

On the geodesics of a homogeneous Finsler space with a special (α, β) -metric

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ABSTRACT. One of the most important concepts in geometry is of geodesics. Geodesic in a manifold is the generalization of notion of a straight line in an Euclidean space. A geodesic in a homogeneous Finsler space $(G/H, F)$ is called homogeneous geodesic if it is an orbit of a one-parameter subgroup of G . Homogeneous geodesics on homogeneous Riemannian manifolds have been studied by many authors. Latifi has extended the concept of homogeneous geodesics in homogeneous Finsler spaces. He has given a criterion for characterization of geodesic vectors. Latifi and Razavi have studied homogeneous geodesics in a 3-dimensional connected Lie group with a left invariant Randers metric and show that all the geodesics on spaces equipped with such metrics are homogeneous. In this paper, we give basic definitions required to define a homogeneous Finsler space. Next, we study geodesics and geodesic vectors for homogeneous Finsler space with infinite series (α, β) -metric. Next, we give a lemma in which the existence of invariant vector field corresponding to 1-form β for a homogeneous Finsler space with infinite series metric is proved. Further, we find necessary and sufficient condition for a non-zero vector in this homogeneous space to be a geodesic vector.

Keywords: Homogeneous Finsler space, homogeneous geodesic, infinite series (α, β) -metric, g.o. space.

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1. INTRODUCTION

According to S. S. Chern ([6]), Finsler geometry is just the Riemannian geometry without the quadratic restriction. Finsler generalized Riemann's theory in his doctoral thesis ([11]), but his name was established in differential geometry by Cartan ([5]). In 1972, M. Matsumoto ([27]) had introduced the concept of (α, β) -metric in Finsler geometry. A Finsler metric of the form $F = \alpha\phi(s)$, $s = \beta/\alpha$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ induced by a Riemannian metric $\tilde{a} = a_{ij}dx^i \otimes dx^j$ on a connected smooth n -manifold M and $\beta = b_i(x)y^i$ a 1-form on M , is called an (α, β) -metric. (α, β) -metrics are the generalizations of the Randers metric, introduced by G. Randers, in ([28]). There are various applications of (α, β) -metrics in information geometry ([1]), physics and biology ([2]). Some notable contributions on Finsler spaces with (α, β) -metrics can be seen in ([17], [18], [30], [31], [32], [35], [36]).

One of the most important concepts in geometry is of geodesics. Geodesic in a manifold is the generalization of notion of a straight line in an Euclidean space. Geodesic can be viewed as a curve that minimizes the distance between two points on the manifold. A geodesic in a homogeneous Finsler space $(G/H, F)$ is called homogeneous geodesic if it is an orbit of a one-parameter subgroup of G . Homogeneous geodesics on homogeneous Riemannian manifolds have been studied by many authors ([12], [20], [22], [38]). The existence of homogeneous geodesics in homogeneous Riemannian manifolds is quite an interesting problem. In ([15]), Kajzer proves that there exists at least one homogeneous geodesic in a Riemannian manifold. Further, in ([33], [34]), Szenthe proves that there exists infinitely many homogeneous geodesics through identity in a Riemannian manifold (G, α) , where G is compact semi-simple Lie group of rank ≥ 2 . Later, in ([21]), Kowalski and Szenthe prove the existence of at least one homogeneous geodesic through each point for any homogeneous Riemannian manifold. Also, Kowalski and Vlasek ([23]) have shown that the result proved in ([21]) is not true in general. They have given some examples of homogeneous Riemannian manifolds of dimension ≥ 4 admitting only one homogeneous geodesic. There are many applications of homogeneous geodesics in mechanics. Arnold ([3]) has studied the geodesics of left invariant Riemannian metrics on Lie groups, extending Euler's theory of rigid body motion and called homogeneous geodesic as "relative equilibria". The equation of motion of many systems in classical mechanics reduces to the geodesic equation in an appropriate Riemannian manifold. For example, Tóth ([37]) has studied trajectories which are orbits of a one-parameter symmetry group in case of Lagrangian and Hamiltonian systems. Also, Lacomba ([24]) uses homogeneous geodesics in the work of Smale's mechanical systems.

Latifi ([25]) has extended the concept of homogeneous geodesics in homogeneous Finsler spaces. In ([25]), he has given a criterion for the characterization

of geodesic vectors. Latifi and Razavi ([26]) study homogeneous geodesics in a 3-dimensional connected Lie group with a left invariant Randers metric and show that all the geodesics on spaces equipped with such metrics are homogeneous. Habibi et al. ([13]) have extended Szenthe's result of homogeneous geodesics for invariant Finsler metrics. Yan and Deng ([40]) generalize this result and prove that there exists at least one homogeneous geodesic through each point for any compact homogeneous Finsler space and also extend Kowalski and Szenthe's result to the Randers space. Yan ([41]) proves the existence of at least one homogeneous geodesic through each point for a homogeneous Finsler space of odd dimension. Hosseini and Moghaddam ([14]) study the existence of homogeneous geodesic in homogeneous (α, β) -spaces. Dušek ([10]) indicates a gap in the proof of main result of ([41]) and reproves this result and also studies homogeneous geodesics on a homogeneous Berwald space or homogeneous reversible Finsler space. Recently in 2018, Yan and Huang ([42]) prove that any homogeneous Finsler space admits at least one homogeneous geodesic through each point.

2. PRELIMINARIES

In this section, we give some basic concepts of Finsler geometry that are required for next sections. For symbols and notations, we refer ([4], [7]) and ([8]).

Definition 2.1. Let M be a smooth manifold of dimension n , T_pM the tangent space at any point $p \in M$. A real valued bilinear function $Q: T_pM \times T_pM \rightarrow [0, \infty)$ is called a Riemannian metric if it is symmetric and positive-definite, i.e., $\forall X, Y \in \mathfrak{X}(M)$,

- (i) $Q(X, Y) = Q(Y, X)$.
- (ii) $Q(X, X) \geq 0$ and $Q(X, X) = 0$ if and only if $X = 0$.

A smooth manifold with a given Riemannian metric is called a Riemannian manifold.

Definition 2.2. Let V be an n -dimensional real vector space. It is called a Minkowski space if there exists a real valued function $F: V \rightarrow \mathbb{R}$ satisfying the following conditions:

- (a) F is smooth on $V \setminus \{0\}$,
- (b) $F(v) \geq 0 \quad \forall v \in V$,
- (c) F is positively homogeneous, i.e., $F(\lambda v) = \lambda F(v), \quad \forall \lambda > 0$,
- (d) For a basis $\{v_1, v_2, \dots, v_n\}$ of V and $y = y^i v_i \in V$, the Hessian matrix $(g_{ij}) = \left(\frac{1}{2} F_{y^i y^j}^2 \right)$ is positive-definite at every point of $V \setminus \{0\}$.

In this case, F is called a Minkowski norm.

Definition 2.3. Let M be a connected smooth manifold. If there exists a function $F: TM \rightarrow [0, \infty)$ such that F is smooth on the slit tangent bundle $TM \setminus \{0\}$ and the restriction of F to any $T_p(M)$, $p \in M$, is a Minkowski norm, then (M, F) is called a **Finsler space** and F is called a Finsler metric.

Let (M, F) be a Finsler space and let (x^i, y^i) be a standard coordinate system in $T_x(M)$. The induced inner product g_y on $T_x(M)$ is given by $g_y(u, v) = g_{ij}(x, y)u^i v^j$, where $u = u^i \frac{\partial}{\partial x^i}$, $v = v^i \frac{\partial}{\partial x^i} \in T_x(M)$. Also note that $F(x, y) = \sqrt{g_y(y, y)}$.

Consider the r^{th} series (α, β) -metric:

$$F(\alpha, \beta) = \beta \sum_{r=0}^{r=\infty} \left(\frac{\alpha}{\beta} \right)^r.$$

If $r = 1$, then it is a Randers metric.

If $r = \infty$, then

$$F = \frac{\beta^2}{\beta - \alpha}.$$

This metric is called an infinite series (α, β) -metric. Interesting fact about this metric is that, it is the difference of a Randers metric and a Matsumoto metric, and satisfies Shen's lemma ([7]) given below:

Lemma 2.4. Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, where α is a Riemannian metric and β is a 1-form whose length with respect to α is bounded above, i.e., $b := \|\beta\|_\alpha < b_0$, where b_0 is a positive real number. Then F is a Finsler metric if and only if the function $\phi = \phi(s)$ is a smooth positive function on $(-b_0, b_0)$ and satisfies the following condition:

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \leq b < b_0.$$

Definition 2.5. A diffeomorphism $\phi: M \rightarrow M$ on a Finsler space (M, F) is called an **isometry** if $F(p, X) = F(\phi(p), d\phi_p(X))$, for any $p \in M$ and $X \in T_p(M)$.

Definition 2.6. A Lie group G is called a Lie transformation group of a smooth manifold M if it has a smooth action on M .

Definition 2.7. Let (M, F) be a connected Finsler space. If the action of the group of isometries of (M, F) , denoted by $I(M, F)$, is transitive on M , then it is called a homogeneous Finsler space.

Let (M, Q) be a Riemannian manifold. A geodesic $\sigma: \mathbb{R} \rightarrow M$ is called homogeneous geodesic if there exists a one-parameter group of isometries $\phi: \mathbb{R} \times M \rightarrow M$ such that $\sigma(t) = \phi(t, \sigma(0))$, $t \in \mathbb{R}$. If all the geodesics of a Riemannian manifold are homogeneous, then it is called a g.o. space (geodesic

orbit space). Every naturally reductive Riemannian manifold is a g.o. space. The first example of a g.o. space which is not naturally reductive was given by Kaplan ([16]).

In 2014, Yan and Deng ([39]) have studied Finsler g.o. spaces defined as follows:

A Finsler space (M, F) is called a Finsler g.o. space if every geodesic of (M, F) is the orbit of a one-parameter subgroup of $G = I(M, F)$, i.e., if $\sigma: \mathbb{R} \rightarrow M$ is a geodesic, then \exists a non-zero vector $Z \in \mathfrak{g} = \text{Lie}(G)$ and $p \in M$ such that $\sigma(t) = \exp(tZ).p$. A Finsler g.o. space has vanishing S-curvature. Also, note that every Finsler g.o. space is homogeneous.

A homogeneous Finsler space is a g.o. space if and only if the projections of all the geodesic vectors cover the set $T_{eH}(G/H) - \{0\}$.

Definition 2.8. Let $(G/H, F)$ be a homogeneous Finsler space and e be the identity of G . A geodesic $\sigma(t)$ through the origin eH of G/H is called homogeneous if it is an orbit of a one-parameter subgroup of G , i.e., there exists a non-zero vector $X \in \mathfrak{g} = \text{Lie}(G)$ such that $\sigma(t) = \exp(tX).eH$, $t \in \mathbb{R}$.

3. GEODESIC VECTOR

The problem of studying homogeneous geodesics of a homogeneous space is basically the study of its geodesic vectors.

Let (M, F) be a homogeneous Finsler space. Then, M can be written as a coset space G/H , where $G = I(M, F)$ is a Lie transformation group of M and H , the compact isotropy subgroup of $I(M, F)$ at some point $x \in M$ ([9]). Let \mathfrak{g} and \mathfrak{h} be the Lie algebras of the Lie groups G and H respectively. Also, let \mathfrak{m} be a subspace of \mathfrak{g} such that $\text{Ad}(h)\mathfrak{m} \subset \mathfrak{m} \quad \forall h \in H$, and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be a reductive decomposition of \mathfrak{g} .

Observe that for any $Y \in \mathfrak{g}$, the vector field $Y^* = \left. \frac{d}{dt} (\exp(tY)H) \right|_{t=0}$ is called the fundamental Killing vector field on G/H generated by Y ([19]). The canonical projection $\pi: G \rightarrow G/H$ induces an isomorphism between the subspace \mathfrak{m} and the tangent space $T_{eH}(G/H)$ through the following map:

$$\begin{aligned} \mathfrak{m} &\longrightarrow T_{eH}(G/H) \\ v &\longrightarrow \left. \frac{d}{dt} (\exp(tv)H) \right|_{t=0}. \end{aligned}$$

We have $d\pi(Y_{\mathfrak{m}}) = Y_{eH}^*$. Using the natural identification and scalar product $g_{Y_{eH}^*}$ on $T_{eH}(G/H)$, we get a scalar product $g_{Y_{\mathfrak{m}}}$ on \mathfrak{m} .

Definition 3.1. Let $(G/H, F)$ be a homogeneous Finsler space and e be the identity of G . A non-zero vector $X \in \mathfrak{g}$ is called a geodesic vector if the curve $\exp(tX).eH$ is a geodesic of $(G/H, F)$.

The following result proved in ([25]) gives a criterion for a non-zero vector to be a geodesic vector in a homogeneous Finsler space.

Lemma 3.2. A non-zero vector $Y \in \mathfrak{g}$ is a geodesic vector if and only if

$$g_{Y_m}(Y_m, [Y, Z]_m) = 0 \quad \forall Z \in \mathfrak{g}.$$

Now, we give the following lemma which was proved in ([31]). In this lemma, we have proved the existence of invariant vector field corresponding to 1-form β for a homogeneous Finsler space with infinite series metric.

Lemma 3.3. Let (M, α) be a Riemannian space. Then the infinite series Finsler metric $F = \frac{\beta^2}{\beta - \alpha}$, $\beta = b_i y^i$, a 1-form with $\|\beta\| = \sqrt{b_i b^i} < 1$ consists of a Riemannian metric α along with a smooth vector field X on M with $\alpha(X|_x) < 1 \quad \forall x \in M$, i.e.,

$$F(x, y) = \frac{\langle X|_x, y \rangle^2}{\langle X|_x, y \rangle - \alpha(x, y)}, \quad x \in M, \quad y \in T_x M,$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $T_x M$ induced by the Riemannian metric α .

Next, we deduce necessary and sufficient condition for a nonzero vector in a homogeneous Finsler space with infinite series (α, β) -metric to be a geodesic vector.

Theorem 3.4. Let G/H be a homogeneous Finsler space with infinite series metric $F = \frac{\beta^2}{\beta - \alpha}$ given by an invariant Riemannian metric $\tilde{\alpha}$ and an invariant vector field \tilde{X} such that $\tilde{X}(H) = X$. Then, a non-zero vector $Y \in \mathfrak{g}$ is a geodesic vector if and only if

$$\begin{aligned} & \frac{\langle X, Y_m \rangle^3}{\left(\langle X, Y_m \rangle - \sqrt{\langle Y_m, Y_m \rangle} \right)^4} \times \\ & \left[\langle X, [Y, Z]_m \rangle \left\{ \langle X, Y_m \rangle^2 - 4 \langle Y_m, Y_m \rangle^{3/2} + \langle X, Y_m \rangle \sqrt{\langle Y_m, Y_m \rangle} + 2 \langle Y_m, Y_m \rangle \right\} \right. \\ & \left. + \langle Y_m, [Y, Z]_m \rangle \left\{ \frac{\langle X, Y_m \rangle^2}{\sqrt{\langle Y_m, Y_m \rangle}} - \langle X, Y_m \rangle \right\} \right] = 0. \end{aligned} \tag{3.1}$$

Proof. Using lemma 3.3, we can write

$$F(Y) = \frac{\langle X, Y \rangle^2}{\langle X, Y \rangle - \sqrt{\langle Y, Y \rangle}}.$$

Also, we know that

$$g_Y(U, V) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} F^2(Y + sU + tV) \Big|_{s=t=0}.$$

After some calculations, we get

$$\begin{aligned} g_Y(U, V) &= \frac{\langle X, Y \rangle^2}{(\langle X, Y \rangle - \sqrt{\langle Y, Y \rangle})^4} \left[\langle X, Y \rangle^2 \langle X, V \rangle \langle X, U \rangle - 4 \langle Y, Y \rangle^{3/2} \langle X, V \rangle \langle X, U \rangle \right. \\ &\quad + 6 \langle Y, Y \rangle \langle X, V \rangle \langle X, U \rangle + \frac{\langle X, Y \rangle^2 \langle X, V \rangle \langle U, Y \rangle}{\sqrt{\langle Y, Y \rangle}} - 4 \langle X, Y \rangle \langle X, V \rangle \langle U, Y \rangle \\ &\quad - \frac{\langle X, Y \rangle^3 \langle U, Y \rangle \langle V, Y \rangle}{\langle Y, Y \rangle^{3/2}} + \frac{\langle X, Y \rangle^3 \langle U, V \rangle}{\sqrt{\langle Y, Y \rangle}} + \frac{4 \langle X, Y \rangle^2 \langle U, Y \rangle \langle V, Y \rangle}{\langle Y, Y \rangle} \\ &\quad \left. - \langle X, Y \rangle^2 \langle U, V \rangle + \frac{\langle X, Y \rangle^2 \langle X, U \rangle \langle V, Y \rangle}{\sqrt{\langle Y, Y \rangle}} - 4 \langle X, Y \rangle \langle X, U \rangle \langle V, Y \rangle \right]. \end{aligned} \quad (3.2)$$

From above equation, we can write

$$\begin{aligned} g_{Y_m}(Y_m, [Y, Z]_m) &= \frac{\langle X, Y_m \rangle^2}{(\langle X, Y_m \rangle - \sqrt{\langle Y_m, Y_m \rangle})^4} \left[\langle X, Y_m \rangle^2 \langle X, [Y, Z]_m \rangle \langle X, Y_m \rangle \right. \\ &\quad - 4 \langle Y_m, Y_m \rangle^{3/2} \langle X, [Y, Z]_m \rangle \langle X, Y_m \rangle + 6 \langle Y_m, Y_m \rangle \langle X, [Y, Z]_m \rangle \langle X, Y_m \rangle \\ &\quad + \frac{\langle X, Y_m \rangle^2 \langle X, [Y, Z]_m \rangle \langle Y_m, Y_m \rangle}{\sqrt{\langle Y_m, Y_m \rangle}} - 4 \langle X, Y_m \rangle \langle X, [Y, Z]_m \rangle \langle Y_m, Y_m \rangle \\ &\quad - \frac{\langle X, Y_m \rangle^3 \langle Y_m, Y_m \rangle \langle [Y, Z]_m, Y_m \rangle}{\langle Y_m, Y_m \rangle^{3/2}} + \frac{\langle X, Y_m \rangle^3 \langle Y_m, [Y, Z]_m \rangle}{\sqrt{\langle Y_m, Y_m \rangle}} \\ &\quad + \frac{4 \langle X, Y_m \rangle^2 \langle Y_m, Y_m \rangle \langle [Y, Z]_m, Y_m \rangle}{\langle Y_m, Y_m \rangle} - \langle X, Y_m \rangle^2 \langle Y_m, [Y, Z]_m \rangle \\ &\quad \left. + \frac{\langle X, Y_m \rangle^2 \langle X, Y_m \rangle \langle [Y, Z]_m, Y_m \rangle}{\sqrt{\langle Y_m, Y_m \rangle}} - 4 \langle X, Y_m \rangle \langle X, Y_m \rangle \langle [Y, Z]_m, Y_m \rangle \right]. \\ &= \frac{\langle X, Y_m \rangle^3}{(\langle X, Y_m \rangle - \sqrt{\langle Y_m, Y_m \rangle})^4} \times \\ &\quad \left[\langle X, [Y, Z]_m \rangle \left\{ \langle X, Y_m \rangle^2 - 4 \langle Y_m, Y_m \rangle^{3/2} + \langle X, Y_m \rangle \sqrt{\langle Y_m, Y_m \rangle} + 2 \langle Y_m, Y_m \rangle \right\} \right. \\ &\quad \left. + \langle Y_m, [Y, Z]_m \rangle \left\{ \frac{\langle X, Y_m \rangle^2}{\sqrt{\langle Y_m, Y_m \rangle}} - \langle X, Y_m \rangle \right\} \right]. \end{aligned} \quad (3.3)$$

Now, from lemma (3.2), $Y \in \mathfrak{g}$ is a geodesic vector if and only if

$$g_{Y_m}(Y_m, [Y, Z]_m) = 0, \quad \forall Z \in \mathfrak{m}.$$

Therefore $g_{Y_m}(Y_m, [Y, Z]_m) = 0$ if and only if equation (3.1) holds. \square

Corollary 3.5. *Let $(G/H, F)$ be a homogeneous Finsler space with infinite series metric $F = \frac{\beta^2}{\beta - \alpha}$ defined by an invariant Riemannian metric $\langle \cdot, \cdot \rangle$ and an invariant vector field \tilde{X} such that $\tilde{X}(H) = X$. Let $Y \in \mathfrak{g}$ be a vector such that $\langle X, [Y, Z]_m \rangle = 0 \forall Z \in \mathfrak{m}$. Then Y is a geodesic vector of $(G/H, \langle \cdot, \cdot \rangle)$ if and only if Y is a geodesic vector of $(G/H, F)$.*

Proof. From equation (3.3), we can write

$$\begin{aligned} g_{Y_m}(Y_m, [Y, Z]_m) &= \frac{\langle X, Y_m \rangle^3}{\left(\langle X, Y_m \rangle - \sqrt{\langle Y_m, Y_m \rangle}\right)^4} \times \\ &\left[\langle X, [Y, Z]_m \rangle \left\{ \langle X, Y_m \rangle^2 - 4\langle Y_m, Y_m \rangle^{3/2} + \langle X, Y_m \rangle \sqrt{\langle Y_m, Y_m \rangle} + 2\langle Y_m, Y_m \rangle \right\} \right. \\ &\quad \left. + \langle Y_m, [Y, Z]_m \rangle \left\{ \frac{\langle X, Y_m \rangle^2}{\sqrt{\langle Y_m, Y_m \rangle}} - \langle X, Y_m \rangle \right\} \right] \\ &= \frac{\langle X, Y_m \rangle^4 \langle Y_m, [Y, Z]_m \rangle}{\left(\langle X, Y_m \rangle - \sqrt{\langle Y_m, Y_m \rangle}\right)^3 \sqrt{\langle Y_m, Y_m \rangle}}, \text{ because } \langle X, [Y, Z]_m \rangle = 0. \end{aligned}$$

Therefore $g_{Y_m}(Y_m, [Y, Z]_m) = 0$ if and only if $\langle Y_m, [Y, Z]_m \rangle = 0$. \square

Theorem 3.6. *Let $(G/H, F)$ be a homogeneous Finsler space with infinite series metric $F = \frac{\beta^2}{\beta - \alpha}$ defined by an invariant Riemannian metric $\langle \cdot, \cdot \rangle$ and an invariant vector field \tilde{X} such that $\tilde{X}(H) = X$. Then X is a geodesic vector of $(G/H, \langle \cdot, \cdot \rangle)$ if and only if X is a geodesic vector of $(G/H, F)$.*

Proof. From equation (3.2), we can write

$$\begin{aligned} g_X(X, [X, Z]_m) &= \frac{\langle X, X \rangle^2}{\left(\langle X, X \rangle - \sqrt{\langle X, X \rangle}\right)^4} \left[\langle X, X \rangle^2 \langle X, [X, Z]_m \rangle \langle X, X \rangle \right. \\ &\quad - 4\langle X, X \rangle^{3/2} \langle X, [X, Z]_m \rangle \langle X, X \rangle + 6\langle X, X \rangle \langle X, [X, Z]_m \rangle \langle X, X \rangle \\ &\quad + \frac{\langle X, X \rangle^2 \langle X, [X, Z]_m \rangle \langle X, X \rangle}{\sqrt{\langle X, X \rangle}} - 4\langle X, X \rangle \langle X, [X, Z]_m \rangle \langle X, X \rangle \\ &\quad \left. - \frac{\langle X, X \rangle^3 \langle X, X \rangle \langle [X, Z]_m, X \rangle}{\langle X, X \rangle^{3/2}} + \frac{\langle X, X \rangle^3 \langle X, [X, Z]_m \rangle}{\sqrt{\langle X, X \rangle}} \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{4 \langle X, X \rangle^2 \langle X, X \rangle \langle [X, Z]_{\mathfrak{m}}, X \rangle}{\langle X, X \rangle} - \langle X, X \rangle^2 \langle X, [X, Z]_{\mathfrak{m}} \rangle \\
& + \frac{\langle X, X \rangle^2 \langle X, X \rangle \langle [X, Z]_{\mathfrak{m}}, X \rangle}{\sqrt{\langle X, X \rangle}} - 4 \langle X, X \rangle \langle X, X \rangle \langle [X, Z]_{\mathfrak{m}}, X \rangle \Big]. \\
& = \frac{\langle X, X \rangle^3 \left(\langle X, X \rangle^2 + \langle X, X \rangle - 2 \langle X, X \rangle \sqrt{\langle X, X \rangle} \right) \langle X, [X, Z]_{\mathfrak{m}} \rangle}{\left(\langle X, X \rangle - \sqrt{\langle X, X \rangle} \right)^4} \\
& = \frac{\langle X, X \rangle^3 \langle X, [X, Z]_{\mathfrak{m}} \rangle}{\left(\langle X, X \rangle - \sqrt{\langle X, X \rangle} \right)^2}.
\end{aligned}$$

Therefore $g_x(X, [X, Z]_{\mathfrak{m}}) = 0$ if and only if $\langle X, [X, Z]_{\mathfrak{m}} \rangle = 0$. \square

Corollary 3.7. *Let $(G/H, F)$ be a homogeneous Finsler space with an invariant infinite series metric $F = \frac{\beta^2}{\beta - \alpha}$ defined by an invariant Riemannian metric $\langle \cdot, \cdot \rangle$ and an invariant vector field \tilde{X} such that $X = \tilde{X}(H)$ be a geodesic vector of Riemannian metric $\langle \cdot, \cdot \rangle$. Then there exists atleast one homogeneous geodesic of $(G/H, F)$ through origin $eH=H$.*

Proof. Since X is a geodesic vector of $(G/H, \langle \cdot, \cdot \rangle)$. Therefore, by above theorem X is a geodesic vector of $(G/H, F)$. Thus, $(G/H, F)$ admits atleast one homogeneous geodesic through origin. \square

Theorem 3.8. *Let (M, F) be a homogeneous Finsler space with an invariant infinite series (α, β) -metric $F = \frac{\beta^2}{\beta - \alpha}$. Then there exists atleast one homogeneous geodesic of (M, F) through each point.*

Proof. Since (M, F) is a homogeneous Finsler space, M can be written as G/H , where $G = I(M, F)$ is a connected Lie group of isometries acting transitively on M , and H is the compact isotropy subgroup of G at some point $x \in M$. Let the Lie algebras of G and H be \mathfrak{g} and \mathfrak{h} respectively. We denote the Killing form of \mathfrak{g} by K and null space(radical) of K by $\text{rad}K$. In ([21]), it is shown that K is nondegenerate on \mathfrak{h} because H is compact. Let \mathfrak{m} be the orthogonal complement of \mathfrak{h} with respect to K , i.e.,

$$\mathfrak{m} = \mathfrak{h}^\perp = \{W \in \mathfrak{g} : K(W, Z) = 0, \forall Z \in \mathfrak{h}\}.$$

Then $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is a reductive decomposition and $\text{rad}K \subset \mathfrak{m}$. Now, using the theorem 1.1 of ([42]), homogeneous Finsler space with infinite series metric admits atleast one homogeneous geodesic through each point. \square

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