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On projectively flat spherically symmetric Finsler metrics

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Abstract. The class of spherically symmetric Finsler metrics is studied and locally dually flat and locally projectively flat spherically symmetric Finsler metrics is classified.

Keywords: Spherically symmetric, Finsler metric, Projectively flat metric.

1. Introduction

Let F be a Finsler metric defined on a convex domain $\Omega \subset \mathbb{R}^n$ and be invariant under any rotation in $\mathbb{R}^n$. Then, F is called spherically symmetric. In [8], by solving the equation of Killing fields, Zhou showed that there exists a positive function ϕ so that F can be written as $F = \phi(r, u, v)$ where

$$r = |x|, \quad u = |y|, \quad v =< x, y >, \quad s = \frac{< x, y >}{|y|}$$

and $|.|$ and $<,>$ denote the Euclidean norm and inner product in $\mathbb{R}^n$, respectively. Many well-known classical Finsler metrics such as Funk metric and Berwald metric are spherically symmetric. Having a nice symmetry makes the

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class of spherically symmetric Finsler metrics very important both in mathematics and applications [5].

A Finsler metric is said to be locally projectively flat if at any point there is a local coordinate system in which the geodesics are straight lines as point sets [3][7]. Recently, motivated by Hilbert's Fourth problem relating to classify the projectively flat Finsler metrics in $\mathbb{R}^n$, Zhou completely classified projectively flat spherically symmetric Finsler metrics [8][4]. According to Rapcsák Lemma, a Finsler metric F on an open subset $\mathcal{U} \in R^n$ is projectively flat on $\mathcal{U}$ if and only if it satisfies $F_{x^k y^l} y^k = F_{x^l}$.

In [1], Amari-Nagaoka introduced the notion of dually flat Riemannian metrics when they study the information geometry on Riemannian manifolds. In Finsler geometry, Shen extends the notion of locally dually flatness for Finsler metrics [6]. A Finsler metric $F = F(x, y)$ on an open subset $U \subset \mathbb{R}^n$ is dually flat if and only if it satisfies

$$[F^2]_{x^m y^l} y^m = 2[F^2]_{x^l}.$$

In this paper, we characterize locally dually flat spherically symmetric Finsler metrics and give complete classification of projectively flat metrics among them. More precisely, we have the following.

Theorem 1.1. *Let $F = \phi(r, u, v)$ be a spherically symmetric Finsler metric on a convex domain $\Omega \subset \mathbb{R}^n$. Then F is projectively flat and locally dually flat if and only if*

$$\phi(r, u, v) = \frac{\sqrt{(k - c^2 r^2)u^2 + c^2 v^2} + cv}{k - c^2 r^2}, \tag{1.1}$$

where k and c are constants. More precisely, every projectively flat and locally dually flat spherically symmetric Finsler metrics is a deformation of the standard Funk metric.

It is remarkable that, the Funk metric F is defined on the standard unite ball B^n as follows

$$F(x, y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle}{1 - |x|^2}.$$

The Funk metric is a special case of the metric defined by (1.1) with $k = c = 1$.

2. **Preliminaries**

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on slit tangent bundle $TM_0 = TM - \{0\}$, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where $G^i = G^i(x, y)$ are called spray coefficients and given by

$$G^i = \frac{1}{4} g^{il} \Big[\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l} \Big].$$

G is called the spray associated to F.

A Finsler metric $F = F(x, y)$ is called locally projectively flat if at any point there is a local coordinate system in which the geodesics are straight lines as point sets. It is known that a Finsler metric $F(x, y)$ on an open domain $U \subset \mathbb{R}^n$ is locally projectively flat if and only if its geodesic coefficients G^i are in the form

$$G^i = Py^i,$$

where $P : TU = U \times \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous with degree one, $P(x, \lambda y) = \lambda P(x, y)$, $\lambda > 0$. We call $P(x, y)$ the projective factor of $F(x, y)$.

A Finsler metric $F = F(x, y)$ on a manifold M is said to be locally dually flat if at any point there is a coordinate system (x^i) in which the spray coefficients are in the following form

$$G^i = -\frac{1}{2} g^{ij} H_{y^j},$$

where $H = H(x, y)$ is a C^∞ homogeneous scalar function on TM_0. Such a coordinate system is called an adapted coordinate system. In [6], Shen proved that the Finsler metric F on an open subset $U \subset \mathbb{R}^n$ is dually flat if and only if it satisfies

$$(F^2)_{x^k y^l} y^k = 2(F^2)_{x^l}.$$

In this case, $H = -\frac{1}{6}[F^2]_{x^m} y^m$.

A Finsler metric F on a domain $\Omega \subseteq \mathbb{R}^n$ is called spherically symmetric if it is invariant under any rotation in $\mathbb{R}^n$. According to the equation of Killing fields, there exists a positive function ϕ depending on two variables so that F can be written as

$$F = |y| \phi \left(|x|, \frac{\langle x, y \rangle}{|y|} \right),$$

where x is a point in the domain Ω, y is a tangent vector at the point x and $\langle , \rangle$, $|\cdot|$ are standard inner product and norm in Euclidean space, respectively.

Lemma 2.1. ([8]) *A Finsler metric F on a convex domain $\Omega \subseteq \mathbb{R}^n$ is spherically symmetric if and only if there exists a positive function $\phi = \phi(r, u, v)$, such that*

$$F(x, y) = \phi(|x|, |y|, \langle x, y \rangle),$$

where

$$|x| = \sqrt{\Sigma_{i=1}^n (x^i)^2}, \quad |y| = \sqrt{\Sigma_{i=1}^n (y^i)^2}, \quad \langle x, y \rangle = \Sigma_{i=1}^n x^i y^i.$$

3. Proof of Theorem 1.1

A Finsler metric $F = F(x, y)$ on an open subset $U \subset \mathbb{R}^n$ is dually flat if and only if it satisfies

$$[F^2]_{x^m y^l} y^m = 2[F^2]_{x^l}. \tag{3.1}$$

In [2], X. Cheng, Z. Shen and Y. Zhou studied and characterized projectively and locally dually flat Finsler metrics on a convex domain $\Omega \subset \mathbb{R}^n$ and found the following PDEs.

Theorem 3.1. *Let $F = F(x, y)$ be a Finsler metric on an open subset $U \subset \mathbb{R}^n$. Then F is dually flat and projectively flat on U if and only if it satisfies*

$$F_{x^k} = cFF_{y^k}, \tag{3.2}$$

where c is a constant.

For a spherically symmetric Finsler metric in $\mathbb{R}^n$, we have the following.

Theorem 3.2. *Let $F = \phi(r, u, v)$ be a spherically symmetric Finsler metric in $\mathbb{R}^n$. Then F is locally dually flat if and only if the following holds*

$$s(\psi_r\psi_s + \psi\psi_{rs}) + r(\psi_s^2 + \psi\psi_{ss}) - 2\psi\psi_r = 0, \tag{3.3}$$

where

$$s := \frac{v}{u} \quad \text{and} \quad \psi(r, s) := \phi(r, 1, \frac{v}{u}).$$

Proof. By direct computations, we have

$$F_{x^l} = \frac{u}{r}\psi_r x^l + \psi_s y^l, \tag{3.4}$$

$$F_{x^k} y^k = u^2\Big[\frac{s}{r}\psi_r + \psi_s\Big], \tag{3.5}$$

$$F_{y^l} = \psi_s x^l + \frac{1}{u}\Big[\psi - s\psi_s\Big] y^l, \tag{3.6}$$

$$F_{x^k y^l} y^k = u\Big[\frac{s}{r}\psi_{rs} + \psi_{ss}\Big] x^l + \Big[\frac{s}{r}\psi_r - \frac{s^2}{r}\psi_{rs} - \psi_{ss}s + \phi_s\Big] y^l. \tag{3.7}$$

By (3.1), F is locally dually flat if and only if

$$F_{y^l} F_{x^k} y^k + FF_{x^k y^l} y^k - 2FF_{x^l} = 0. \tag{3.8}$$

Plugging (3.4), (3.5), (3.6) and (3.7) in (3.8) imply (3.1). This completes the proof. □

In [8], L. Zhou gave the following classification of projectively spherically symmetric Finsler metric.

Theorem 3.3. *Suppose that F is a spherically symmetric Finsler metric on a convex domain $\Omega \subset \mathbb{R}^n$, F is projectively flat if and only if there exist smooth functions $f = f(t) > 0$ and $g = g(r)$ such that*

$$\phi(r,u,v) = \int f(\frac{v^2}{u^2} - r^2)du + g(r)v, \tag{3.9}$$

where $F(x,y) = \phi(|x|, |y|, \langle x,y\rangle)$.

Now, we are going to prove the main result.

Proof of Theorem 1.1: Using the identities

$$F_{x^l} = \frac{\phi_r}{r}x^l + \phi_v y^l, \quad \text{and} \quad F_{y^l} = \phi_v x^l + \frac{\phi_u}{u}y^l$$

and by Theorem 3.1, we get

$$\frac{\phi_r}{r} = c\phi\phi_v, \quad \phi_v = \frac{c\phi\phi_u}{u}. \tag{3.10}$$

By (3.9) we have

$$\phi_u(r,u,v) = f(\frac{v^2}{u^2} - r^2). \tag{3.11}$$

Plugging (3.11) into (3.10) yields

$$\phi_v(r,u,v) = \frac{c}{u}\phi(r,u,v)f\Big(\frac{v^2}{u^2} - r^2\Big). \tag{3.12}$$

Using 1-homogeneity of ϕ with respect to (u,v) and Euler's theorem, we have $\phi = \phi_u u + \phi_v v$. Thus, from (3.12) we conclude that

$$\phi(r,u,v) = \frac{f(\frac{v^2}{u^2} - r^2)u^2}{u - cf(\frac{v^2}{u^2} - r^2)v}. \tag{3.13}$$

Therefore, it suffices to find explicit formula of f. Taking derivative of (3.13) with respect to r implies that

$$\frac{\phi_r(r,u,v)}{r} = \frac{-2f'(\frac{v^2}{u^2} - r^2)u^3}{\Big[u - cf(\frac{v^2}{u^2} - r^2)v\Big]^2} \tag{3.14}$$

Substituting (3.10) and (3.12) into (3.14), we obtain the following ODE on f:

$$2f' + c^2 f^3 = 0 \tag{3.15}$$

Solving (3.15), we have

$$f(t) = \frac{1}{\sqrt{c^2 t + k}},$$

where k is a constant. The proof follows from (3.13). □

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