial y^j}\right]$$

is positive definite at every point $(x,y) \in TM_0$.

The following bilinear symmetric form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ is positive definite:

$$\mathbf{g}_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(x, y + su + tv)\Big]|_{s=t=0}.$$

By the homogeneity of F, we have

$$\mathbf{g}_y(u,v) = g_{ij}(x,y)u^iv^j, \qquad F = \sqrt{g_{ij}(x,y)u^iu^j}.$$

Let M be an n-dimensional manifold. A special (α,β)-metric L is a Finsler structure L on TM that has the form

$$L(x,y) := \sqrt{c_1\alpha^2(x,y) + 2c_2\alpha(x,y)\beta(x,y) + c_3\beta(x,y)^2},$$

where

$$\alpha(x,y) := \sqrt{\tilde{a}_{ij} y^i y^j} \quad , \quad \beta(x,y) := \tilde{b}_i(x) y^i, \qquad c_i, \quad i = 1,2,3 \;\; \textit{constant}.$$

The a_{ij} are the components of a Riemannian metric and the b_i are those of a 1-form. Due to the presence of the β term, L metrics do not satisfy

$$L(x,-y) = -L(x,y)$$

when $\tilde{b} \neq 0$. In fact, the Finsler function of a L- space is absolutely homogeneous if and only if it is Riemannian. Also, in order for L to be positive if and only if

$$\| \tilde{b} \| := \sqrt{\tilde{b}_i \tilde{b}^i} < b_0, \quad \text{where} \;\; \tilde{b}^i := \tilde{a}^{ij} \tilde{b}_j.$$

The Riemannian metric $\tilde{a} = \tilde{a}_{ij} dx^i \otimes dx^j$ induces the musical bijection between 1-forms and vector fields on M, namely $\flat : T_xM \to T_x^*M$ given by $X \to \tilde{a}_x(X, \circ)$ and its inverse $\sharp : T_x^*M \to T_xM$. (see [2]) In the local coordinates we have

$$(X^\flat)_i = \tilde{a}_{ij} y^j \qquad (\theta^*)^i = \tilde{a}^{ij} \theta_j \qquad \forall X \in T_xM \quad , \quad \forall \theta \in T_x^*M.$$

Now the vector field corresponding to 1-form $\tilde{b}$ will be denoted by $\tilde{b}^\sharp$. Obviously, we have

$$\| \tilde{b} \| = \| \tilde{b}^\sharp \|$$

and

$$\beta(x,y) = (b^\sharp)^\flat(y) = \tilde{a}_x(b^\sharp, y).$$

Thus a special (α, β)-metric L with Riemannian metric $\tilde{a} = \tilde{a}_{ij} dx^i \otimes dx^j$ and 1-form $\tilde{b}$ can be shown by

$$L(x,y) = \sqrt{c_1 \tilde{a}_x(y,y) + 2c_2 \sqrt{\tilde{a}_x(y,y)}\, \tilde{a}(\tilde{b}^\sharp, y) + c_3 \tilde{a}_x(\tilde{b}^\sharp, y)^2}, \tag{2.1}$$
$$x \in M \quad , \quad y \in T_xM \quad , \quad c_i : constant \qquad i = 1,2,3.$$

where

$$\tilde{a}_x(\tilde{b}^\sharp, \tilde{b}^\sharp) < b_0, \quad \forall x \in M.$$

Let π^*TM be the pull-back of the tangent bundle TM by $\pi : TM_0 \to M$. Unlike the Levi-Civita connection in Riemannian geometry, there is no unique natural connection in the Finsler case. Among these connections on π^*TM, we choose the Chern connection whose coefficients are denoted by Γ^i_{jk} (see [2], p. 38). This connection is almost g-compatible and has no torsion. Since, in general, the Chern connection coefficients Γ^i_{jk} in natural coordinates have a directional dependence, we must define a fixed reference vector.

Let $\sigma = \sigma(t)$ be a smooth regular curve in M, with velocity field T. Let

$$W(t) := W^i(t) \frac{\partial}{\partial x^i}$$

be a vector field along σ. The expression

$$\left[\frac{dW^i}{dt} + W^j T^k (\Gamma^i_{jk})_{\sigma,T}\right] \frac{\partial}{\partial x^i}|_{\sigma(t)}$$

would have defined the covariant derivative $D_T W$ with reference vector T. A curve $\sigma(t)$, with velocity $T = \dot{\sigma}(t)$ is a Finslerian geodesic if

$$D_T\left[\frac{T}{F(T)}\right] = 0 \qquad \text{with reference vector } T$$

that the constant speed geodesics are precisely the solution of

$$D_T T = 0 \qquad \text{with reference vector } T.$$

Since

$$T := \frac{d\sigma^i}{dt}\frac{\partial}{\partial x^i},$$

then the differential equations that describe constant speed geodesics are:

$$\frac{d^2\sigma^i}{dt^2} + \frac{d\sigma^j}{dt}\frac{d\sigma^k}{dt}(\Gamma^i_{jk})_{(\sigma,T)} = 0.$$

Before defining homogeneous Finsler spaces, we discuss here some basic concepts required.

Definition 2.1. Let G be a smooth manifold having the structure of an abstract group. G is called a Lie group, if the maps $i : G \to G$ and $\mu : G \times G \to G$ defined as

$$i(g) = g^{-1}, \quad \text{and} \quad \mu(g,h) = gh,$$

respectively, are smooth.

Let G be a Lie group and M, a smooth manifold. Then a smooth map $f : G \times M \to M$ satisfying

$$f\big(g_2, f(g_1,x)\big) = f(g_2g_1,x), \qquad \forall g_1, g_2 \in G, \quad x \in M.$$

is called a smooth action of G on M.

Definition 2.2. Let M be a smooth manifold and G, a Lie group. If G acts smoothly on M, then G is called a Lie transformation group of M.

The following theorem gives us a differentiable structure on the coset space of a Lie group.

Theorem 2.3. *Let G be a Lie group and H, its closed subgroup. Then there exists a unique differentiable structure on the left coset space G/H with the induced topology that turns $\frac{G}{H}$ into a smooth manifold such that G is a Lie transformation group of G/H .*

Definition 2.4. Let (M, L) be a connected Finsler space and $I(M, L)$ the group of isometries of (M, L). If the action of $I(M, L)$ is transitive on M, then (M, L) is said to be a homogeneous Finsler space.

Let G be a Lie group acting transitively on a smooth manifold M. Then for $a \in M$, the isotropy subgroup G_a of G is a closed subgroup and by (2.3), G is a Lie transformation group of G/G_a. Further, G/G_a is diffeomorphic to M.

Theorem 2.5. [5] *Let (M, L) be a Finsler space. Then $G = I(M, L)$, the group of isometries of M is a Lie transformation group of M. Let $a \in M$ and $I_a(M, L)$ be the isotropy subgroup of $I(M, L)$ at a. Then $I_a(M, L)$ is compact.*

Let (M, L) be a homogeneous Finsler space, i.e. $G = I(M, L)$ acts transitively on M. For $a \in M$, let $H = I_a(M, L)$ be a closed isotropy subgroup of G which is compact. Then H is a Lie group itself being a closed subgroup of G. Write M as the quotient space G/H.

Definition 2.6. [17] Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of the Lie groups G and H respectively. Then the direct sum decomposition of $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m},$$

where $\mathfrak{m}$ is a subspace of $\mathfrak{g}$ such that $Ad(h)(\mathfrak{m}) \subset \mathfrak{m}$, $\forall h \in H$, is called a reductive decomposition of $\mathfrak{g}$, and if such decomposition exists, then $(G/H, L)$ is called reductive homogeneous space.

Therefore, we can write any homogeneous Finsler space as a coset space of a connected Lie group with an invariant Finsler metric. Here, the Finsler metric L is viewed as G invariant Finsler metric on M.

Definition 2.7. A one-parameter subgroup of a Lie group G is a homomorphism $\psi : \mathbb{R} \to G$, such that $\psi(0) = e$, where e is the identity of G.

Recall [5] the following result which gives us the existence of one-parameter subgroup of a Lie group.

Theorem 2.8. *Let G be a Lie group having Lie algebra $\mathfrak{g}$. Then for any $Y \in \mathfrak{g}$, there exists a unique one-parameter subgroup ψ such that $\psi(0) = Y_e$, where e is the identity element of G.*

Definition 2.9. Let G be a Lie group with identity element e and $\mathfrak{g}$ its Lie algebra. The exponential map $exp : \mathfrak{g} \to G$ is defined by

$$exp(tY) = \psi(t), \qquad \forall t \in \mathbb{R}$$

where $\psi : \mathbb{R} \to G$ is unique one-parameter subgroup of G with

$$\dot{\psi}(0) = Y_e.$$

In the case of reductive homogeneous manifold, we can identify the tangent space $T_H(G/H)$ of G/H at the origin $eH = H$ with $\mathfrak{m}$ through the map

$$Y \to \frac{d}{dt} exp(tX)H\bigg|_{t=0}, \qquad Y \in \mathfrak{m},$$

since M is identified with G/H and Lie algebra of any Lie group G is viewed as T_eG.

3. **Invariant vector field**

Lemma 3.1. *Let (M, L) be a Finsler space with a special (α, β)-metric L satisfying*

$$L^2(\alpha, \beta) = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2,$$

where c_i are constant. Let $I(M, \tilde{a})$ and $I(M, L)$ denote the isometry groups of Riemannian manifold $(M, \tilde{a})$ and Finsler manifold (M, L) respectively. Then, $I(M, L)$ is a closed subgroup of the $I(M, \tilde{a})$.

Proof. Let $\eta \in I(M, L)$ and $q \in M$, we have

$$L(q, Y_q) = L\big(\eta(q), d\eta_q Y_q\big).$$

So we have

$$\begin{aligned} &\sqrt{c_1\tilde{a}(Y_q, Y_q) + 2c_2\sqrt{\tilde{a}(Y_q, Y_q)}\tilde{a}(X_q, Y_q) + c_3\tilde{a}(X_q, Y_q)^2} \\ &= \sqrt{c_1\tilde{a}(d\eta_q Y_q, d\eta_q Y_q) + 2c_2W\tilde{a}(X_{\eta(q)}, d\eta_q Y_q) + c_3\tilde{a}(X_{\eta(q)}, d\eta_q Y_q)^2}, \end{aligned} \tag{3.1}$$

where

$$W := \sqrt{\tilde{a}(d\eta_q Y_q, d\eta_q Y_q)}.$$

Replacing y by $-y$ in above equation, we get

$$\begin{aligned} &\sqrt{c_1\tilde{a}(Y_q, Y_q) - 2c_2\sqrt{\tilde{a}(Y_q, Y_q)}\tilde{a}(X_q, Y_q) + c_3\tilde{a}(X_q, Y_q)^2} \\ &= \sqrt{c_1\tilde{a}(d\eta_q Y_q, d\eta_q Y_q) - 2c_2W\tilde{a}(X_{\eta(q)}, d\eta_q Y_q) + c_3\tilde{a}(X_{\eta(q)}, d\eta_q Y_q)^2}. \end{aligned} \tag{3.2}$$

Subtracting equation (3.2) from equation (3.1), we get

$$\sqrt{\tilde{a}(Y_q, Y_q)}\tilde{a}(X_q, Y_q) = \sqrt{\tilde{a}(d\eta_q Y_q, d\eta_q Y_q)}\tilde{a}\big(X_{\eta(q)}, d\eta_q Y_q\big). \tag{3.3}$$

Combining the above equations implies that

$$\Big\{\tilde{a}(d\eta_q Y_q, d\eta_q Y_q) - \tilde{a}(Y_q, Y_q)\Big\} \cdot \Big\{c_1\tilde{a}(Y_q, Y_q) - c_3\tilde{a}(X_{\eta(q)}, d\eta_q Y_q)^2\Big\} = 0. \tag{3.4}$$

We suppose that

$$c_1\tilde{a}(Y_q, Y_q) - c_3\tilde{a}(X_{\eta(q)}, d\eta_q Y_q)^2 \neq 0$$

that is $c_1 \neq 0$ and $c_2 \neq 0$. So we have

$$\tilde{a}(d\eta_q Y_q, d\eta_q Y_q) - \tilde{a}(Y_q, Y_q) = 0, \tag{3.5}$$

which leads to

$$\tilde{a}(Y_q, Y_q) = \tilde{a}(d\eta_q Y_q, d\eta_q Y_q), \tag{3.6}$$

Adding equations (3.2) and (3.1) and using equation (3.6), we get

$$\tilde{a}(X_{\eta(q)}, d\eta_q Y_q) = \tilde{a}(X_q, Y_q). \tag{3.7}$$

Thus $\eta \in I(M, \tilde{a})$ and for any $q \in M$ we have

$$d\eta_q X_q = X_{\eta(q)}.$$

This completes the proof. □

By Lemma (3.1), we conclude that if (M, L) is a homogeneous Finsler space with metric

$$L^2(\alpha, \beta) = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2,$$

then the Riemannian space $(M, \tilde{a})$ is homogeneous. Further, M can be written as a coset space $\frac{G}{H}$, where $G = I(M, L)$ is a Lie transformation group of M and H, the compact isotropy subgroup $I_a(M, L)$ of $I(M, L)$ at some point $a \in M$ [6]. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of the Lie groups G and H, respectively. If $\mathfrak{g}$ can be written as a direct sum of subspaces $\mathfrak{h}$ and $\mathfrak{m}$ of $\mathfrak{g}$ such that $Ad(h)\mathfrak{m} \subset \mathfrak{m}\ \ \forall h \in H$, then from Definition (2.6), $(G/H, L)$ is a reductive homogeneous space.

Theorem 3.2. *Let $L^2(\alpha, \beta) = c\alpha^2 + 2c_2\alpha\beta + c_3\beta^2$ be a G-invariant (α, β)-metric on G/H. Then α is a G-invariant Riemannian metric and the vector X corresponding to the 1-form β is also G-invariant.*

Proof. Let L be G-invariant metric on $\frac{G}{H}$, we have

$$L(Ad(h)(y)) = L(y) \qquad \forall\ h \in H\ \ ,\ y \in \mathfrak{m}$$

By (2.1), we get

$$\sqrt{c_1\tilde{a}(Ad(h)Y, Ad(h)Y) + 2c_2 Z\tilde{a}(X, Ad(h)Y) + c_3\tilde{a}(X, Ad(h)Y)^2}$$
$$= \sqrt{c_1\tilde{a}(Y, Y) + 2c_2\sqrt{\tilde{a}(Y, Y)}\tilde{a}(X, Y) + c_3\tilde{a}(X, Y)^2},$$

where

$$Z := \sqrt{\tilde{a}(Ad(h)Y, Ad(h)Y)}.$$

After simplification, we get

$$\begin{aligned} c_1\tilde{a}(Ad(h)Y, Ad(h)Y) + 2c_2 Z\tilde{a}(X, Ad(h)Y) + c_3\tilde{a}(X, Ad(h)Y)^2 \\ = c_1\tilde{a}(Y,Y) + 2c_2\sqrt{\tilde{a}(Y,Y)}\tilde{a}(X,Y) + c_3\tilde{a}(X,Y)^2. \end{aligned} \quad (3.8)$$

Replacing y by $-y$ in (3.8) implies that

$$\begin{aligned} c_1\tilde{a}(Ad(h)Y, Ad(h)Y) - 2c_2 Z\tilde{a}(X, Ad(h)Y) + c_3\tilde{a}(X, Ad(h)Y)^2 \\ = c_1\tilde{a}(Y,Y) - 2c_2\sqrt{\tilde{a}(Y,Y)}\tilde{a}(X,Y) + c_3\tilde{a}(X,Y)^2. \end{aligned} \quad (3.9)$$

(3.9)-(3.8) yields

$$\sqrt{\tilde{a}(Y,Y)}\tilde{a}(X,Y) = \sqrt{\tilde{a}(Ad(h)Y, Ad(h)Y)}\ \tilde{a}\big(X, Ad(h)Y\big).$$

Combining the above equations we have

$$\Big\{\tilde{a}(Ad(h)Y, Ad(h)Y) - \tilde{a}(Y,Y)\Big\} \cdot \Big\{c_1\tilde{a}(Y,Y) - c_3\tilde{a}(X, Ad(h)Y)^2\Big\} = 0.$$

We suppose that

$$c_1\tilde{a}(Y,Y) - c_3\tilde{a}(X, Ad(h)Y)^2 \neq 0$$

that is $c_1 \neq 0$, $c_3 \neq 0$. So we get

$$\tilde{a}\big(Ad(h)Y, Ad(h)Y\big) - \tilde{a}\big(Y,Y\big) = 0,$$

which leads to

$$\tilde{a}\big(Y,Y\big) = \tilde{a}\big(Ad(h)Y, Ad(h)Y\big), \quad (3.10)$$

Adding equations (3.9) and (3.8) and using equation (3.10), we get

$$\tilde{a}(X,Y) = \tilde{a}(X, Ad(h)Y). \quad (3.11)$$

Therefore, α is a G-invariant Riemannian metric and

$$Ad(h)X = X$$

which proves that X is also G-invariant. □

The following theorem gives us a complete description of invariant vector fields.

Theorem 3.3. [7] *There exists a bijection between the set of invariant vector fields on G/H and the subspace*

$$V = \Big\{Y \in \mathfrak{m} : Ad(h)Y = Y, \quad \forall h \in H\Big\}. \quad (3.12)$$

4. Homogeneous Geodesics

In this section, we study the homogeneous geodesics of Finsler spaces equipped with the following metric

$$L(x,y)=\sqrt{c_1\tilde{a}(y,y)+2c_2\sqrt{\tilde{a}(y,y)}\tilde{a}(X,y)+c_3\tilde{a}(X,y)^2}$$

Definition 4.1. A Finsler space (M,L) is called a homogeneous Finsler space if the group of isometries of (M,L), $I(M,L)$ acts transitively on M

Also, we have the following.

Remark 4.2. *Any homogeneous Finsler manifold $M=G/H$ is a reductive homogeneous space.*

Definition 4.3. For a homogeneous Riemannian manifold $(G/H,\tilde{a})$, or a homogeneous Finsler $(G/H,L)$ manifold a non-zero vector $X\in\mathfrak{g}$ is called a geodesic vector if the curve

$$\gamma(t)=exp(tZ)(o)$$

is a geodesic on $(G/H,\tilde{a})$, or on $(G/H,L)$, respectively.

Suppose that $(G/H,\tilde{a})$ is a homogeneous Riemannian manifold, and $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{h}$ is a reductive decomposition. In [12], it is proved that a vector $X\in\mathfrak{g}$ is a geodesic vector if and only if

$$\tilde{a}([X,Y]_{\mathfrak{m}},X_{\mathfrak{m}})=0,\qquad \forall Y\in\mathfrak{m}. \tag{4.1}$$

In [14], Latifi proved a similar theorem for Finslerian case as follows.

Theorem 4.4. *A vector $X\in\mathfrak{g}-\{0\}$ is a geodesic vector if and only if*

$$g_{X_{\mathfrak{m}}}([X,Z]_{\mathfrak{m}},X_{\mathfrak{m}})=0,\qquad \forall Z\in\mathfrak{g}.$$

Also as a corollary of the above theorem he proved the following corollary:

Corollary 4.5. *A vector $X\in\mathfrak{g}-\{0\}$ is a geodesic vector if and only if*

$$g_{X_{\mathfrak{m}}}([X,Z]_{\mathfrak{m}},X_{\mathfrak{m}})=0,\qquad \forall Z\in\mathfrak{m}.$$

Now, we are going to study Theorem 4.4 for the mentioned Finsler metric. More precisely, we prove the following.

Theorem 4.6. *Let $(G/H, L)$ be a homogeneous Finsler space with*

$$L(x,y) = \sqrt{c_1\tilde{a}(y,y) + 2c_2\sqrt{\tilde{a}(y,y)}\tilde{a}(X,y) + c_3\tilde{a}(X,y)^2}$$

defined by the Riemannian metric $\tilde{a}$ and the vector field X. Then, X is a geodesic vector of $(G/H, \tilde{a})$ if and only if X is a geodesic vector of $(G/H, L)$.

Proof. By using the formula

$$g_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial t \partial s}\Big[L^2(y + su + tv)\Big] \,|_{s=t=0}$$

and some computations, for the metric L, we have

$$\begin{aligned} g_y(u,v) =& c_1\tilde{a}(u,v) + c_3\tilde{a}(X,u)\tilde{a}(X,v) - c_2\frac{\tilde{a}(y,v)\tilde{a}(u,y)\tilde{a}(X,y)}{\tilde{a}(y,y)^{\frac{3}{2}}} \\ &+ c_2\frac{\tilde{a}(u,v)\tilde{a}(X,y) + \tilde{a}(X,v)\tilde{a}(u,y) + \tilde{a}(u,X)\tilde{a}(y,v)}{\tilde{a}(y,y)^{\frac{1}{2}}}. \end{aligned} \tag{4.2}$$

So for all $Z \in \mathfrak{m}$, we have

$$g_{X_\mathfrak{m}}(X_\mathfrak{m}, [X,Z]_\mathfrak{m}) = \tilde{a}(X_\mathfrak{m}, [X,Z]_\mathfrak{m})\Big\{c_1 + 2c_2\sqrt{\tilde{a}(X,X)} + c_3\tilde{a}(X,X)\Big\}.$$

Thus, $g_{X_\mathfrak{m}}(X_\mathfrak{m}, [X,Z]_\mathfrak{m}) = 0$ if and only if the following holds

$$\tilde{a}(X_\mathfrak{m}, [X,Z]_\mathfrak{m}) = 0.$$

This completes the proof. □

Theorem 4.7. *Let $(G/H, L)$ be a homogeneous Finsler space with*

$$L(x,y) = \sqrt{c_1\tilde{a}(y,y) + 2c_2\sqrt{\tilde{a}(y,y)}\tilde{a}(X,y) + c_3\tilde{a}(X,y)^2}$$

defined by the Riemannian metric $\tilde{a}$ and the vector field X. Let $y \in \mathfrak{g} - \{0\}$ be a vector which $\tilde{a}(X, [y,z]\mathfrak{m}) = 0$ for all $z \in \mathfrak{m}$. Then, y is a geodesic vector of $(G/H, L)$ if and only if y is a geodesic vector of $(G/H, \tilde{a})$.

Proof. By using the relation (4.2) and some computations, we have

$$\begin{aligned} g_{y_\mathfrak{m}}(y_\mathfrak{m}, [y,z]_\mathfrak{m}) = \tilde{a}(y_\mathfrak{m}, [y,z]_\mathfrak{m})\Big(c_1 + c_2\frac{\tilde{a}(X,y)}{\tilde{a}(y,y)^{\frac{1}{2}}}\Big) \\ +\tilde{a}(X, [y,z]_\mathfrak{m})\Big(c_3\tilde{a}(X,y) + c_2\tilde{a}(y,y)^{\frac{1}{2}}\Big). \end{aligned}$$

This completes the proof. □

5. Geodesic Vectors On Four Dimensional Real Group

Suppose that M is a $4n$-dimensional manifold. Also let J_i, $i = 1, 2, 3$, be three fiberwise endomorphism of TM such that

$$J_1 J_2 = -J_2 J_1 = J_3, \tag{5.1}$$

$$J_i^2 = -Id_{TM}, \qquad i = 1, 2, 3, \tag{5.2}$$

$$N_i = 0, \qquad i = 1, 2, 3. \tag{5.3}$$

where N_i is the Nijenhuis tensor (torsion) corresponding to J_i defined as follows:

$$N_i(X, Y) = [J_i X, J_i Y] - [X, Y] - J_i([X, J_i Y] + [J_i X, Y]), \tag{5.4}$$

for all vector fields X, Y on M. Then the family $\mathbb{H} = \{J_i\}_{i=1,2,3}$ is called a hypercomplex structure on M.

In fact three complex structures J_1, J_2 and J_3 on a $4n$-dimensional manifold M form a hypercomplex structure if they satisfy in the relation (5.1) (since an almost complex structure is a complex structure if and only if it has no torsion, see [13] page 124.). A Riemannian metric $\tilde{a}$ on a hypercomplex manifold $(M, \mathbb{H})$ is called hyper-Hermitian if

$$\tilde{a}(J_i X, J_i Y) = \tilde{a}(X, Y)$$

for all vector fields X, Y on M and $i = 1, 2, 3$. A hypercomplex structure $\mathbb{H} = \{J_i\}_{i=1,2,3}$ on a Lie group G is said to be left invariant if for any $a \in G$,

$$J_i = Tl_a \circ J_i \circ Tl_{a^{-1}}, \tag{5.5}$$

where Tl_a is the differential function of the left translation l_a.

In this section, we consider left invariant hyper-Hermitian Riemannian metrics on left invariant hypercomplex 4-dimensional simply connected Lie groups. These spaces have been classified by M. L. Barberis as follows (see [3]).

She has shown that $\mathfrak{g}$ is either Abelian or isomorphic to one of the following Lie algebras:
(1) $[e_2, e_3] = e_4$, $[e_3, e_4] = e_2$, $[e_4, e_2] = e_3$, e_1 : *central*,
(2) $[e_1, e_3] = e_1$, $[e_2, e_3] = e_2$, $[e_1, e_4] = e_2$, $[e_2, e_4] = -e_1$,
(3) $[e_1, e_2] = e_2$, $[e_1, e_3] = e_3$, $[e_1, e_4] = e_4$,
(4) $[e_1, e_2] = e_2$, $[e_1, e_3] = \frac{1}{2}e_2$, $[e_1, e_4] = \frac{1}{2}e_4$, $[e_3, e_4] = \frac{1}{2}e_2$.
where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis.

The case (1) is diffeomorphic to $R \times S^3$ and the other cases are diffeomorphic to R^4 (see [3] and [4]).

Now we discuss about left invariant Finsler metrics L satisfying

$$L(x, y) = \sqrt{c_1 \tilde{a}(y, y) + 2c_2 \sqrt{\tilde{a}(y, y)}\tilde{a}(X, y) + c_3 \tilde{a}(X, y)^2},$$

where $c_i = constant$, $i = 1, 2, 3$ which can arise from these Riamannian metrics and left invariant vector field $X = \sum_{i=1}^{4} x_i e_i$ on these spaces. We want to describe all geodesics vectors of (G, L) in any of the above cases.

By using the relation (4.2) and some computations, we have

$$g_y(y,[y,z]) = \tilde{a}(y,[y,z])\Big(c_1 + c_2\frac{\tilde{a}(X,y)}{\tilde{a}(y,y)^{\frac{1}{2}}}\Big) \\ +\tilde{a}(X,[y,z])\Big(c_3\tilde{a}(X,y) + c_2\tilde{a}(y,y)^{\frac{1}{2}}\Big). \tag{5.6}$$

By using Theorem 4.4, and (5.6), a vector $y = \sum_{i=1}^{4} y_i e_i$ of $\mathfrak{g}$ is a geodesic vector if and only if

$$\tilde{a}(y,[y,z]) = 0 \qquad \text{and} \qquad \tilde{a}(X,[y,z]) = 0 \qquad \forall z \in \mathfrak{g},$$

therefore

$$\tilde{a}\Big(\sum_{i=1}^{4} y_i e_i, [\sum_{i=1}^{4} y_i e_i, e_j]\Big) = 0,$$
$$\tilde{a}\Big(\sum_{i=1}^{4} x_i e_i, [\sum_{i=1}^{4} y_i e_i, e_j]\Big) = 0$$

for each $j = 1, 2, 3, 4$. So we get the following system of equations in different cases.

Case (1)

$$\begin{cases} x_3y_4 - x_4y_3 = 0, \\ x_4y_2 - x_2y_4 = 0, \\ x_2y_3 - x_3y_2 = 0. \end{cases}$$

As a special case, if $X = x_1e_1 + x_2e_2$, then a vector y of $\mathfrak{g}$ is a geodesic vector if and only if $y \in Span\{e_1, e_2\}$.

Case (2)

$$j = 1 \Rightarrow \begin{cases} y_1y_3 + y_2y_4 = 0, \\ x_1y_3 + x_2y_4 = 0. \end{cases}$$

$$j = 2 \Rightarrow \begin{cases} y_1y_4 - y_2y_3 = 0, \\ x_1y_4 - x_2y_3 = 0. \end{cases}$$

$$j = 3 \Rightarrow \begin{cases} y_1^2 + y_2^2 = 0, \\ x_1y_1 + x_2y_2 = 0. \end{cases}$$

$$j = 4 \Rightarrow \Big\{ x_2y_1 - x_1y_2 = 0.$$

As a special case, if $X = x_3e_3 + x_4e_4$, then a vector y of $\mathfrak{g}$ is a geodesic vector if and only if $y \in Span\{e_3, e_4\}$.

Case (3)

$$j = 1 \Rightarrow \begin{cases} y_2^2 + y_3^2 + y_4^2 = 0, \\ x_2y_2 + x_3y_3 + x_4y_4 = 0. \end{cases}$$

$$j = 2 \Rightarrow \begin{cases} y_1y_2 = 0, \\ x_2y_1 = 0. \end{cases}$$

$$j = 3 \Rightarrow \begin{cases} y_3y_1 = 0, \\ x_3y_1 = 0. \end{cases}$$

$$j = 4 \Rightarrow \begin{cases} y_1y_4 = 0, \\ x_4y_1 = 0. \end{cases}$$

As a special case, if $X = x_1e_1$, then a vector y of $\mathfrak{g}$ is a geodesic vector if and only if $y \in Span\{e_1\}$.

Case (4)

$$j = 1 \Rightarrow \begin{cases} 2y_2^2 + y_3^2 + y_4^2 = 0, \\ 2x_2y_2 + x_3y_3 + x_4y_4 = 0. \end{cases}$$

$$j = 2 \Rightarrow \begin{cases} y_2y_1 = 0, \\ x_2y_1 = 0. \end{cases}$$

$$j = 3 \Rightarrow \begin{cases} y_3y_1 - y_2y_4 = 0, \\ x_3y_1 - x_2y_4 = 0. \end{cases}$$

$$j = 4 \Rightarrow \begin{cases} y_4y_1 + y_2y_3 = 0, \\ x_2y_3 + x_4y_1 = 0. \end{cases}$$

As a special case, if $X = x_1e_1$, then a vector y of $\mathfrak{g}$ is a geodesic vector if and only if $y \in Span\{e_1\}$.

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