

Characterization of Finsler spaces of scalar curvature

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ABSTRACT. The aim of the present paper is to provide an intrinsic investigation of two special Finsler spaces whose defining properties are related to Berwald connection, namely, Finsler space of scalar curvature and of constant curvature. Some characterizations of a Finsler space of scalar curvature are proved. Necessary and sufficient conditions under which a Finsler space of scalar curvature reduces to a Finsler space of constant curvature are investigated.

Keywords: Berwald connection, Deviation tensor, indicatory tensor, Finsler manifold of scalar curvature, Finsler manifold of constant curvature.

1. INTRODUCTION

Special Finsler spaces are investigated locally by many authors (cf., for example, [2]-[5], [7], [8], [10]). On the other hand, the global or intrinsic investigation of such spaces is rare in the literature. Some contributions in this direction are found in [9] and [12].

In the present paper, we treat intrinsically two types of special Finsler spaces: Finsler space of scalar curvature and Finsler space of constant curvature. Some characterizations of Finsler spaces of scalar curvature are proved. Necessary

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AMS 2020 Mathematics Subject Classification: 53C60, 53C30.

and sufficient conditions for Finsler space of scalar curvature to reduce to a Finsler space of constant curvature are investigated.

It should be noted that some important results of [7], [8] and [10] are retrieved from the obtained global results, when localized.

2. NOTATION AND PRELIMINARIES

In this section, we give a brief account of the basic concepts of the pullback approach to intrinsic Finsler geometry necessary for this work. For more details, we refer to [1], [6] and [9]. The following notation will be used throughout this paper:

M : a real differentiable manifold of finite dimension n and class C^∞ ,
 $\mathfrak{F}(M)$: the $\mathfrak{F}(M)$ -algebra of differentiable functions on M ,
 $\pi : M \rightarrow M$: the subbundle of nonzero vectors tangent to M ,
 $P : \pi^{-1}(TM) \rightarrow M$: the pullback of the tangent bundle TM by π ,
 $\mathfrak{X}(\pi(M))$: the $\mathfrak{F}(M)$ -module of differentiable sections of $\pi^{-1}(TM)$,
 i_X : the interior product with respect to $X \in \mathfrak{X}(M)$,

Elements of $\mathfrak{X}(\pi(M))$ will be called π -vector fields and will be denoted by barred letters \bar{X} . Tensor fields on $\pi^{-1}(TM)$ will be called π -tensor fields. The fundamental π -vector field is the π -vector field $\bar{\eta}$ defined by $\bar{\eta}(u) = (u, u)$ for all $u \in M$. We have the following short exact sequence of vector bundles:

$$0 \rightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(M) \xrightarrow{\rho} \pi^{-1}(TM) \rightarrow 0,$$

the bundle morphisms ρ and γ being defined as usual [11].

Let D be a linear connection (or simply a connection) on the pullback bundle $\pi^{-1}(TM)$. The connection (or the deflection) map associated with D is defined by

$$K : TM \rightarrow \pi^{-1}(TM) : X \mapsto D_X \bar{\eta}.$$

A tangent vector $X \in T_u(M)$ at $u \in M$ is horizontal if $K(X) = 0$. The vector space $H_u(M) = \{X \in T_u(M) : K(X) = 0\}$ is the horizontal space at u . A connection D is said to be regular if $T_u(M) = V_u(M) \oplus H_u(M) \forall u \in M$, where $V_u(M)$ is the vertical space at u . Let $\beta := (\rho|_{H(M)})^{-1}$, called the horizontal map of the connection D , then

$$\rho \circ \beta = id_{\pi^{-1}(TM)}, \quad \beta \circ \rho = id_{H(M)} \text{ on } H(M).$$

For a regular connection D , the horizontal and vertical covariant derivatives $\overset{1}{D}$ and $\overset{2}{D}$ are defined, for a vector (1) π -form A , for example, by

$$(\overset{1}{D} A)(\bar{X}, \bar{Y}) := (D_{\beta \bar{X}} A)(\bar{Y}), \quad (\overset{2}{D} A)(\bar{X}, \bar{Y}) := (D_{\gamma \bar{X}} A)(\bar{Y}).$$

The horizontal ((h)h-) and mixed ((h)hv-) torsion tensors of the connection D are defined respectively by

$$Q(\bar{X}, \bar{Y}) := \mathbf{T}(\beta \bar{X}, \beta \bar{Y}), \quad T(\bar{X}, \bar{Y}) := \mathbf{T}(\gamma \bar{X}, \beta \bar{Y}) \quad \forall \bar{X}, \bar{Y} \in \mathfrak{X}(\pi(M)),$$

where $\mathbf{T}(X, Y) = D_X \rho Y - D_Y \rho X - \rho[X, Y]$ is the (classical) torsion of the connection D

The horizontal (h-), mixed (hv-) and vertical (v-) curvature tensors of D are defined respectively by

$$\begin{aligned} R(\bar{X}, \bar{Y})\bar{Z} &:= \mathbf{K}(\beta\bar{X}, \beta\bar{Y})\bar{Z}, & P(\bar{X}, \bar{Y})\bar{Z} &:= \mathbf{K}(\beta\bar{X}, \gamma\bar{Y})\bar{Z}, \\ S(\bar{X}, \bar{Y})\bar{Z} &:= \mathbf{K}(\gamma\bar{X}, \gamma\bar{Y})\bar{Z}, \end{aligned}$$

where $\mathbf{K}(X, Y)\rho Z = -D_X D_Y \rho Z + D_Y D_X \rho Z + D_{[X, Y]}\rho Z$ is the (classical) curvature tensor of the connection D .

The cotracted curvatures of D or the (v)h-, (v)hv- and (v)v-torsion tensors are defined respectively by

$$\widehat{R}(\bar{X}, \bar{Y}) := R(\bar{X}, \bar{Y})\bar{\eta}, \quad \widehat{P}(\bar{X}, \bar{Y}) := P(\bar{X}, \bar{Y})\bar{\eta}, \quad \widehat{S}(\bar{X}, \bar{Y}) := S(\bar{X}, \bar{Y})\bar{\eta}.$$

Theorem 2.1. [11] *Let (M, L) be a Finsler manifold. There exists a unique regular connection D° on $\pi^{-1}(TM)$ such that*

- (a): $D_{h^\circ X} L = 0$,
- (b): D° is torsion-free: $\mathbf{T}^\circ = 0$,
- (c): The (v)hv-torsion tensor \widehat{P}° of D° vanishes: $\widehat{P}^\circ(\bar{X}, \bar{Y}) = 0$.

Such a connection is called *Berwald connection associated with (M, L)* .

Throughout the paper R° , \widehat{R}° and $H := i_{\bar{\eta}} \widehat{R}^\circ$ will denote respectively the h -curvature, the (v) h -torsion and the deviation tensor of Berwald connection D° . Moreover, $\overset{1}{D}^\circ$ and $\overset{2}{D}^\circ$ will denote respectively the horizontal covariant derivative and the vertical covariant derivative associated with D° .

3. FINSLER SPACE OF SCALAR CURVATURE

In this section, we establish intrinsically some characterizations of the property of being of scalar curvature.

Let (M, L) be a Finsler manifold and g the Finsler metric defined by L . Denote $\ell := L^{-1} i_{\bar{\eta}} g$, $\phi(\bar{X}) := \bar{X} - L^{-1} \ell(\bar{X})\bar{\eta}$ and $h(\bar{X}, \bar{Y}) := g(\phi(\bar{X}), \bar{Y}) = g(\bar{X}, \bar{Y}) - \ell(\bar{X})\ell(\bar{Y})$, the angular metric tensor.

Definition 3.1. [12] *A Finsler manifold (M, L) of dimension $n \geq 3$ is said to be of scalar curvature if the deviation tensor H satisfies*

$$H(\bar{X}) = kL^2\phi(\bar{X}),$$

where k is a scalar function on M , positively homogenous of degree zero in y ($h(0)$), called *scalar curvature*

In particular, if the scalar curvature k is constant, then (M, L) is called a Finsler manifold of constant curvature.

ω is $h(r)$ in y iff $D_{\gamma\bar{\eta}}^\circ \omega = r\omega$.

Definition 3.2. [12] *An operator \mathcal{P} , called the projection operator of indicatrix, is defined as follows:*

(a) *If ω is a π -tensor field of type $(1, p)$, then*

$$(\mathcal{P} \cdot \omega)(\bar{X}_1, \dots, \bar{X}_p) := \phi(\omega(\phi(\bar{X}_1), \dots, \phi(\bar{X}_p))).$$

(b) *If ω is a π -tensor field of type $(0, p)$, then*

$$(\mathcal{P} \cdot \omega)(\bar{X}_1, \dots, \bar{X}_p) := \omega(\phi(\bar{X}_1), \dots, \phi(\bar{X}_p)).$$

(c) *In particular, a π -tensor field ω is said to be an indicatory tensor if it satisfies $\mathcal{P} \cdot \omega = \omega$.*

Remark 3.3. *The projection \mathcal{P} preserves tensor type. The π -tensor fields ϕ and \bar{h} are indicatory. Moreover, for any π -tensor field ω , $\mathcal{P} \cdot \omega$ is indicatory.*

The following result provides some characterizations of Finsler spaces of scalar curvature.

Theorem 3.4. *Let (M, L) be a Finsler manifold of dimension $n \geq 3$. The following assertion are equivalent:*

- (a): *(M, L) is of scalar curvature k ,*
- (b): *The $(v)h$ -torsion tensor \widehat{R}° satisfies*

$$\widehat{R}^\circ(\bar{X}, \bar{Y}) = \mathfrak{A}_{\bar{X}, \bar{Y}}\{L\phi(\bar{Y})[k\ell(\bar{X}) + \frac{1}{3}C(\bar{X})]\}. \quad (3.1)$$

- (c): *The h -curvature tensor R° has the form*

$$\begin{aligned} R^\circ(\bar{X}, \bar{Y})\bar{Z} &= \mathfrak{A}_{\bar{X}, \bar{Y}}\left\{\phi(\bar{Y})\left[\ell(\bar{Z})(k\ell(\bar{X}) + \frac{1}{3}C(\bar{X})) + \frac{1}{3}B(\bar{Z}, \bar{X})\right.\right. \\ &\quad \left.+\frac{2}{3}\ell(\bar{X})C(\bar{Z}) + k\bar{h}(\bar{Z}, \bar{X})\right] + \frac{1}{3}\ell(\bar{X})C(\bar{Y})\phi(\bar{Z}) \\ &\quad \left.+\ L^{-1}\bar{h}(\bar{X}, \bar{Z})\bar{\eta}\left[k\ell(\bar{Y}) + \frac{1}{3}C(\bar{Y})\right]\right\}, \end{aligned} \quad (3.2)$$

where $\mathfrak{A}_{\bar{X}, \bar{Y}}\omega(\bar{X}, \bar{Y}) := \omega(\bar{X}, \bar{Y}) - \omega(\bar{Y}, \bar{X})$, and

$$C(\bar{X}) := L(\overset{2}{D}^\circ k)(\bar{X}), \quad B(\bar{X}, \bar{Y}) := L(\mathcal{P} \cdot \overset{2}{D}^\circ C)(\bar{X}, \bar{Y}) \quad (3.3)$$

To prove this theorem, we need the following three lemmas, which can easily be proved.

Lemma 3.5. *For a Finsler manifold (M, L) , we have:*

- (a): $i_{\bar{\eta}}\ell = L, \quad i_{\bar{\eta}}\phi = 0, \quad i_{\bar{\eta}}\bar{h} = 0.$
- (b): $\overset{1}{D}^\circ L = 0, \quad \overset{1}{D}^\circ \ell = 0.$
- (c): $\overset{2}{D}^\circ L = \ell, \quad \overset{2}{D}^\circ \ell = L^{-1}\bar{h}.$
- (d): $\overset{2}{D}^\circ \phi = -L^{-2}\bar{h} \otimes \bar{\eta} + L\phi \otimes \ell.$

$$(e): \mathcal{P} \cdot \ell = 0, \quad \mathcal{P} \cdot \hbar = \hbar.$$

Lemma 3.6. *The π -scalar form C , defined by (3.3), has the following properties:*

- (a): $i_{\bar{\eta}} C = 0$
- (b): $\mathcal{P} \cdot C = C$ (C is indicatory)
- (c): $(D^\circ C)(\bar{\eta}, \bar{X}) = 0$ (C is $h(0)$),
- (d): $(D^\circ C)(\bar{X}, \bar{\eta}) = -C(\bar{X})$.

Lemma 3.7. *The π -scalar form B , defined by (3.3), has the following properties:*

- (a): $i_{\bar{\eta}} B = 0$
- (b): $\mathcal{P} \cdot B = B$ (B is indicatory)
- (c): $(D^\circ B)(\bar{\eta}, \bar{X}, \bar{Y}) = 0$ (B is $h(0)$),
- (d): $B(\bar{X}, \bar{Y}) = L(D^\circ C)(\bar{X}, \bar{Y}) + C(\bar{X})\ell(\bar{Y})$,
- (e): B is symmetric.
- (f): $(D^\circ B)(\bar{X}, \bar{\eta}, \bar{Y}) = (D^\circ B)(\bar{X}, \bar{Y}, \bar{\eta}) = -B(\bar{X}, \bar{Y})$.

Proof of Theorem 3.4:

(a) \Rightarrow (b): Let (M, L) be a Finsler manifold of scalar curvature k . Then, by Definition 3.1, the deviation tensor H has the form

$$H(\bar{X}) = kL^2\phi(\bar{X}). \quad (3.4)$$

On the other hand, from Theorem 4.6 of [13], we have

$$\widehat{R}^\circ(\bar{X}, \bar{Y}) = \frac{1}{3}\mathfrak{A}_{\bar{X}, \bar{Y}}(D^\circ H)(\bar{X}, \bar{Y}). \quad (3.5)$$

From which, together with (3.4) and Lemma 3.5, we get

$$\begin{aligned} \widehat{R}^\circ(\bar{X}, \bar{Y}) &= \frac{1}{3}\mathfrak{A}_{\bar{X}, \bar{Y}}[L^2(D^\circ k) \otimes \phi + k(D^\circ L^2) \otimes \phi + kL^2(D^\circ \phi)](\bar{X}, \bar{Y}) \\ &= \frac{1}{3}\mathfrak{A}_{\bar{X}, \bar{Y}}[LC \otimes \phi + 2Lk\ell \otimes \phi - k\hbar \otimes \bar{\eta} + L\phi \otimes \ell](\bar{X}, \bar{Y}) \\ &= \frac{1}{3}\mathfrak{A}_{\bar{X}, \bar{Y}}[LC \otimes \phi + 3Lk\ell \otimes \phi - k\hbar \otimes \bar{\eta}](\bar{X}, \bar{Y}). \end{aligned}$$

Hence, the result follows.

(b) \Rightarrow (c): Suppose that the $(v)h$ -torsion tensor \widehat{R}° satisfies (3.1):

$$\widehat{R}^\circ(\bar{X}, \bar{Y}) = \mathfrak{A}_{\bar{X}, \bar{Y}}\{L\phi(\bar{Y})[k\ell(\bar{X}) + \frac{1}{3}C(\bar{X})]\}. \quad (3.6)$$

In view of Theorem 4.6 of [13], we have

$$R^\circ(\bar{X}, \bar{Y})\bar{Z} = (D^\circ \widehat{R}^\circ)(\bar{Z}, \bar{X}, \bar{Y}).$$

From which, taking into account (3.6) and Lemmas 3.5, 3.6 and 3.7, the result follows.

(c) \Rightarrow (a): Suppose that the h -curvature tensor R° has the form (3.2):

$$\begin{aligned} R^\circ(\bar{X}, \bar{Y})\bar{Z} &= \mathfrak{A}_{\bar{X}, \bar{Y}} \left\{ \phi(\bar{Y}) \left[\ell(\bar{Z})(k\ell(\bar{X}) + \frac{1}{3}C(\bar{X})) + \frac{1}{3}B(\bar{Z}, \bar{X}) \right. \right. \\ &\quad \left. \left. + \frac{2}{3}\ell(\bar{X})C(\bar{Z}) + k\hbar(\bar{Z}, \bar{X}) \right] + \frac{1}{3}\ell(\bar{X})C(\bar{Y})\phi(\bar{Z}) \right. \\ &\quad \left. + L^{-1}\hbar(\bar{X}, \bar{Z})\bar{\eta} \left[k\ell(\bar{Y}) + \frac{1}{3}C(\bar{Y}) \right] \right\}, \end{aligned}$$

Setting $\bar{X} = \bar{\eta}$ and $\bar{Z} = \bar{\eta}$ into the above equation, taking into account Lemmas 3.5, 3.6 and 3.7, the result follows. \square

Let $R^\circ(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) := g(R^\circ(\bar{X}, \bar{Y})\bar{Z}, \bar{W})$, then we have:

Corollary 3.8. *For a Finsler manifold of scalar curvature k , the h -curvature tensor R° satisfies:*

$$\begin{aligned} \text{(a)} \quad R^\circ(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) - R^\circ(\bar{X}, \bar{Y}, \bar{W}, \bar{Z}) &= \mathfrak{A}_{\bar{X}, \bar{Y}} \{ \hbar(\bar{Z}, \bar{X})N(\bar{W}, \bar{Y}) + \hbar(\bar{W}, \bar{Y})N(\bar{Z}, \bar{X}) \}. \\ \text{(b)} \quad R^\circ(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) + R^\circ(\bar{X}, \bar{Y}, \bar{W}, \bar{Z}) &= \mathfrak{A}_{\bar{X}, \bar{Y}} \{ \hbar(\bar{W}, \bar{Y})F(\bar{Z}, \bar{X}) + \hbar(\bar{Z}, \bar{Y})F(\bar{W}, \bar{X}) \\ &\quad + \hbar(\bar{W}, \bar{Z})F(\bar{Y}, \bar{X}) \}. \end{aligned}$$

where N and F are the π -tensor fields of type $(0, 2)$ defined respectively by

$$\begin{aligned} N(\bar{X}, \bar{Y}) &:= kg(\bar{X}, \bar{Y}) + \ell(\bar{X})\ell(\bar{Y}) \\ &\quad + \frac{1}{3}B(\bar{X}, \bar{Y}) + 2\ell(\bar{X})C(\bar{Y}) + 2C(\bar{X})\ell(\bar{Y}), \\ F(\bar{X}, \bar{Y}) &:= \frac{1}{3}B(\bar{X}, \bar{Y}) + 2C(\bar{X})\ell(\bar{Y}). \end{aligned}$$

We end this section by the following result.

Proposition 3.9. *For a Finsler manifold of scalar curvature, $\mathcal{P} \cdot R^\circ$ vanishes if and only if $\mathcal{P} \cdot N$ vanishes, where N is the π -form defined by Corollary 3.8.*

Proof. Let (M, L) be a Finsler manifold of scalar curvature. Then, by Theorem 3.4, we have

$$(\mathcal{P} \cdot R^\circ)(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = \mathfrak{A}_{\bar{X}, \bar{Y}} \{ \hbar(\bar{Y}, \bar{W}) \left[\frac{1}{3}B(\bar{Z}, \bar{X}) + k\hbar(\bar{Z}, \bar{X}) \right] \}. \quad (3.7)$$

On the other hand, by Corollary 3.8, we obtain

$$(\mathcal{P} \cdot N)(\bar{X}, \bar{Y}) = \frac{1}{3}B(\bar{Z}, \bar{X}) + k\hbar(\bar{Z}, \bar{X}).$$

From which together with (3.7), we get

$$(\mathcal{P} \cdot R^\circ)(\bar{X}, \bar{Y}, \bar{Z}) = \mathfrak{A}_{\bar{X}, \bar{Y}} \{ \phi(\bar{Y})(\mathcal{P} \cdot N)(\bar{Z}, \bar{X}) \}. \quad (3.8)$$

It is clear, from (3.8), that if $\mathcal{P} \cdot N = 0$, then $\mathcal{P} \cdot R^\circ = 0$.

Conversely, if $\mathcal{P} \cdot R^\circ = 0$, it follows again from (3.8) that

$$\mathfrak{A}_{\bar{X}, \bar{Y}}\{\phi(\bar{Y})(\mathcal{P} \cdot N)(\bar{Z}, \bar{X})\} = 0.$$

Taking the contracted trace with respect to \bar{Y} , the above relation reduces to

$$(n-2)(\mathcal{P} \cdot N)(\bar{Z}, \bar{X}) = 0.$$

Consequently, as $n \geq 3$, $\mathcal{P} \cdot N$ vanishes. \square

4. FINSLER SPACE OF CONSTANT CURVATURE

In this section, we investigate intrinsically necessary and sufficient conditions under which a Finsler manifold of scalar curvature reduces to a Finsler manifold of constant curvature.

The following lemma is useful for subsequent use.

Lemma 4.1. *For a Finsler manifold (M, L) of constant curvature, we have*

$$\mathfrak{A}_{\bar{X}, \bar{Y}}\left\{A(\bar{X}, \bar{Y}, \bar{Z}) + C(\bar{X})h(\bar{Y}, \bar{Z})\right\} = 0,$$

where A is the π -tensor field of type $(0,3)$ defined by

$$A(\bar{X}, \bar{Y}, \bar{Z}) := L(\mathcal{P} \cdot \overset{2}{D}^\circ B)(\bar{X}, \bar{Y}, \bar{Z})$$

and C, B are the π -tensor fields defined by (3.3).

Proof. We have

$$\begin{aligned} A(\bar{X}, \bar{Y}, \bar{Z}) &= L(\mathcal{P} \cdot \overset{2}{D}^\circ B)(\bar{X}, \bar{Y}, \bar{Z}). \\ &= L(\overset{2}{D}^\circ B)(\phi(\bar{X}), \phi(\bar{Y}), \phi(\bar{Z})). \\ &= L(\overset{2}{D}^\circ B)(\bar{X}, \bar{Y}, \bar{Z}) - \ell(\bar{Z})(\overset{2}{D}^\circ B)(\bar{X}, \bar{Y}, \bar{\eta}) \\ &\quad - \ell(\bar{Y})(\overset{2}{D}^\circ B)(\bar{X}, \bar{\eta}, \bar{Z}) + L^{-1}\ell(\bar{Y})\ell(\bar{Z})(\overset{2}{D}^\circ B)(\bar{X}, \bar{\eta}, \bar{\eta}) \\ &\quad - \ell(\bar{X})(\overset{2}{D}^\circ B)(\bar{\eta}, \bar{Y}, \bar{Z}) + L^{-1}\ell(\bar{X})\ell(\bar{Z})(\overset{2}{D}^\circ B)(\bar{\eta}, \bar{Y}, \bar{\eta}) \\ &\quad + L^{-1}\ell(\bar{X})\ell(\bar{Y})(\overset{2}{D}^\circ B)(\bar{\eta}, \bar{\eta}, \bar{Z}) - L^{-2}\ell(\bar{X})\ell(\bar{Y})\ell(\bar{Z})(\overset{2}{D}^\circ B)(\bar{\eta}, \bar{\eta}, \bar{\eta}). \end{aligned}$$

In view Lemmas 3.5 and 3.7, the above Equation reduces to

$$A(\bar{X}, \bar{Y}, \bar{Z}) = L(\overset{2}{D}^\circ B)(\bar{X}, \bar{Y}, \bar{Z}) + \ell(\bar{Z})B(\bar{X}, \bar{Y}) + \ell(\bar{Y})B(\bar{X}, \bar{Z}). \quad (4.1)$$

On the other hand, from (3.3) and Lemmas 3.5, 3.6 and 3.7, we have

$$\begin{aligned}
(D^\circ B)(\bar{X}, \bar{Y}, \bar{Z}) &= \ell(\bar{X})(D^\circ C)(\bar{Y}, \bar{Z}) + L(D^\circ D^\circ C)(\bar{X}, \bar{Y}, \bar{Z}) \\
&\quad + L^{-1}h(\bar{X}, \bar{Z})C(\bar{Y}) + \ell(\bar{Z})(D^\circ C)(\bar{X}, \bar{Y}) \\
&= \ell(\bar{X})(D^\circ C)(\bar{Y}, \bar{Z}) + L(D^\circ D^\circ C)(\bar{X}, \bar{Y}, \bar{Z}) \\
&\quad + L^{-1}h(\bar{X}, \bar{Z})C(\bar{Y}) + L^{-1}\ell(\bar{Z})\ell(\bar{X})C(\bar{Y}) \\
&\quad + L\ell(\bar{Z})(D^\circ D^\circ k)(\bar{X}, \bar{Y}) \\
&= L^{-1}\ell(\bar{X})B(\bar{Y}, \bar{Z}) + L(D^\circ D^\circ C)(\bar{X}, \bar{Y}, \bar{Z}) \\
&\quad + L^{-1}h(\bar{X}, \bar{Z})C(\bar{Y}) + L\ell(\bar{Z})(D^\circ D^\circ k)(\bar{X}, \bar{Y}).
\end{aligned}$$

From which, together (4.1), we obtain

$$\begin{aligned}
A(\bar{X}, \bar{Y}, \bar{Z}) &= \mathfrak{S}_{\bar{X}, \bar{Y}, \bar{Z}}\{\ell(\bar{X})B(\bar{Y}, \bar{Z})\} + h(\bar{X}, \bar{Z})C(\bar{Y}) \\
&\quad + L^2\{(D^\circ D^\circ C)(\bar{X}, \bar{Y}, \bar{Z}) + \ell(\bar{Z})(D^\circ D^\circ k)(\bar{X}, \bar{Y})\}. \quad (4.2)
\end{aligned}$$

On the other hand, for every (1) π -form ω , one can show that

$$(D^\circ D^\circ \omega)(\bar{X}, \bar{Y}, \bar{Z}) - (D^\circ D^\circ \omega)(\bar{Y}, \bar{X}, \bar{Z}) = 0 \quad (4.3)$$

Then the result follows from (4.2) and (4.3). \square

Theorem 4.2. *A Finsler manifold (M, L) of scalar curvature k reduces to a Finsler manifold of constant curvature k if and only if the π -scalar form $C = L D^\circ k$ vanishes.*

Proof. Firstly, suppose that (M, L) is Finsler manifold of scalar curvature k . If (M, L) reduces to a Finsler manifold of constant curvature k , then the π -scalar form C vanishes immediately.

Conversely, suppose that (M, L) is a Finsler manifold of scalar curvature k such that the π -scalar form C vanishes. Hence

$$D^\circ k = 0. \quad (4.4)$$

By (3.1), together with $C = 0$, we obtain

$$\widehat{R}^\circ(\bar{X}, \bar{Y}) = kL\ell(\bar{X})\bar{Y} - \ell(\bar{Y})\bar{X}. \quad (4.5)$$

On the other hand, we have [13]:

$$\mathfrak{S}_{\bar{X}, \bar{Y}, \bar{Z}}\{(D^\circ_{\beta\bar{X}}R^\circ)(\bar{Y}, \bar{Z}, \bar{W}) + P^\circ(\widehat{R}^\circ(\bar{X}, \bar{Y}), \bar{Z})\bar{W}\} = 0.$$

From which, noting that the $(v)hv$ -torsion \widehat{P}° vanishes [13], it follows that

$$\mathfrak{S}_{\bar{X}, \bar{Y}, \bar{Z}}(D^\circ_{\beta\bar{X}}\widehat{R}^\circ)(\bar{Y}, \bar{Z}) = 0. \quad (4.6)$$

Now, from (4.5) and (4.6), using (3.4) and $D_{\beta\bar{X}}^\circ\ell = 0$, we get

$$\begin{aligned} &L(D_{\beta\bar{X}}^\circ k)(\ell(\bar{Y})\bar{Z} - \ell(\bar{Z})\bar{Y}) + L(D_{\beta\bar{Y}}^\circ k)(\ell(\bar{Z})\bar{X} - \ell(\bar{X})\bar{Z}) \\ &+ L(D_{\beta\bar{Z}}^\circ k)(\ell(\bar{X})\bar{Y} - \ell(\bar{Y})\bar{X}) = 0. \end{aligned}$$

Setting $\bar{Z} = \bar{\eta}$ into the above equation, noting that $\ell(\bar{\eta}) = L$ (Lemma 3.5), we obtain

$$\begin{aligned} &L(D_{\beta\bar{X}}^\circ k)(\ell(\bar{Y})\bar{\eta} - L\bar{Y}) + L(D_{\beta\bar{Y}}^\circ k)(L\bar{X} - \ell(\bar{X})\bar{\eta}) \\ &+ L(D_{\beta\bar{\eta}}^\circ k)(\ell(\bar{X})\bar{Y} - \ell(\bar{Y})\bar{X}) = 0. \end{aligned}$$

Taking the trace of both sides with respect to \bar{Y} , it follows that

$$D_{\beta\bar{X}}^\circ k = L^{-1}(D_{\beta\bar{\eta}}^\circ k)\ell(\bar{X}). \quad (4.7)$$

Applying the v -covariant derivative with respect to \bar{Y} on both sides of (4.7), yields

$$\ell(\bar{Y})D_{\beta\bar{X}}^\circ k + L(\overset{2}{D}^\circ\overset{1}{D}^\circ k)(\bar{X}, \bar{Y}) = L^{-1}h(\bar{X}, \bar{Y})(D_{\beta\bar{\eta}}^\circ k) + \ell(\bar{X})(\overset{2}{D}^\circ\overset{1}{D}^\circ k)(\bar{\eta}, \bar{Y}).$$

From (4.4), noting that $(\overset{2}{D}^\circ\overset{1}{D}^\circ k)(\bar{X}, \bar{Y}) = (\overset{1}{D}^\circ\overset{2}{D}^\circ k)(\bar{Y}, \bar{X})$, the above relation reduces to (provided that $n \geq 3$)

$$\ell(\bar{Y})D_{\beta\bar{X}}^\circ k = L^{-1}h(\bar{X}, \bar{Y})(D_{\beta\bar{\eta}}^\circ k).$$

Setting $\bar{Y} = \bar{\eta}$ into the above equation, noting that $\ell(\bar{\eta}) = L$ and $h(\cdot, \bar{\eta}) = 0$, it follows that $D_{\beta\bar{X}}^\circ k = 0$. Consequently,

$$\overset{1}{D}^\circ k = 0. \quad (4.8)$$

Now, (4.4) and (4.8) imply that k is a constant. \square

Theorem 4.3. *A Finsler manifold (M, L) of scalar curvature k reduces to a Finsler manifold of constant curvature k if and only if the π -scalar form $B = L(\mathcal{P} \cdot \overset{2}{D}^\circ C)$ vanishes.*

Proof. Let (M, L) be a Finsler manifold of scalar curvature k .

If (M, L) reduces to a Finsler manifold of constant curvature k , then, by Theorem 4.2, the π -scalar form C vanishes. Consequently, the π -scalar form B vanishes.

Conversely, suppose that (M, L) has the property that the π -scalar form B vanishes. Hence, the π -scalar form A of Lemma 4.1 vanishes. Consequently by Lemma 4.1, we have

$$C(\bar{X})\phi(\bar{Y}) - C(\bar{Y})\phi(\bar{X}) = 0$$

Taking the trace of both sides of the above equation with respect to \bar{Y} , noting that $Tr(\phi) = n - 1$ [12], it follows that

$$(n - 2)C(\bar{X}) = 0.$$

From which, the π -scalar form C vanishes as $n \geq 3$. Consequently, by Theorem 4.2, (M, L) is of constant curvature k . \square

Theorem 4.4. *A Finsler manifold (M, L) of scalar curvature k reduces to a Finsler manifold of constant curvature k if and only if the π -scalar form $A = L(\mathcal{P} \cdot \overset{2}{D}^\circ B)$ vanishes.*

Proof. The proof is similar to that of the above theorem. \square

Summing up, we have.

Theorem 4.5. *Let (M, L) be a Finsler manifold of scalar curvature k . The following assertions are equivalent:*

- (a): (M, L) is of constant curvature k ,
- (b): The $(1)\pi$ -scalar form $C = L \overset{2}{D}^\circ k$ vanishes,
- (c): The $(2)\pi$ -scalar form $B = L(\mathcal{P} \cdot \overset{2}{D}^\circ C)$ vanishes,
- (d): The $(3)\pi$ -scalar form $A = L(\mathcal{P} \cdot \overset{2}{D}^\circ B)$ vanishes.

Corollary 4.6. *A Finsler manifold of scalar curvature is of constant curvature if and only if $\mathcal{P} \cdot F = 0$, where F is the π -form defined by Corollary 3.8.*

Proof. The proof follows from the identity

$$\mathcal{P} \cdot F(\bar{X}, \bar{Y}) = \frac{1}{3}B(\bar{X}, \bar{Y}).$$

which can easily be proved. \square

REFERENCES

1. H. Akbar-Zadeh, *Initiation to global Finsler geometry*, Elsevier, 2006.
2. H. Izumi and T. N. Srivastava, *On R_3 -like Finsler spaces*, Tensor, N. S., **32** (1978), 339–349.
3. H. Izumi and M. Yoshida, *On Finsler spaces of perpendicular scalar curvature*, Tensor, N. S., **32** (1978), 219–224.
4. M. Matsumoto, *On Finsler spaces with curvature tensors of some special forms*, Tensor, N. S., **22** (1971), 201–204.
5. M. Matsumoto, *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Press, Otsu, Japan, 1986.
6. R. Miron and M. Anastasiei, *The geometry of Lagrange spaces: Theory and applications*, Kluwer Acad. Publ., 1994.
7. H. Rund, *The differential geometry of Finsler spaces*, Springer-Verlag, Berlin, 1959.
8. T. Sakaguchi, *On Finsler spaces of scalar curvature*, Tensor, N. S., **38** (1982), 211–219.
9. A. A. Tamim, *Special Finsler manifolds*, J. Egypt. Math. Soc., **10(2)** (2002), 149–177.
10. M. Yoshida, *On Finsler spaces of H_P -scalar curvature*, Tensor, N. S., **38** (1982), 205–210.
11. Nabil L. Youssef, S. H. Abed and A. Soleiman, *Cartan and Berwald connections in the pullback formalism*, Algebras, Groups and Geometries, **25**, 4 (2008), 363–386. arXiv: 0707.1320 [math. DG].

12. Nabil L. Youssef, S. H. Abed and A. Soleiman, *A global approach to the theory of special Finsler manifolds*, J. Math. Kyoto Univ., **48**, 4 (2008), 857–893. arXiv: 0704.0053 [math.DG].
13. Nabil L. Youssef, S. H. Abed and A. Soleiman, *Geometric objects associated with the fundamental connections in Finsler geometry*, J. Egypt. Math. Soc., **18**, 1 (2010), 67–90. arXiv: 0805.2489 [math.DG].

Received: 07.02.2020

Accepted: 28.06.2020