Journal of Finsler Geometry and its Applications
Vol. 7, No. 2 (2026), pp 137-149
https://doi.org/10.22098/jfga.2026.19259.1202

Construction of new left-invariant Randers metrics on Lie groups

Azar Fatahi[a], Masoumeh Hosseini[a] and Hamid Reza Salimi Moghaddam[a*]

[a]Department of Pure Mathematics, Faculty of Mathematics and Statistics, University of Isfahan, Isfahan, 81746-73441-Iran.

E-mail: azi.f3000@yahoo.com
E-mail: hoseini_masomeh@ymail.com
E-mail: hr.salimi@sci.ui.ac.ir

Abstract. In this short article, using a left-invariant Randers metric F on a real Lie group G, we define a new left-invariant Randers metric $\tilde{F}$. We show that F is of Berwald (Douglas) type if and only if $\tilde{F}$ is of Berwald (Douglas) type. In the case of Berwaldian metrics, we give the relation between their flag curvatures. Also, we have studied the relations between their base Riemannian metrics. Finally, as examples, the results are studied in the Heisenberg group and almost Abelian Lie groups.

Keywords: Invariant Riemannian metric, invariant Randers metric, Lie group.

1. Introduction

(α, β)-metrics are a type of Finsler metric that balances simplicity with rich geometric structure, making them widely studied in both pure and applied contexts (see [1], [3], and [4]). Geometrically, an (α, β)-metric can be viewed as a deformation of a Riemannian metric α by a 1-form β, which allows one to model anisotropic phenomena while retaining tractable curvature computations.

*Corresponding Author
AMS 2020 Mathematics Subject Classification: 53C30, 53C60, 53C25, 22E60

This work is licensed under a Creative Commons Attribution-NonCommercial 4.0 International License.

Copyright © 2026 The Author(s). Published by University of Mohaghegh Ardabili

Matsumoto formally defined these metrics in [7], but the first instance appeared in 1941, when G. Randers introduced the Randers metric for applications in physics, particularly in general relativity (see [12] and [13]). From a geometric perspective, a Randers metric represents a solution to Zermelo's navigation problem, describing the shortest travel time under a wind, a clear physical motivation for adding a vector field (or 1-form) to a Riemannian background.

Similar to the Riemannian case, left-invariant Finsler metrics on Lie groups are of special significance because they encode the geometry of the group in a homogeneous way, allowing one to study the interplay between algebraic structure and curvature. On Lie groups, left-invariant (α, β)-metrics are particularly natural, the Riemannian metric α can be chosen as a left-invariant Riemannian metric (e.g., derived from an inner product on the Lie algebra), and the 1-form β can be taken as left-invariant, corresponding to a fixed covector at the identity. This setting bridges Finsler geometry, Lie theory, and control theory. Thus, the study of left-invariant (α, β)-metrics on Lie groups is a very interesting and fertile field in Finsler geometry (see for example [5], [9], [11] , [14] and [16]). Deforming a Finsler metric while preserving important geometric properties, such as being Berwald or Douglas, provides a valuable method for constructing explicit examples in this field.

In this article, we start with a left-invariant Randers metric F and define a new left-invariant Randers metric $\tilde{F}$. First, let us consider the general case on an arbitrary smooth manifold. Suppose that h is a Riemannian metric on a smooth manifold M and X is a nowhere zero vector field on M (if there exists such a vector field) with the assumption $\sqrt{\mathsf{h}(X,X)} < 1$, then $F(x,y) = \sqrt{\mathsf{h}(y,y)} + \mathsf{h}(X(x),y)$ is a Randers metric on M. For any $y \in T_xM \setminus \{0\}$ the fundamental tensor g_y is an inner product on T_xM. Hence, we can consider g_X as a Riemannian metric on M. According to [10], for any $v, z \in T_xM$, we have:

$$g_X(v,z) = \Big(\mathsf{h}(v,z) + \frac{1}{\|X(x)\|}\mathsf{h}(X(x),v)\mathsf{h}(X(x),z)\Big)(1 + \|X(x)\|), \tag{1.1}$$

where the norm $\|X(x)\|$ is computed by the Riemannian metric h.

Suppose that for any $x \in M$ we have $\sqrt{g_X(X(x),X(x))} < 1$. Now, we define a new Randers metric $\tilde{F}$ on M as follows:

$$\tilde{F}(x,y) = \sqrt{g_X(y,y)} + g_X(X(x),y). \tag{1.2}$$

We mention that one may use g_y and X, g_X and another vector field Z, or g_y and Z to define different Randers metrics or other (α, β)-metrics. However, we have considered g_X and the vector field X for simplicity. Using the same way one may define a sequence of new metrics.

If G is a Lie group equipped with a left-invariant Riemannian metric h, and X is an arbitrary non-zero left-invariant vector field on G with the condition

$\|X\| < 1$, then we see that F, which is defined as above, is a left-invariant Randers metric on G. Now, using (1.1), we see that the Riemannian metric g_X is a left-invariant Riemannian metric on G. On the other hand, if $g_X(X,X) < 1$, (for example it happens if $\sqrt{\mathsf{h}(X,X)} < \frac{1}{2}$) then the Finsler metric $\tilde{F}$ defined by (1.2) is a left-invariant Randers metric on G.

In this paper, we study the geometry of the left-invariant Riemannian metric g_X and the left-invariant Randers metric $\tilde{F}$ and their relations with the metrics h and F. More precisely, in the next section, we focus on the Riemannian metrics g_X and h. We compute the Levi-Civita connection and the sectional curvature of g_X. Also, we give a necessary and sufficient condition for a geodesic (Killing) vector field of h to be a geodesic (Killing) vector field with respect to g_X. In section 3, the left-invariant Randers metric $\tilde{F}$ which is defined by (1.2), is considered, and it is shown that $\tilde{F}$ is of Douglas (Berwald) type if and only if F is of Douglas (Berwald) type. In the last section, we give some examples of such left-invariant Randers metrics and compute their curvatures on the Lie groups which J. Milnor studied in [8] (almost Abelian Lie groups), and the Heisenberg group (see [15]).

2. The geometry of the Riemannian metric g_X

In this section, we have considered the left-invariant Riemannian metric g_X defined by (1.1), where h is a left-invariant Riemannian metric and $X \in \mathfrak{g}$ is a left-invariant vector field on a Lie group G and $\mathfrak{g}$ denotes the Lie algebra of G. We will denote the inner products defined by g_X and h on $\mathfrak{g}$ with g_X and $\langle , \rangle$, respectively. We know that there exists a linear mapping $\phi : \mathfrak{g} \to \mathfrak{g}$ such that for any $v, z \in \mathfrak{g}$ we have:

$$g_X(v,z) = \langle v, \phi(z) \rangle. \tag{2.1}$$

Now, using equations (1.1) and (2.1), we can easily write:

$$\phi(z) = (1 + \|X\|)z + \frac{1 + \|X\|}{\|X\|}\langle X, z\rangle X. \tag{2.2}$$

Since ϕ is a symmetric mapping, there exists an orthonormal basis concerning the inner product $\langle , \rangle$, of the Lie algebra $\mathfrak{g}$ consisting of eigenvectors of ϕ. Suppose that $\{X_1, \cdots, X_n\}$ is the above basis consisting of eigenvectors and for $i = 1, \cdots, n$, λ_i is the eigenvalue corresponded to the eigenvector X_i, i.e. $\phi(X_i) = \lambda_i X_i$.

We know that $X = \sum_{j=1}^{n} \langle X, X_j \rangle X_j$, so for any $i = 1 \cdots n$ we have:

$$\lambda_i X_i = (1 + \|X\|)X_i + \frac{1 + \|X\|}{\|X\|}\langle X, X_i\rangle \sum_{j=1}^{n} \langle X, X_j\rangle X_j. \tag{2.3}$$

For any $1 \leq i, j \leq n$, we have:

$$\lambda_i = (1 + \|X\|)\Big(1 + \frac{\langle X, X_i\rangle^2}{\|X\|}\Big),$$

and

$$g_X(X_i, X_j) = \lambda_j \delta_{ij}.$$

On the other hand, equation (2.3) shows that if for some i_0 $(1 \leq i_0 \leq n)$ we have $\langle X, X_{i_0} \rangle \neq 0$, then for any $i \neq i_0$ we have $\langle X, X_i \rangle = 0$, which shows that $X_{i_0} = \frac{X}{\|X\|}$. So we have $\lambda_{i_0} = (1 + \|X\|)^2$ and $\lambda_i = 1 + \|X\|$, where $i \neq i_0$.

Notation 2.1. *From now on the set $\{X_1, X_2, \cdots, X_n\}$ denotes the orthonormal basis relative to the inner product $\langle , \rangle$ for $\mathfrak{g}$ consisting of eigenvectors of ϕ and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are their corresponding eigenvalues.*

Proposition 2.2. *Let (G, h) be a Lie group with a left-invariant Riemannian metric h. Suppose that X is a left-invariant vector field on G such that $\|X\| < 1$, where the norm is computed with respect to h. Then the Levi-Civita connection of the Riemannian metric g_X, defined by the equation (1.1) is obtained as follows:*

$$\begin{aligned}\nabla_{X_i} X_j &= \sum_{k=1}^{n} \frac{1}{2}\Big(\frac{\lambda_i}{\lambda_k}\alpha_{kji} - \frac{\lambda_j}{\lambda_k}\alpha_{ikj} - \alpha_{jik}\Big) X_k \\ &= \sum_{k=1}^{n} \frac{1}{2}\Big(\frac{\|X\| + \langle X, X_i\rangle^2}{\|X\| + \langle X, X_k\rangle^2}\alpha_{kji} - \frac{\|X\| + \langle X, X_j\rangle^2}{\|X\| + \langle X, X_k\rangle^2}\alpha_{ikj} - \alpha_{jik}\Big) X_k,\end{aligned}$$

where α_{ijk} are structural coefficients with respect to the basis $\{X_1, X_2, \cdots, X_n\}$.

Proof. Using the Koszul formula, we have:

$$g_X(\nabla_{X_i} X_j, X_k) = \frac{1}{2}\Big(g_X(X_i, [X_k, X_j]) - g_X(X_j, [X_i, X_k]) - g_X(X_k, [X_j, X_i])\Big). \tag{2.4}$$

It follows from $[X_i, X_j] = \sum_{k=1}^{n} \alpha_{ijk} X_k$ and (2.1) that

$$g_X(\nabla_{X_i} X_j, X_k) = \frac{1}{2}(\lambda_i \alpha_{kji} - \lambda_j \alpha_{ikj} - \lambda_k \alpha_{jik}). \tag{2.5}$$

Now, the fact that,

$$\nabla_{X_i} X_j = \sum_{k=1}^{n} \langle \nabla_{X_i} X_j, X_k \rangle X_k = \sum_{k=1}^{n} \frac{1}{\lambda_k} g_X(\nabla_{X_i} X_j, X_k) X_k, \tag{2.6}$$

together with a direct computation, completes the proof. □

Proposition 2.3. *Under the same assumptions as in Proposition 2.2, if K is the sectional curvature of Riemannian metric g_X, then the formula for K is as follows:*

$$\begin{aligned}K(X_i, X_j) \;=\; & \frac{1}{4}\sum_{l=1}^{n}\Big(- \frac{4}{\lambda_l}\alpha_{ljj}\alpha_{lii} - \frac{2}{\lambda_l}\alpha_{lji}\alpha_{ilj} + \frac{2}{\lambda_j}\alpha_{ijl}\alpha_{jli} + \frac{2}{\lambda_i}\alpha_{ilj}\alpha_{jil} \\ & - \frac{3\lambda_l}{\lambda_i\lambda_j}\alpha_{ijl}^2 + \frac{\lambda_j}{\lambda_i\lambda_l}\alpha_{ilj}^2 + \frac{\lambda_i}{\lambda_j\lambda_l}\alpha_{lji}^2\Big).\end{aligned} \tag{2.7}$$

Proof. It is easy to see that, the sectional curvature of Riemannian metric g_X is as follows:

$$K(X_i, X_j) = \frac{1}{\lambda_j}\langle R(X_i, X_j)X_j, X_i\rangle. \tag{2.8}$$

On the other hand, for the curvature tensor of the Riemannian metric we have:

$$R(X_i, X_j)X_k = \nabla_{X_i}\nabla_{X_j}X_k - \nabla_{X_j}\nabla_{X_i}X_k - \nabla_{[X_i,X_j]}X_k. \tag{2.9}$$

Now, a direct computation gives:

$$\nabla_{X_i}\nabla_{X_j}X_k = \frac{1}{4}\sum_{l,h=1}^{n}\Big(\frac{\lambda_j}{\lambda_l}\alpha_{lkj} - \frac{\lambda_k}{\lambda_l}\alpha_{jlk} - \alpha_{kjl})(\frac{\lambda_i}{\lambda_h}\alpha_{hli} - \frac{\lambda_l}{\lambda_h}\alpha_{ihl} - \alpha_{lih}\Big)X_h. \tag{2.10}$$

$$\nabla_{X_j}\nabla_{X_i}X_k = \frac{1}{4}\sum_{l,h=1}^{n}\Big(\frac{\lambda_i}{\lambda_l}\alpha_{lki} - \frac{\lambda_k}{\lambda_l}\alpha_{ilk} - \alpha_{kil})(\frac{\lambda_j}{\lambda_h}\alpha_{hlj} - \frac{\lambda_l}{\lambda_h}\alpha_{jhl} - \alpha_{ljh}\Big)X_h. \tag{2.11}$$

Also due to the equation $[X_i, X_j] = \sum_{k=1}^{n}\alpha_{ijk}X_k$, we have:

$$\nabla_{[X_i,X_j]}X_k = \frac{1}{2}\sum_{l,h=1}^{n}\alpha_{ijl}\Big(\frac{\lambda_l}{\lambda_h}\alpha_{hkl} - \frac{\lambda_k}{\lambda_h}\alpha_{lhk} - \alpha_{klh}\Big)X_h. \tag{2.12}$$

Hence we obtain

$$\begin{aligned} R(X_i, X_j)X_k = \frac{1}{4}\sum_{l,h=1}^{n}\Big\{&(\frac{\lambda_j}{\lambda_l}\alpha_{lkj} - \frac{\lambda_k}{\lambda_l}\alpha_{jlk} - \alpha_{kjl})(\frac{\lambda_i}{\lambda_h}\alpha_{hli} - \frac{\lambda_l}{\lambda_h}\alpha_{ihl} - \alpha_{lih}) \\ &- (\frac{\lambda_i}{\lambda_l}\alpha_{lki} - \frac{\lambda_k}{\lambda_l}\alpha_{ilk} - \alpha_{kil})(\frac{\lambda_j}{\lambda_h}\alpha_{hlj} - \frac{\lambda_l}{\lambda_h}\alpha_{jhl} - \alpha_{ljh}) \\ &- 2\alpha_{ijl}(\frac{\lambda_l}{\lambda_h}\alpha_{hkl} - \frac{\lambda_k}{\lambda_h}\alpha_{lhk} - \alpha_{klh})\Big\}X_h. \end{aligned} \tag{2.13}$$

So the equations (2.8) and (2.13), complete the proof. □

Proposition 2.4. *Under the same assumptions as in Proposition 2.2 we have:*

(1) *If v is a geodesic vector field with respect to the Riemannian metric h, then v is a geodesic vector field with respect to the Riemannian metric g_X, if and only if*

$$ad_v^*X = 0 \quad or \quad \langle X, v\rangle = 0. \tag{2.14}$$

(2) *If v is a Killing vector field with respect to the Riemannian metric h, then v is a Killing vector field with respect to the Riemannian metric g_X if and only if*

$$ad_v^*X = 0, \tag{2.15}$$

where ad_v^* denotes the transpose of ad_v with respect to $\langle , \rangle$.

Proof. For (1), we recall that, using [6], v is a geodesic vector field with respect to g_X, if and only if:

$$g_X(v,[v,y]) = 0 \quad \forall y \in \mathfrak{g}. \tag{2.16}$$

Now, according to (1.1) and because v is the geodesic vector field concerning the Riemannian metric $\langle , \rangle$, we have:

$$\langle X, v\rangle\langle X, [v,y]\rangle = 0,$$

which completes the proof.
For (2), we mention that v is a Killing vector field concerning g_X if and only if:

$$g_X([v,y],z) + g_X(y,[v,z]) = 0 \quad \forall y,z \in \mathfrak{g}. \tag{2.17}$$

Now, the equation (1.1), together with the fact that v is the Killing vector field for the Riemannian metric h, shows that the above equation is equivalent to the following equation:

$$\langle X,z\rangle\langle X,[v,y]\rangle + \langle X,y\rangle\langle X,[v,z]\rangle = 0 \qquad \forall y,z. \tag{2.18}$$

Now, if we put $z = y$, then the result is concluded. □

3. The geometry of the Randers metric defined by g_X and X

In this section, using a Randers metric F, we define a new Randers metric $\tilde{F}$ and study some of their geometric relations. Suppose that (M, F) is a Randers space, where F is defined by a Riemannian metric h and a vector field X. Let g_y be the fundamental tensor of the Randers metric F. As mentioned in the previous section, g_X is a Riemannian metric on M. Suppose that $\sqrt{g_X(X,X)} < 1$ (we recall that, for example, this condition holds if $\sqrt{\mathsf{h}(X,X)} < \frac{1}{2}$). We can easily see the equation 1.2 defines a Randers metric on M. A direct computation shows that

$$\tilde{F}(x,y) = \Big(\|y\|^2 + \frac{\mathsf{h}(X(x),y)^2}{\|X\|}\Big)^{\frac{1}{2}}\Big(1+\|X\|\Big)^{\frac{1}{2}} + \mathsf{h}(X(x),y)\Big)\Big(1+\|X\|\Big)^2, \tag{3.1}$$

where the norm $\|\|$ is computed by h.

We note that, during this article, when we use the Randers metric $\tilde{F}$, we have considered $\sqrt{g_X(X,X)} < 1$.

Proposition 3.1. *Let G be a Lie group with a left-invariant Randers metric F arising from a left-invariant Riemannian metric h and a left-invariant vector field X.*

(1) *F is of Douglas type if and only if $\tilde{F}$ is of Douglas type.*
(2) *F is of Berwald type if and only if $\tilde{F}$ is of Berwald type.*

Proof. For (1), we know that the Randers metric F is of Douglas type if and only if $\langle X, [\mathfrak{g},\mathfrak{g}]\rangle = 0$, where similar to the previous section $\langle , \rangle$ is the inner product induced by the Riemannian metric h on $\mathfrak{g}$. On the other hand, the equations

(2.1) and (2.2) together with a little computations show that $\langle X,[\mathfrak{g},\mathfrak{g}]\rangle = 0$ if and only if $g_X(X,[\mathfrak{g},\mathfrak{g}]) = 0$, so the proof is completed.
For the part (2), by Proposition 4.1 in [5], the Randers metric F is of Berwald type if and only if

$$\langle X,[y,z]\rangle = 0, \tag{3.2}$$
$$\langle [y,X],z\rangle + \langle [z,X],y\rangle = 0. \tag{3.3}$$

Consider the basis defined in Proposition 2.2. Suppose that F is of Berwald type. So F (and also $\tilde{F}$) is of Douglas type and for any $1 \leq i,j \leq n$ we have $\langle [X_i,X],X_j\rangle + \langle [X_j,X],X_i\rangle = 0$, which proves that $g_X([X_i,X],X_j) + g_X([X_j,X],X_i) = 0$.

Conversely, let $\tilde{F}$ be of Berwald type. Therefore, $\tilde{F}$ (and also F) is of Douglas type and

$$g_X([X_i,X],X_j)+g_X([X_j,X],X_i) = (1+\|X\|)(\langle X_j,[X_i,X]\rangle+\langle X_i,[X_j,X]\rangle) = 0. \tag{3.4}$$

The last equation completes the proof. □

If F (or equivalently $\tilde{F}$) is of Douglas type, then the formulas of Levi-Civita connections of h and g_X are simple. In this case, we have the following proposition.

Proposition 3.2. *Suppose that ∇ and $\bar{\nabla}$ denote the Levi-Civita connections of g_X and h, respectively. Under the same assumptions as in Proposition (3.1), if F is of Douglas type then for ∇ and $\bar{\nabla}$ we have:*

$$\begin{aligned} \nabla_{X_i}X_j &= \sum_{k=1}^{n} \frac{1+\|X\|}{2\lambda_k}(\alpha_{kji}-\alpha_{ikj}-\alpha_{jik})X_k \qquad (3.5)\\ &= \bar{\nabla}_{X_i}X_j + \frac{1}{2\|X\|(1+\|X\|)}\Big(\langle [X_i,X],X_j\rangle + \langle [X_j,X],X_i\rangle\Big)X. \end{aligned}$$

Proof. The Koszul formula

$$g_X(\nabla_{X_i}X_j,X_k) = \frac{1}{2}\Big(g_X(X_i,[X_k,X_j]) - g_X(X_j,[X_i,X_k]) - g_X(X_k,[X_j,X_i])\Big), \tag{3.6}$$

together with the equations (2.1) and (3.6) show that

$$g_X(\nabla_{X_i}X_j,X_k) = \frac{1+\|X\|}{2}(\alpha_{kji}-\alpha_{ikj}-\alpha_{jik}). \tag{3.7}$$

As well as

$$g_X(\nabla_{X_i}X_j,X_k) = (1+\|X\|)\langle \bar{\nabla}_{X_i}X_j,X_k\rangle. \tag{3.8}$$

Now the equation

$$\nabla_{X_i}X_j = \sum_{k=1}^{n}\langle \nabla_{X_i}X_j,X_k\rangle X_k \tag{3.9}$$

completes the proof of the first equation.

For the second equation, we recall that the Randers metric $\tilde{F}$ is of Douglas type if and only if $g_X(X,[y,z])=0$. Now according to (3.8) we have

$$\phi(\nabla_{X_i}X_j)=(1+\|X\|)\bar{\nabla}_{X_i}X_j. \tag{3.10}$$

On the other hand, it is known that (see [2]):

$$\nabla_{X_i}X_j=\frac{1}{2}([X_i,X_j]-ad^*_{X_i}X_j-ad^*_{X_j}X_i), \tag{3.11}$$

where the transpose is computed with respect to g_X. Due to the (1.1) we have

$$g_X(X,ad^*_{X_i}X_j)=(1+\|X\|)^2\langle X,ad^*_{X_i}X_j\rangle, \tag{3.12}$$

and

$$\langle\phi([X_i,X]),X_j\rangle=(1+\|X\|)\langle[X_i,X],X_j\rangle. \tag{3.13}$$

Now according to (3.11), (3.12)) and (3.13) we have:

$$\langle X,\nabla_{X_i}X_j\rangle=-\frac{1}{2(1+\|X\|)}\Big(\langle[X_i,X],X_j\rangle+\langle[X_j,X],X_i\rangle\Big). \tag{3.14}$$

On the other hand, we know

$$\phi(\nabla_{X_i}X_j)=(1+\|X\|)\nabla_{X_i}X_j+\frac{1+\|X\|}{\|X\|}\langle X,\nabla_{X_i}X_j\rangle X. \tag{3.15}$$

Now, substituting the relations (3.14) and (3.10) in the relation (3.15), completes the proof. □

Corollary 3.3. *In the previous proposition, if X belongs to the center of Lie algebra of G or X is a Killing vector field of the Riemannian metric h, then the Levi-Civita connection of h and the Levi-Civita connection of g_X coincide. In this case, the sectional curvature of the Riemannian metric g_X is as follows:*

$$K(X_i,X_j)=\frac{1}{1+\|X\|}\bar{K}(X_i,X_j), \tag{3.16}$$

where K and $\bar{K}$ denote the sectional curvature of g_X and h, respectively.

Proof. A direct computation together with (3.5), concludes the proof. □

Corollary 3.4. *With the assumptions of Proposition 3.1, if F (or equivalently $\tilde{F}$) is of Berwald type, then the Levi-Civita connection of h and the Levi-Civita connection of g_X coincide.*

Proposition 3.5. *Under the same assumptions as in Corollary 3.4, the sectional curvature of the Riemannian metric g_X is given by*

$$K(X_i,X_j)=\frac{1}{1+\|X\|}\bar{K}(X_i,X_j). \tag{3.17}$$

Proof. Using Corollary (3.4), we have

$$R(X_i, X_j)X_j = \bar{R}(X_i, X_j)X_j,$$

where $\bar{R}$ denotes the curvature tensor of the Riemannian metric h. Now it is sufficient to recall that, by equation (2.1), we have:

$$K(X_i, X_j) = \frac{1}{\lambda_j}\langle R(X_i, X_j)X_j, X_i\rangle. \tag{3.18}$$

We get the proof. □

Here we mention that $K(X_{i_0}, X_j) = \bar{K}(X_{i_0}, X_j) = 0$.

Proposition 3.6. *With the same assumptions as in Proposition 3.2, the sectional curvature of the Riemannian metric g_X is given by*

$$K(X_i, X_j) = \frac{1}{1+\|X\|}\Big\{\bar{K}(X_i, X_j) + \frac{1}{\|X\|(1+\|X\|)}\big(\langle[X_i, X], X_i\rangle\langle[X_j, X], X_j\rangle - \frac{1}{4}(\langle[X_i, X], X_j\rangle + \langle[X_j, X], X_i\rangle)^2\big)\Big\}.$$

Proof. We know the curvature tensor of the Riemannian metric g_X is as follows:

$$R(X_i, X_j)X_k = \nabla_{X_i}\nabla_{X_j}X_k - \nabla_{X_j}\nabla_{X_i}X_k - \nabla_{[X_i,X_j]}X_k. \tag{3.19}$$

Now, using Proposition 3.2 we have:

$$\begin{aligned} R(X_i, X_j)X_k = \bar{R}(X_i, X_j)X_k + \frac{1}{2\|X\|(1+\|X\|)}\Big\{\Big(&\langle[X_i, X], \bar{\nabla}_{X_j}X_k\rangle \\ &+ \langle[\bar{\nabla}_{X_j}X_k, X], X_i\rangle - \langle[X_j, X], \bar{\nabla}_{X_i}X_k\rangle \\ &- \langle[\bar{\nabla}_{X_i}X_k, X], X_j\rangle - \langle[[X_i, X_j], X], X_k\rangle \\ &- \langle[X_k, X], [X_i, X_j]\rangle\Big)X + \Big(\langle[X_j, X], X_k\rangle + \langle[X_k, X], X_j\rangle\Big)\bar{\nabla}_{X_i}X \\ &- \Big(\langle[X_i, X], X_k\rangle + \langle[X_k, X], X_i\rangle\Big)\bar{\nabla}_{X_j}X\Big\}, \end{aligned}$$

where $\bar{R}$ denotes the curvature tensor of the Riemannian metric h. We also know that the sectional curvature of the Riemannian metric g_X is as follows:

$$K(X_i, X_j) = \frac{1}{\lambda_j}\langle R(X_i, X_j)X_j, X_i\rangle \tag{3.20}$$

Then with a direct computation, we have:

$$\begin{aligned} K(X_i, X_j) = \frac{1}{\lambda_j}\Bigg\{&\bar{K}(X_i, X_j) - \frac{1}{\|X\|(1+\|X\|)}\Big(\langle\bar{R}(X_i, X_j)X_j, X\rangle\langle X, X_i\rangle \\ &- \langle[X_j, X], X_j\rangle\langle[X_i, X], X_i\rangle\Big) \\ &- \frac{1}{4\|X\|(1+\|X\|)}\Big(\langle[X_i, X], X_j\rangle + \langle[X_j, X], X_i\rangle\Big)^2\Bigg\}. \end{aligned}$$

We get the proof. □

In the following, we compute the flag curvature of the Randers metric $\tilde{F}$, in terms of the flag curvature of the Randers metric F, where they are of Berwald type.

Proposition 3.7. *Assume that G is a Lie group with a left-invariant Berwaldian Randers metric F arising from a left-invariant Riemannian metric h and a left-invariant vector field X. Then the flag curvature $K^{\tilde{F}}(X_j, P)$ of $\tilde{F}$ is given by*

$$K^{\tilde{F}}(X_j, P) = \frac{(1+\langle X, X_j\rangle)^2}{\lambda_j(1+\lambda_j^{\frac{1}{2}}\langle X, X_j\rangle)^2} K^F(X_j, P), \tag{3.21}$$

where $P = span\{X_i, X_j\}$ and $K^F(X_i, X_j)$ denotes the flag curvature of F.

Proof. By Proposition 3.1, $\tilde{F}$ is of Berwald type. According to the Proposition 3.1 in [5] , the flag curvatures K^F and $K^{\tilde{F}}$ of F and $\tilde{F}$, respectively, are as follows:

$$K^{\tilde{F}}(X_j, P) = \frac{g_X(X_j, X_j)}{\tilde{F}^2(X_j)} K(X_i, X_j), \tag{3.22}$$

$$K^F(X_j, P) = \frac{\mathsf{h}(X_j, X_j)}{F^2(X_j)} \bar{K}(X_i, X_j), \tag{3.23}$$

where, as above, K is the sectional curvature of the Riemannian metric g_X and $\bar{K}$ is the sectional curvature of the Riemannian metric h.
Using Proposition 3.5, we have

$$K(X_j, X_j) = \frac{1}{\lambda_j} \bar{K}(X_j, X_j). \tag{3.24}$$

On the other hand, we have

$$\mathsf{h}(X_j, X_j) = 1, \qquad F^2(X_j) = (1+\langle X, X_j\rangle)^2,$$

and

$$g_X(X_j, X_j) = \lambda_j, \qquad \tilde{F}^2(X_j) = \lambda_j(1+\lambda_j^{\frac{1}{2}}\langle X, X_j\rangle)^2.$$

Now, by (3.24), (3.23) and (3.22), we conclude that (3.21) holds. □

4. Examples

4.1. **The Heisenberg group H_3.** Let H_3 be the simply connected 3-dimensional Heisenberg Lie group, and $\langle , \rangle$ be the inner product on its Lie algebra $\mathfrak{h}_3$, induced by the left-invariant Riemannian metric h on H_3 introduced in [15]. Suppose that $\{x, y, z\}$ is an orthogonal basis of $\mathfrak{h}_3$ and

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the matrix of the inner product $\langle , \rangle$ with respect to this basis, where $\lambda > 0$. Suppose that X is a left-invariant vector field belonging to the center of the

Lie algebra $\mathfrak{h}_3$. So X is a scalar multiple of z. Let $X = cz$ and $\|X\| = |c| < \frac{1}{2}$. We can see that the linear mapping $\phi : \mathfrak{h}_3 \to \mathfrak{h}_3$ is as follows:

$$\phi(v) = (1 + |c|)(v + |c|\langle z, v\rangle z). \tag{4.1}$$

It is easy to see that the matrix representation of ϕ with respect to the basis $\{x, y, z\}$ is of the form

$$\begin{pmatrix} 1+|c| & 0 & 0 \\ 0 & 1+|c| & 0 \\ 0 & 0 & (1+|c|)^2 \end{pmatrix}.$$

Therefore, its eigenvalues are $\lambda_1 = 1+|c|$, $\lambda_2 = 1+|c|$, $\lambda_3 = (1+|c|)^2$ with the corresponding eigenvectors $X_1 = \frac{x}{\sqrt{\lambda}}$, $X_2 = \frac{y}{\sqrt{\lambda}}$ and $X_3 = z$, which construct an orthonormal basis for $\mathfrak{h}_3$. So we have

$$[X_1, X_2] = \frac{1}{\lambda} z \tag{4.2}$$

$$[X_1, X_3] = [X_2, X_3] = 0. \tag{4.3}$$

Therefore the non-zero structure constants are $\alpha_{123} = -\alpha_{213} = \frac{1}{\lambda}$.

Now according to Proposition 2.2 the Levi-Civita connection of the Riemannian metric g_X is as follows:

$$\begin{aligned} &\nabla_x y = \frac{1}{2} z, \quad \nabla_x z = -\frac{1+|c|}{2\lambda} y, \\ &\nabla_y z = \frac{1+|c|}{2\lambda} x, \quad \nabla_y x = -\frac{1}{2} z, \\ &\nabla_z x = -\frac{1+|c|}{2\lambda} y, \quad \nabla_z y = \frac{1+|c|}{2\lambda} x, \\ &\nabla_x x = \nabla_y y = \nabla_z z = 0. \end{aligned}$$

Due to the (1.1) and (4.1) we have

$$\begin{aligned} &g_X(x, x) = g_X(y, y) = \lambda(1 + |c|), \quad g_X(z, z) = (1 + |c|)^2, \\ &g_X(x, y) = g_X(x, z) = g_X(z, y) = 0. \end{aligned}$$

Therefore, according to Proposition 2.3 the sectional curvature of the Riemannian metric g_X is as follow

$$\begin{aligned} K(x, y) &= \frac{-3}{4\lambda^2}, \\ K(x, z) = K(y, z) &= \frac{1}{4\lambda^2}. \end{aligned}$$

4.2. **The almost Abelian Lie groups.** A non-Abelian Lie group G is called almost Abelian if it has a codimension one Abelian subgroup. Suppose that G is an almost Abelian Lie group with Lie algebra $\mathfrak{g}$. Let $\langle , \rangle$ be a left-invariant Riemannian metric on G. Then there exist $b \in \mathfrak{g}$ and a commutative ideal $\mathfrak{u}$ of codimention one such that $[b, z] = z$, for any $z \in \mathfrak{u}$ (see [8] and [5]). Suppose

that $\{u_1 = b, u_2, u_3, \cdots, u_n\}$ is an orthonormal basis for $\mathfrak{g}$ with respect to $\langle , \rangle$, where $\{u_2, u_3, \cdots, u_n\}$ is a basis for $\mathfrak{u}$. In this case we have,

$$[b, u_i] = u_i, \quad i = 2, \cdots, n, \quad \text{and} \quad [u_i, u_j] = 0, \quad i, j \neq 1.$$

Therefore, the non-zero structural constants are $\alpha_{1ii} = -\alpha_{i1i} = 1, \quad i = 2, 3, \cdots, n$. Let $X = \xi b$, where in $\|X\| = \xi < \frac{1}{2}$. Then the linear mapping $\phi : \mathfrak{g} \to \mathfrak{g}$ is as follows:

$$\phi(v) = (1 + \xi)v + (1 + \xi)\xi\langle b, v\rangle b. \tag{4.4}$$

Then, the matrix representation of ϕ with respect to the ordered basis $\{u_1 = b, u_2, u_3, \cdots, u_n\}$ is as follows

$$\begin{pmatrix} (1+\xi)^2 & 0 & 0 & \dots & \dots & 0 \\ 0 & (1+\xi) & 0 & \dots & \dots & 0 \\ \vdots & 0 & \ddots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \dots & 0 & (1+\xi). \end{pmatrix}.$$

Hence, the eigenvalues of ϕ are $\lambda_1 = (1 + \xi)^2$, $\lambda_2 = \cdots = \lambda_n = (1 + \xi)$ and the corresponding eigenvectors are $\{b, u_2, \cdots, u_n\}$.

Now according to the Proposition 2.2 the Levi-Civita connection of the Riemannian metric g_X is as follows:

$$\begin{aligned} &\nabla_{u_j} b = -u_j, \quad j \neq 1, \\ &\nabla_{u_j} u_j = \frac{1}{1+\xi} b, \quad j \neq 1, \\ &\nabla_{u_i} u_j = 0, i \neq j \quad \textit{and} \quad i, j = 2, 3, \cdots, n, \\ &\nabla_b b = 0 \\ &\nabla_b u_j = 0, \quad j = 2, 3, \cdots, n. \end{aligned}$$

Therefore, according to the Proposition 2.3 the sectional curvature of the Riemannian metric g_X is as follows:

$$K(u_i, u_j) = -\frac{1}{(1+\xi)^2}.$$

Remark 4.1. *Finally, we mention that using formula (2.8) of* [10]*, the results of this paper can be generalized in an almost similar way of Randers metrics to the other* (α, β)*-metrics.*

Acknowledgment: The authors are grateful to the anonymous reviewer for their careful reading of the manuscript and for their helpful feedback, which strengthened this paper. We are grateful to the office of Graduate Studies of the University of Isfahan for their support.

References

1. P. L. Antonelli, R. S. Ingarden and M. Matsumoto, *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*, Kluwer, Dordrecht, 1993.
2. A. Arvanitoyeorgos, *An introduction to Lie groups and the geometry of Homogeneous spaces*, American Mathematical Society, Athen, 2003.
3. G. S. Asanov, *Finsler Geometry, Relativity and Gauge Theories*, D. Reidel, Dordrecht, 1985.
4. S. S. Chern, and Z. Shen, *Riemann-Finsler Geometry*, World Scientific, Singapore, 2005.
5. S. Deng, M. Hosseini, H. Liu and H. R. Salimi Moghaddam, *On Left Invariant* (α, β)*-metrics on Some Lie Groups*, Houst. J. Math. **45**(2019), 1071-1088.
6. D. Latifi, *Homogeneous geodesics in homogeneous Finsler spaces*, J. Geom. Phys. **57**(2007), 1421-1433.
7. M. Matsumoto, *Theory of Finsler spaces with* (α, β)*-metric*, Rep. Math. Phys. **31** (1992), 43-83.
8. J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Adv. Math. **21**(1976), 293-329.
9. M. Nejadahmad and H. R. Salimi Moghaddam, *The relation between automorphism group and isometry group of left invariant* (α, β)*-metrics*, J. Finsler Geom. Appl. **4**(1) (2023), 81-87.
10. M. Parhizkar and H. R. Salimi Moghaddam, *Naturally reductive homogeneous* (α, β)*-metric spaces*, Arch. Math. **57**(2021), 1-11.
11. M. Parhizkar and D. Latifi, On the flag curvature of invariant (α, β)-metrics,Int. J. Geom. Methods Mod. Phys. **13** (2016), 1650039, 1-11.
12. C. Pfeifer, *Finsler spacetime geometry in physics*, Int. J. Geom. Methods Mod. Phys. **16**, supp02(2019), 1941004.
13. G. Randers, *On an asymmetrical metric in the four-space of general relativity*, Phys. Rev. **59**(1941), 195-199.
14. H. R. Salimi Moghaddam, *Invariant Matsumoto metrics on homogeneous spaces*, Osaka J. Math. **51**(2014), 39-45.
15. H. R. Salimi Moghaddam, *On the left invariant Randers and Matsumoto metrics of Berwald type on 3-dimensional Lie groups*, Monash J. Math. **177**(2015), 649-658.
16. M. Talebie, *On the flag curvature of left invariant generalized m-Kropina metrics on some Lie groups*, J. Finsler Geom. Appl. **5**(2) (2024), 62-69.

Received: 24.01.2026
Accepted: 17.04.2026