Journal of Finsler Geometry and its Applications
Vol. 7, No. 2 (2026), pp 121-136
https://doi.org/10.22098/jfga.2026.18968.1195

# Bi-Ricci soliton structure on three dimensional Lie groups

Ghodratallah Fasihi-Ramandi[a*], Majid Ali Chaudhary[b], Shahroud Azami[a] and Sahar Pour-Farrokh[a]

[a]Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran.
[b]Department of Mathematics, Maulana Azad National Urdu University, Hyderabad, India.

E-mail: fasihi@sci.ikiu.ac.ir
E-mail: majid_alichoudhary@yahoo.co.in
E-mail: azami@sci.ikiu.ac.ir
E-mail: saharpf@gmail.com

**Abstract.** This research establishes the framework for bi-Ricci soliton structures and studies $3D$ Lorentzian homogeneous bi-Ricci solitons. Their existence and types in terms of shrinking, expanding, and steady will be considered. Our results show that, in almost all possible cases, there is a non-trivial bi-Ricci soliton on $3D$ homogeneous manifolds.

**Keywords:** Lorentzian manifold, Homogeneous space, Lie groups.

### Abbreviations

This manuscript employs the following abbreviations:

*Corresponding Author
AMS 2020 Mathematics Subject Classification: 53B40, 53C30

This work is licensed under a Creative Commons Attribution-NonCommercial 4.0 International License.
Copyright © 2026 The Author(s). Published by University of Mohaghegh Ardabili

| | |
|---|---|
| $b-\mathcal{RS}$ | bi-Ricci soliton |
| $\mathcal{RS}$ | Ricci soliton |
| $\mathcal{RF}$ | Ricci flow |
| $\mathcal{LG}$ | Lie groups |
| $\mathcal{LM}$ | Lorentzian manifold |
| $\mathcal{RM}$ | Riemannian manifold |
| $b-\mathcal{RC}$ | bi-Ricci curvature |

## 1. Introduction

A $\mathcal{RS}$ is defined as a $\mathcal{RM}$ $(M, g)$, along with a vector field $Y$ on $M$ that fulfills

$$\frac{1}{2}\mathcal{L}_Y g + \mathrm{Ric} = \mu g.$$

In the above relation, $\mathcal{L}_Y$ denotes the Lie derivative along $Y$, the symbol Ric represents the Ricci tensor, while $\mu$ signifies a real number. A $\mathcal{RS}$ is classified as shrinking, steady, or expanding when $\mu$ is greater than 0, equal to 0, or less than 0, respectively [2][27].

$\mathcal{RS}$ acts as a preliminary step towards understanding the $\mathcal{RF}$ because they are fixed points of the $\mathcal{RF}$. Furthermore, $\mathcal{RS}$ aids in comprehending the singular points of the $\mathcal{RF}$.

The study of the $\mathcal{RS}$ has also been extended to string theory by theoretical physicists. The fundamental advancement in this domain is attributed to Friedan (for additional insights into the physical aspects of $\mathcal{RF}$, see [1, 20].) While the $\mathcal{RS}$ was first defined and investigated in the Riemannian case, it is currently being investigated in the field of quasi-$\mathcal{RM}$, especially in the Lorentzian case [10, 24]. Also, the $\mathcal{RS}$ equation can provide clarity regarding the overall analysis of Einstein's field equations, in which this soliton equation is a special case. It has been shown in [12] that $3D$ $\mathcal{LG}$ do not admit left-invariant Riemannian-$\mathcal{RS}$ (see also [19, 23]). The existence of Ricci-Lorentzian solitons in $3D$ $\mathcal{LG}$ has been demonstrated in [4], which shows that these manifolds admit $\mathcal{RS}$ of all types.

$3D$ locally homogeneous $\mathcal{LM}$ are either locally symmetric or locally isometric with respect to a $3D$ $\mathcal{LG}$ equipped with a left-invariant Lorentzian metric.

As we know, the Ricci operator $Q$ is always self-adjoint and therefore diagonalizable in $\mathcal{RM}$, while in $\mathcal{LM}$ four distinct cases may arise, which are termed as Segre types [18]:

$(Ia)$: Segre type $\{11, 1\}$: $Q$ is symmetric; thus, it can be diagonalized. The comma distinguishes between the space-like and time-like vectors. In the degenerate scenario, at least two Ricci eigenvectors are identical.

$(Ib)$ Segre type $\{1z\overline{z}\}$: $Q$ has one real eigenvalue and two complex conjugate eigenvalues.

Type (II) Segre type {21}: $Q$ possesses two real eigenvalues (that coincide when degenerate), one of which has a multiplicity of two and is linked with the $1D$ eigenspace.

Type (III) Segre type {3}: $Q$ possesses three identical eigenvalues, which are related with its $1D$ eigenspace.

On the other hand, the study of the differential geometry of quasi-$\mathcal{RM}$ relies closely on the concept of curvature. While Ricci curvature and scalar curvature are fundamental tools, in modern research we often need generalized definitions or combinations of these quantities to reveal subtle features of the geometric structure of the manifold. $b-\mathcal{RC}$ is one such advanced definition.

If $(M,g)$ is a quasi-$\mathcal{RM}$ and $(v,w)$ is a pair of vectors, the $b-\mathcal{RC}$ tensor is described by:

$$\mathrm{biRic}(v,w)=\mathrm{Ric}(v,v)+\mathrm{Ric}(w,w)-\mathrm{Rm}(v,w,v,w).$$

It can be seen that in three dimensions, this concept is equal to half the scalar curvature for each pair of orthogonal unit vectors $(v,w)$. In higher dimensions, the positivity of $b-\mathcal{RC}$ is a stronger condition, and this curvature in the special case coincides with the mean curvature defined by Brandel et al. [3] This concept of curvature is naturally used in the study of minimal weighted cuts [3]. The $b-\mathcal{RC}$ tensor is also essential in understanding the stable Bernstein problem [11, 22].

With this idea, we induced the concept of $b-\mathcal{RS}$ as follows.

A quasi-$\mathcal{RM}$ $(M,g)$ with vector field $Y$ on $M$ and a real number $\mu$, is referred to as a $b-\mathcal{RS}$ if it satisfies

$$\mathrm{biRic}+\frac{1}{2}\mathcal{L}_Y g=\mu g, \tag{1.1}$$

where biRic is the $b-\mathcal{RC}$ of $(M,g)$.

In the present discussion, we aim to study the existence of the $b-\mathcal{RS}$ in $3D$ Lorentzian $\mathcal{LG}$.

## 2. Three-Dimensional Homogeneous Lorentzian Manifolds

This section is devoted to the presentation of our main results.

According to [6], any connected, complete, and simply connected $3D$ homogeneous $\mathcal{LM}$ is a $\mathcal{LG}$. These spaces have been classified, and we study them case by case.

### 2.1. Unimodular Three-Dimensional Lie groups.

Let $\times$ denote the vector product in the Lorentzian space $\mathbb{R}^3_1$ induced by the para-quaternion multiplication (i.e. $e_1\times e_2=-e_3$, $e_2\times e_3=e_1$, $e_3\times e_1=e_2$ where $\{e_1,e_2,e_3\}$ is a unitary orthogonal basis with the signature $(++-)$. A Lie bracket $[,]$ related to a Lie algebra $\mathfrak{g}$ is referred to as unimodular iff the endomorphism $L$, characterized by $[Z,Y]=L(Z\times Y)$, is self-adjoint as described in [25]. Let

us consider the various forms of $L$ that we defined previously. There are four categories of unimodular $3D$ Lie algebras.

2.2. **Segre type** $\{11,1\}$**.** If $L$ is diagonalizable with eigenvalues $\{a,b,c\}$ corresponding to the orthonormal basis $\{e_1,e_2,e_3\}$ of the sign $(++-)$, then we have:

$$(\mathfrak{g}_{Ia}): \quad [e_1,e_2]=-ce_3,\ [e_1,e_3]=-be_2,\ [e_2,e_3]=ae_1. \tag{2.1}$$

By the symmetry property, the only non-zero components of the curvature tensor are expressed as:

$$\begin{aligned} R_{1221} &= \frac{1}{4}\left(a^2+b^2-3c^2-2ab+2ac+2bc\right),\\ R_{1313} &= \frac{1}{4}(a^2-3b^2+c^2+2ab-2ac+2bc),\\ R_{2332} &= \frac{1}{4}(3a^2-b^2-c^2-2ab-2ac+2bc). \end{aligned}$$

Under these conditions, $Q$ is diagonalizable with eigenvalues given by:

$$\begin{cases} \lambda_1 = \frac{1}{2}\left((b-c)^2-a^2\right),\\ \lambda_2 = \frac{1}{2}\left((a-c)^2-b^2\right),\\ \lambda_3 = \frac{1}{2}\left((a-b)^2-c^2\right). \end{cases} \tag{2.2}$$

The components of the bi-Ricci tensor are given as follows:

$$\begin{aligned} \mathrm{biRic}_{11} &= (b-c)^2-a^2,\\ \mathrm{biRic}_{22} &= (a-c)^2-b^2,\\ \mathrm{biRic}_{33} &= -\left((a-b)^2-c^2\right),\\ \mathrm{biRic}_{12} &= \tfrac{1}{4}\left((a-b)^2+c^2-2c(a+b)\right),\\ \mathrm{biRic}_{13} &= -\tfrac{7}{4}a^2+\tfrac{1}{2}ab+\tfrac{1}{2}ac+\tfrac{3}{4}b^2-\tfrac{3}{2}bc+\tfrac{5}{4}c^2,\\ \mathrm{biRic}_{23} &= \tfrac{3}{4}a^2+\tfrac{1}{2}ab-\tfrac{3}{2}ac-\tfrac{5}{4}b^2+\tfrac{1}{2}bc+\tfrac{3}{4}c^2. \end{aligned}$$

For any $Y=\Sigma Y_i e_i$, from (2.1) we have:

$$(\mathcal{L}_Y g)=\begin{pmatrix} 0 & (a-b)Y_3 & (c-a)Y_2\\ (a-b)Y_3 & 0 & (b-c)Y_1\\ (c-a)Y_2 & (b-c)Y_1 & 0 \end{pmatrix}.$$

Therefore, according to the relation (1.1), a $b-\mathcal{RS}$ of this type exists iff the following holds:

$$\begin{cases} 1)\ (b-c)^2-a^2=\mu,\\ 2)\ (a-c)^2-b^2=\mu,\\ 3)\ (a-b)^2-c^2=\mu,\\ 4)\ \frac{1}{4}\left((a-b)^2+c^2-2c(a+b)\right)+\frac{1}{2}(a-b)Y_3=0,\\ 5)\ -\frac{7}{4}a^2+\frac{1}{2}ab+\frac{3}{4}b^2-\frac{3}{2}bc+\frac{5}{4}c^2+\frac{1}{2}(c-a)Y_2=0,\\ 6)\ \frac{3}{4}a^2+\frac{1}{2}ab-\frac{3}{2}ac-\frac{5}{4}b^2+\frac{1}{2}bc+\frac{3}{4}c^2+\frac{1}{2}(b-c)Y_1=0. \end{cases} \tag{2.3}$$

From the first and second equations of the system (2.3), we obtain:

$$b^2 + c^2 - a^2 - 2bc = a^2 + c^2 - b^2 - 2ac$$
$$\Rightarrow (b-a)(b+a) = c(b-a).$$

Hence $b = a$ or $b + a = c$.

**Case** $(a)$: If $b + a = c$, then $b = c - a$. Substituting into the first equation (2.3), we have $\mu = 0$. Now, we put the equality $c = a + b$ into the third equation (2.3):

$$(a-b)^2 + (b+a)^2 = 0 \Rightarrow 2a^2 + 2b^2 = 0 \Rightarrow 2(a^2 + b^2) = 0.$$

Therefore $a = b = c = 0$.

**Case** $(b)$: By substituting $b = a$ into the third equation, we have $\mu = -c^2$. Also, by substituting into the second equation relation (2.3), we get:

$$(a-c)^2 - a^2 = -c^2 \Rightarrow 2c(-a + c) = 0,$$

which shows $c = 0$ or $a = c$. Both of these conditions yield $a = b = c = 0$.

**Theorem 2.1.** *3D $\mathcal{LG}$ of Segre type $\{11, 1\}$, does not admit $b-\mathcal{RS}$ structure.*

2.2.1. $\{1z\bar{z}\}$**.** Let $L$ has a complex eigenvalue. In this case,

$$L = \begin{pmatrix} a & 0 & 0 \\ 0 & c & -b \\ 0 & b & c \end{pmatrix}, \qquad b \neq 0,$$

corresponding to $\{e_1, e_2, e_3\}$ of signature $(++-)$.

The corresponding $\mathfrak{g}$ is described as:

$$(\mathfrak{g}_{Ib}): \quad [e_1, e_2] = be_2 - ce_3,\ [e_1, e_3] = -ce_2 - be_3,\ [e_2, e_3] = ae_1.$$

The nonzero elements of $R$ are as follows:

$$R_{1221} = R_{1313} = \frac{1}{4}(a^2 + 4b^2), \quad R_{2332} = \frac{3}{4}a^2 + b^2 - ac, \quad R_{1231} = b(a - 2c).$$

$Q$ corresponding to $\{e_1, e_2, e_3\}$ is given as:

$$Q = \begin{pmatrix} -\frac{1}{2}(a^2 + 4b^2) & 0 & 0 \\ 0 & \frac{1}{2}a(a-2c) & -b(a-2c) \\ 0 & b(a-2c) & \frac{1}{2}a(a-2c) \end{pmatrix}, \quad b \neq 0.$$

Hence, $Q$ is of the Segre type $\{1z\overline{z}\}$ if $a \neq 2c$ and of the type $\{11,1\}$ if $a = 2c$. Hence, the components of the bi Ricci tensor are specified by:

$$\begin{aligned}
\mathrm{biRic}_{11} &= -(a^2+4b^2),\\
\mathrm{biRic}_{22} &= a(a-2c),\\
\mathrm{biRic}_{33} &= -a(a-2c),\\
\mathrm{biRic}_{12} &= \tfrac{1}{4}a^2 - b^2 - ac,\\
\mathrm{biRic}_{13} &= ac - \tfrac{5}{4}a^2 - 3b^2,\\
\mathrm{biRic}_{23} &= \tfrac{3}{4}a^2 + b^2 - ac.
\end{aligned}$$

For a vector $Y = \Sigma Y_i e_i$, considering the above cases, we have:

$$(\mathcal{L}_Y g) = \begin{pmatrix} 0 & bY_2 + (a-c)Y_3 & bY_3 + (c-a)Y_2 \\ bY_2 + (a-c)Y_3 & -2bY_1 & 0 \\ bY_3 + (c-a)Y_2 & 0 & -2bY_1 \end{pmatrix}.$$

Therefore, a homogeneous $b - \mathcal{RS}$ of the type $\{1z\overline{z}\}$ exists iff

$$\begin{cases}
1)\ a^2 - 4b^2 = \mu,\\
2)\ a^2 - 2ac - bY_1 = \mu,\\
3)\ +a^2 - 2ac + bY_1 = \mu,\\
(4)\ \frac{1}{4}a^2 - b^2 - ac + \frac{1}{2}bY_2 + \frac{1}{2}(a-c)Y_3 = 0,\\
(5)\ ac - \frac{5}{4}a^2 - 3b^2 + \frac{1}{2}Y_3 b + \frac{1}{2}(c-a)Y_2 = 0,\\
(6)\ \frac{3}{4}a^2 + b^2 - ac = 0.
\end{cases} \tag{2.4}$$

From the equality of the second and third equations in (2.4), one gets:

$$a^2 - 2ac - Y_1 b = a^2 - 2ac + Y_1 b \Rightarrow -Y_1 b = Y_1 b.$$

Since $b \neq 0$, then $Y_1 = 0$. From the equality of the first and second equations (2.4), we can get:

$$a^2 - 2ac = -a^2 - 4b^2 \tag{2.5}$$

According to the sixth equation (2.4), we obtain: $ac = \frac{3}{4}a^2 + b^2$. Now, by using the relation (2.5), we can write:

$$a^2 - \left(\frac{3}{4}a^2 + b^2\right) + 2b^2 = 0.$$

This results $a = b = 0$ and this is a contradiction.

**Theorem 2.2.** *There are no homogeneous $b - \mathcal{RS}$ of Segre type $\{1z\overline{z}\}$.*

2.2.2. *Segre type* $\{21\}$. Suppose that $L$ has a double root of a minimal polynomial. Then, corresponding to the orthogonal basis of $\{e_1, e_2, e_3\}$, with $(++-)$, we have

$$L = \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1}{2} + b & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} + b \end{pmatrix}$$

and the respective $\mathfrak{g}$ is as follows:

$$(\mathfrak{g}_{II}): \ [e_1,e_2]=\frac{1}{2}e_2-\left(b-\frac{1}{2}\right)e_3, \ [e_1,e_3]=-\left(b+\frac{1}{2}\right)e_2-\frac{1}{2}e_3, \ \ [e_2,e_3]=ae_1.$$

The nonzero components of $R$ are given as:

$$R_{1221}=\frac{1}{4}(a^2-2a+4b), \quad R_{1313}=\frac{1}{4}(a^2+2a-4b),$$
$$R_{2332}=\frac{1}{4}a(3a-4b), \qquad R_{1231}=\frac{1}{2}a-b.$$

So $Q$ is of the following form:

$$Q=\begin{pmatrix} -\frac{1}{2}a^2 & 0 & 0 \\ 0 & \frac{1}{2}(a+1)(a-2b) & -\frac{1}{2}a+b \\ 0 & \frac{1}{2}a-b & \frac{1}{2}(a-1)(a-2b) \end{pmatrix}, \tag{2.6}$$

which have the eigenvalues $\lambda_1=-\frac{1}{2}a^2$ and $\lambda_2=\lambda_3=\frac{1}{2}a(a-2b)$. Consequently, $Q$ is of type $\{21\}$ iff $a=0$ or $a=b$.
Thus, the elements of the bi-Ricci tensor are given as follows:

$$\begin{aligned}
&\mathrm{biRic}_{11}=-a^2,\\
&\mathrm{biRic}_{22}=(a+1)(a-2b),\\
&\mathrm{biRic}_{33}=-\left((a-1)(a-2b)\right),\\
&\mathrm{biRic}_{12}=\tfrac{1}{4}a^2-ab,\\
&\mathrm{biRic}_{13}=-\tfrac{5}{4}a^2+\tfrac{3}{2}ab-\tfrac{1}{2}a+b,\\
&\mathrm{biRic}_{23}=-\tfrac{3}{4}a^2-ab+a-2b.
\end{aligned}$$

For the vector field $Y=\Sigma Y_i e_i$, we have

$$(\mathcal{L}_Y g)=\begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{12} & -Y_1 & Y_1 \\ a_{13} & Y_1 & -Y_1 \end{pmatrix}$$

which $a_{12}=\frac{1}{2}(Y_2+(2a-2b-1)Y_3)$ and $a_{13}=\frac{1}{3}(Y_3+(2b-2a-1)Y_2)$. The conditions required and adequate for the existence of a homogeneous $b-\mathcal{RS}$ of the type $\{21\}$, are that the following equations hold:

$$\begin{cases}
1) \ -a^2=\mu,\\
2) \ (a+1)(a-2b)-\frac{1}{2}Y_1=\mu,\\
3) \ (a-1)(a-2b)+\frac{1}{2}Y_1=\mu,\\
4) \ \frac{1}{4}a^2-ab+\frac{1}{4}\left(Y_2+Y_3(2a-2b-1)\right)=0,\\
5) \ -\frac{5}{4}a^2+\frac{3}{2}ab-\frac{1}{2}a+b+\frac{1}{4}\left(Y_3+Y_2(2b-2a-1)\right)=0,\\
6) \ \frac{3}{4}a^2-ab+a-2b+\frac{1}{2}Y_1=0.
\end{cases} \tag{2.7}$$

According to the first equation (2.7), we have $\mu = -a^2$. Substituting the value of $\mu$ into the second and third equations of (2.7), we obtain:

$$a^2 - 2ab + a - 2b - \frac{1}{2}Y_1 = -a^2 \Rightarrow 2a^2 - 2ab + a - 2b - \frac{1}{2}Y_1 = 0,$$
$$a^2 - 2ab - a + 2b + \frac{1}{2}Y_1 = -a^2 \Rightarrow 2a^2 - 2ab - a + 2b + \frac{1}{2}Y_1 = 0.$$

From these equations, it follows that $Y_1 = 2(a - 2b)$ and $a(a - b) = 0$. Thus $a = 0 \neq b$ or $a = b \neq 0$ (we excluded the case $a = b = 0$ that relates to a flat manifold).

**First Case:** If $a = 0$, in the fourth and fifth equations in (2.7) we have:

$$Y_2 - (2b + 1)Y_3 = 0,$$
$$(2b - 1)Y_2 + Y_3 = -4b.$$

Therefore,

$$Y_1 = -4b,\ Y_2 = -2b^2 - b, \quad Y_3 = -b,\ \mu = 0.$$

**Second Case:** By substituting $a = b$ into the second and third equations (2.7), we get $Y_1 = -2a$. Substituting into the sixth equation (2.7), we obtain:

$$\frac{3}{4}a^2 - a^2 + a^2 - 2a + \frac{1}{4}(-2a) = 0 \Rightarrow a(a + 8) = 0,$$

provided that $a = b = -8$. Also, from the fourth and fifth equations, we can see that $Y_2 - Y_3 = -48$ and $Y_2 - Y_3 = 4(-48)$, which is impossible. Therefore, this leads to the subsequent theorem

**Theorem 2.3.** *The 3D $\mathcal{LG}$ of Lorentzian Segre type $\{21\}$ has a steady $b-\mathcal{RS}$ structure with the following properties:*

$$(a = 0, b \neq 0, Y_1 = -4b, Y_2 = Y_3 = 2, \mu = 0)\,.$$

2.2.3. *Segre type* $\{3\}$. Let $L$ has a triple root in the minimal polynomial. Then

$$L = \begin{pmatrix} a & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & a & 0 \\ -\frac{1}{\sqrt{2}} & 0 & a \end{pmatrix},$$

corresponds to the $\{e_1, e_2, e_3\}$ of $(++-)$ and the related $\mathfrak{g}$ is:

$$(\mathfrak{g}_{III}) : \begin{cases} [e_1, e_2] = -\frac{1}{\sqrt{2}}e_1 - ae_3, \quad [e_1, e_3] = -\frac{1}{\sqrt{2}}e_1 - ae_2, \\ [e_2, e_3] = ae_1 + \frac{1}{\sqrt{2}}e_2 - \frac{1}{\sqrt{2}}e_3. \end{cases}$$

Therefore, the nonzero components of $R$ are as follows:

$$R_{1221} = \tfrac{1}{4}(a^2 + 4), \qquad R_{1331} = 1 - \tfrac{1}{4}a^2, \quad R_{2323} = \tfrac{1}{4}a^2,$$

$$R_{1231} = 1, \qquad R_{1223} = R_{1323} = \tfrac{1}{\sqrt{2}}a.$$

$Q$, expressed in $\{e_1, e_2, e_3\}$, is as follows with a single eigenvalue $-\frac{1}{2}a^2$:

$$Q = \begin{pmatrix} -\frac{1}{2}a^2 & -\frac{1}{\sqrt{2}}a & -\frac{1}{\sqrt{2}}a \\ -\frac{1}{\sqrt{2}}a & -\frac{1}{2}(a^2+2) & -1 \\ \frac{1}{\sqrt{2}}a & 1 & 1-\frac{1}{2}a^2 \end{pmatrix},$$

If $a \neq 0$, then $Q$ will be of type $\{3\}$, while $a = 0$, then $\mathfrak{g}$ is two-step nilpotent. Here, the components of the bi-Ricci tensor are described as follows:

$$\begin{aligned} \mathrm{biRic}_{11} &= -a^2, \\ \mathrm{biRic}_{22} &= -(a^2+2), \\ \mathrm{biRic}_{33} &= a^2-2, \\ \mathrm{biRic}_{12} &= -\tfrac{3}{4}a^2, \\ \mathrm{biRic}_{13} &= \tfrac{1}{4}a^2, \\ \mathrm{biRic}_{23} &= -\tfrac{1}{4}a^2-2. \end{aligned}$$

For the vector $Y = \Sigma Y_i e_i$, the Lie derivative has the subsequent representation:

$$(\mathcal{L}_Y g) = \frac{1}{\sqrt{2}} \begin{pmatrix} -2(Y_2+Y_3) & Y_1 & Y_1 \\ Y_1 & 2Y_3 & Y_3-Y_2 \\ Y_1 & Y_3-Y_2 & -2Y_2 \end{pmatrix}.$$

So the condition for the $b-\mathcal{RS}$ (1.1), on $\mathfrak{g}_{III}$ will be as follows:

$$\begin{cases} 1) \ -a^2 - \frac{1}{\sqrt{2}}(Y_2+Y_3) = \mu, \\ 2) \ -a^2 - 2 + \frac{1}{\sqrt{2}}Y_3 = \mu, \\ 3) \ a^2 - 2 - \frac{1}{\sqrt{2}}Y_2 = -\mu, \\ 4) \ -\frac{3}{4}a^2 + \frac{1}{2\sqrt{2}}Y_1 = 0, \\ 5) \ -\frac{1}{4}a^2 + \frac{1}{2\sqrt{2}}Y_1 = 0, \\ 6) \ -\frac{1}{4}a^2 - 2 + \frac{1}{2\sqrt{2}}(Y_3-Y_2) = 0. \end{cases} \tag{2.8}$$

From the fourth equation (2.8), we can get:

$$Y_1 = \frac{3\sqrt{2}}{2}a^2,$$

and from the fifth equation (2.8), we have:

$$Y_1 = -\frac{\sqrt{2}}{2}a^2.$$

Therefore, $a = 0$, we can write

$$\begin{aligned} -\frac{1}{\sqrt{2}}(Y_2+Y_3) &= \mu, \\ -2 + \frac{1}{\sqrt{2}}Y_3 &= \mu, \\ -2 - \frac{1}{\sqrt{2}}Y_2 &= -\mu. \end{aligned}$$

By solving these equations, we obtain:

$$Y_2 = -2\sqrt{2}, \qquad Y_3 = 2\sqrt{2}, \qquad \mu = 0.$$

**Theorem 2.4.** *Any Segre type* $\{3\}$ *unimodular* $\mathcal{LG}$ *is a homogeneous* $b - \mathcal{RS}$ *with*

$$(a = 0,\ Y_1 = 0,\ Y_2 = -2\sqrt{2},\ Y_3 = 2\sqrt{2},\ \mu = 0).$$

2.3. **Non-unimodular Lie groups.** From [14], the non-unimodular Lorentzian $\mathfrak{g}$ of non-constant sectional curvature can be expressed, for an appropriate $e_1, e_2, e_3$, as

$$(\mathfrak{g}_{IV}): \quad [e_1, e_2] = 0,\ [e_1, e_3] = ae_1 + be_2,\ [e_2, e_3] = ce_1 + de_2, \tag{2.9}$$

where $a + d \neq 0$ and one of the subsequent is true:

IV.1: $\{e_1, e_2, e_3\}$ is orthonormal with $\langle e_1, e_1\rangle = -\langle e_2, e_2\rangle = -\langle e_3, e_3\rangle = -1$ and the $ac - bd = 0$.

IV.2: $\{e_1, e_2, e_3\}$ is orthonormal with $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = -\langle e_3, e_3\rangle = 1$ and $ac + bd = 0$.

IV: $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis with

$$\langle \cdot, \cdot \rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

and $ac = 0$.

The three cases are examined individually.

2.3.1. *Non-unimodular Lie algebra of type* IV $\cdot$ 1. The non-zero components of $R$ are given as follows:

$$R_{1212} = \frac{1}{4}\left(b^2 + c^2 + 4ad - 2bc\right),$$
$$R_{1313} = \frac{1}{4}\left(4a^2 - 3b^2 + c^2 + 2bc\right),$$
$$R_{2332} = \frac{1}{4}\left(b^2 - 3c^2 + 4d^2 + 2bc\right).$$

Hence, $Q$ is diagonalizable with eigenvalues

$$\lambda_1 = \frac{1}{2}\left(b^2 - c^2 - 2a(a + d)\right),$$
$$\lambda_2 = \frac{1}{2}\left(c^2 - b^2 - 2d(a + d)\right),$$
$$\lambda_3 = \frac{1}{2}\left((b - c)^2 - 2(a^2 + d^2)\right).$$

Here, the components of the bi-Ricci tensor are given as follows:

$$\begin{aligned}
\mathrm{biRic}_{11} &= -\left(b^2 - c^2 - 2a(a+d)\right), \\
\mathrm{biRic}_{22} &= -\left(c^2 - b^2 - 2d(a+d)\right), \\
\mathrm{biRic}_{33} &= (b-c)^2 - 2(a^2+d^2), \\
\mathrm{biRic}_{12} &= a^2 - d^2 - ad - \tfrac{5}{4}b^2 + \tfrac{3}{4}c^2 + \tfrac{1}{2}bc, \\
\mathrm{biRic}_{13} &= -a^2 - d^2 + ad + \tfrac{3}{4}b^2 + \tfrac{3}{4}c^2 - \tfrac{3}{2}bc, \\
\mathrm{biRic}_{23} &= \tfrac{1}{4}(b-c)^2 - (a^2 + ad + d^2).
\end{aligned}$$

Also, we can see that the derivative of the meter for any $Y = \Sigma Y_i e_i$ is as follows:

$$(\mathcal{L}_Y g) = \begin{pmatrix} -2aY_3 & (b-c)Y_3 & aY_1 + cY_2 \\ (b-c)Y_3 & 2dY_3 & -bY_1 - dY_2 \\ aY_1 + cY_2 & -bY_1 - dY_2 & 0 \end{pmatrix}.$$

Therefore, the conditions required and adequate for a homogeneous $b - \mathcal{RS}$ (1.1), on $\mathfrak{g}_{IV.1}$ are given as follows:

$$\begin{cases}
1)\ -b^2 + c^2 + 2a^2 + 2ad - aY_3 = -\mu, \\
2)\ -b^2 + c^2 - 2d^2 - 2ad - dY_3 = \mu, \\
3)\ b^2 - 2bc + c^2 - 2a^2 - 2d^2 = \mu, \\
4)\ a^2 - d^2 - ad - \tfrac{5}{4}b^2 + \tfrac{3}{4}c^2 + \tfrac{1}{2}bc + \tfrac{1}{2}(b-c)Y_3 = 0, \\
5)\ -a^2 - d^2 + ad + \tfrac{3}{4}b^2 + \tfrac{3}{4}c^2 - \tfrac{3}{2}bc + \tfrac{1}{2}(aY_1 + cY_2) = 0, \\
6)\ \tfrac{1}{4}(b-c)^2 - (a^2 + ad + d^2) - \tfrac{1}{2}(bY_1 + dY_2) = 0.
\end{cases} \tag{2.10}$$

From the equality of the first and third equations in (2.10) we obtain:

$$aY_3 = 2(c^2 - d^2 + ad - bc).$$

Also, from the second and the third equations in (2.10) we have:

$$dY_3 = 2(a^2 - b^2 - ad + bc).$$

Hence, we get

$$Y_3 = \frac{2(a^2 + c^2) - 2(b^2 + d^2)}{a + d}.$$

According to the fifth equation in (2.10), we have:

$$aY_1 + cY_2 = 2(a^2 + d^2 - ad) - \frac{3}{2}(b-c)^2. \tag{2.11}$$

From the sixth equation, we have:

$$bY_1 + dY_2 = \frac{1}{2}(b-c)^2 - 2(a^2 + ad + d^2). \tag{2.12}$$

Therefore, assuming $ad - bc \neq 0$, we deduce that

$$Y_1 = \frac{Ad - Bc}{ad - bc}, \quad Y_2 = \frac{Ba - Ab}{ad - bc},$$

where

$$A = 2(a^2 + d^2 - ad) - \frac{3}{2}(b-c)^2,$$
$$B = \frac{1}{2}(b-c)^2 - 2(a^2 + ad + d^2).$$

Also, let

$$C = a^2 - d^2 - ad - \frac{5}{4}b^2 + \frac{3}{4}c^2 + \frac{1}{2}bc,$$

if the parameters $a, b, c, d$ satisfy the subsequent conditions

$$ad \neq bc, \quad C + (b-c)\frac{(a^2+c^2)-(b^2+d^2)}{a+d} = 0,$$

then, according to the above arguments, we can present the next theorem.

**Theorem 2.5.** *The 3D Lorentzian be group with Lie algebra IV.3 admits a steady, shrinking, and expanding $b - \mathcal{RS}$ structure.*

2.3.2. *Non unimodular Lie algebra* $\mathrm{IV} \cdot 2$. Consider non-monomodular Lie algebra $\mathfrak{g}_{IV}$ with the given basis in $IV \cdot 2$. A direct computation yields the subsequent nonzero components of $R$:

$$R_{1212} = ad - \frac{1}{4}(b+c)^2,$$
$$R_{1331} = \frac{1}{4}\left(4a^2 + 3b^2 - c^2 + 2bc\right),$$
$$R_{2323} = \frac{1}{4}\left(b^2 - 3c^2 - 4d^2 - 2bc\right).$$

Hence, $Q$ is diagonalizable with the following eigenvalues:

$$\lambda_1 = \frac{1}{2}\left(b^2 - c^2 + 2a(a+d)\right),$$
$$\lambda_2 = \frac{1}{2}\left(c^2 - b^2 + 2d(a+d)\right),$$
$$\lambda_3 = \frac{1}{2}\left((b+c)^2 + 2(a^2+d^2)\right).$$

Thus, the components of the bi-Ricci tensor are given as follows:

$$\begin{aligned}
\mathrm{biRic}_{11} &= b^2 - c^2 + 2a(a+d),\\
\mathrm{biRic}_{22} &= c^2 - b^2 + 2d(a+d),\\
\mathrm{biRic}_{33} &= -(b+c)^2 - 2(a^2+d^2),\\
\mathrm{biRic}_{12} &= a^2 + ad - d^2 + \tfrac{1}{4}(b+c)^2,\\
\mathrm{biRic}_{13} &= -a^2 - \tfrac{5}{4}c^2 - d^2 - ad + \tfrac{3}{4}b^2 - \tfrac{1}{2}bc,\\
\mathrm{biRic}_{23} &= -a^2 - \tfrac{5}{4}b^2 + \tfrac{3}{4}c^2 - d^2 + ad - \tfrac{1}{2}bc.
\end{aligned}$$

A simple calculation of the relation (2.9) and using the fact that the structure of constants holds in the properties $ac + bd = 0$ and $a + d \neq 0$, shows that the

Lie derivative along the vector $Y = \Sigma Y_i e_i$ is given by:

$$(\mathcal{L}_Y g) = \begin{pmatrix} 2aY_3 & (b+c)Y_3 & -aY_1 - cY_2 \\ (b+c)Y_3 & 2dY_3 & -bY_1 - dY_2 \\ -aY_1 - cY_2 & -bY_1 - dY_2 & 0 \end{pmatrix}.$$

Therefore, a $b - \mathcal{RS}$ must satisfy the following conditions:

$$\begin{cases} 1)\ b^2 - c^2 + 2a(a+d) + aY_3 = \mu \\ 2)\ c^2 - b^2 + 2d(a+d) + dY_3 = \mu, \\ 3)\ (b+c)^2 + 2(a^2+d^2) = \mu, \\ 4)\ a^2 + ad - d^2 + \frac{1}{4}(b+c)^2 + \frac{1}{2}(b+c)Y_3 = 0, \\ 5)\ -a^2 - \frac{5}{4}c^2 - d^2 - ad + \frac{3}{4}b^2 - \frac{1}{2}bc - \frac{1}{2}(aY_1 + cY_2) = 0, \\ 6)\ -a^2 - \frac{5}{4}b^2 + \frac{3}{4}c^2 + d^2 + ad - \frac{1}{2}bc - \frac{1}{2}(bY_1 + dY_2) = 0. \end{cases} \tag{2.13}$$

From the third equation in (2.13), we obtain the value of $\mu$:

$$\mu = (b+c)^2 + 2(a^2+d^2).$$

Now, from the first and second equations, one can see

$$Y_3 = \frac{2(b+c)^2 - 2(a+d)^2 + 4(a^2+d^2)}{a+d}.$$

From the fifth equation of (2.13) we have:

$$aY_1 + cY_2 = -2\left(a^2 + \frac{5}{4}c^2 + d^2 + ad - \frac{3}{4}b^2 + \frac{1}{2}bc\right). \tag{2.14}$$

From the sixth equation of (2.13), we also have:

$$bY_1 + dY_2 = -2\left(a^2 + \frac{5}{4}b^2 - \frac{3}{4}c^2 - d^2 - ad + \frac{1}{2}bc\right). \tag{2.15}$$

If $ad - bc \neq 0$, we deduce that

$$Y_1 = \frac{Ad - Bc}{ad - bc}, \quad Y_2 = \frac{Ba - Ab}{ad - bc},$$

where

$$A = 2(a^2 + d^2 - ad) - \frac{3}{2}(b-c)^2,$$
$$B = \frac{1}{2}(b-c)^2 - 2(a^2 + ad + d^2).$$

Also, let

$$C = a^2 + ad - d^2 + \frac{1}{4}(b+c)^2,$$

if the parameters $a, b, c, d$ satisfy the following conditions;

$$ad - bc \neq 0, \quad C + (b+c)\frac{(b+c)^2 - (a+d)^2 + 2(a^2+d^2)}{a+d} = 0,$$

then, according to the above arguments, we can present the next theorem.

**Theorem 2.6.** *The 3D Lorentzian Lie algebra of type* IV $\cdot$ 3 *admits shrinking* $b - \mathcal{RS}$ *structures.*

2.4. **Non-unimodular Lie algebra** IV $\cdot$ 3**.** Now suppose that $\mathfrak{g}_{IV}$ admits the unitary pseudo-orthogonal basis described in IV $\cdot$ 2, given. Consider the following unitary orthogonal basis:

$$\widetilde{e}_1 := e_1, \quad \widetilde{e}_2 := \frac{1}{\sqrt{2}}(e_2 - e_3), \quad \widetilde{e}_3 := \frac{1}{\sqrt{2}}(e_2 + e_3),$$

with signature $(+ + -)$. So, the nonzero components of $R$ are given as:

$$\begin{aligned} R_{1212} &= \tfrac{1}{4}\left(2ad - 2a^2 - c(2b + c)\right), \quad R_{1213} = \tfrac{1}{2}\left(a^2 + bc - ad\right), \\ R_{1313} &= \tfrac{1}{4}\left(2ad - 2a^2 + c(c - 2b)\right), \quad R_{2323} = -\tfrac{3}{4}c^2. \end{aligned}$$

$Q$ in the new basis $\{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3\}$ is:

$$Q = \begin{pmatrix} -\frac{1}{2}c^2 & 0 & 0 \\ 0 & \frac{1}{2}(a(d-a) + c(c-b)) & \frac{1}{2}(a(a-d) + bc) \\ 0 & -\frac{1}{2}(a(a-d) + bc) & \frac{1}{2}(a(a-d) + c(b+c)) \end{pmatrix},$$

which has the eigenvalues $\lambda_1 = -\frac{1}{2}c^2$ and $\lambda_2 = \lambda_3 = \frac{1}{2}c^2$. So, $Q$ is of type $\{21\}$.
Here, the components of the bi-Ricci tensor are given as follows:

$$\begin{aligned} \mathrm{biRic}_{11} &= -c^2, \\ \mathrm{biRic}_{22} &= a(d-a) + c(c-b), \\ \mathrm{biRic}_{33} &= -\left(a(a-d) + c(b+c)\right), \\ \mathrm{biRic}_{12} &= \tfrac{1}{4}c^2, \\ \mathrm{biRic}_{13} &= -\tfrac{5}{4}c^2, \\ \mathrm{biRic}_{23} &= ad - a^2 - cb + \tfrac{3}{4}c^2. \end{aligned}$$

For any $Y = \Sigma Y_i \widetilde{e}_i$, the derivative of the meter, is as follows:

$$(\mathcal{L}_Y g) = \begin{pmatrix} \sqrt{2}a(Y_3 - Y_2) & a_{12} & a_{13} \\ a_{12} & bY_1 + \sqrt{2}dY_3 & -bY_1 - \frac{1}{\sqrt{2}}d(Y_2 + Y_3) \\ a_{13} & -bY_1 - \frac{1}{\sqrt{2}}d(Y_2 + Y_3) & bY_1 + \sqrt{2}dY_2 \end{pmatrix},$$

in which

$$\begin{aligned} a_{12} &= \frac{1}{2}\left(Y_3(b + 2c) + \sqrt{2}Y_1 a - Y_2 b\right), \\ a_{13} &= \frac{1}{2}\left(Y_2(b - 2c) - \sqrt{2}Y_1 a - Y_3 b\right). \end{aligned}$$

Hence, the formula (1.1) holds if and only if

$$\begin{cases} 1)\ -c^2+\frac{\sqrt{2}}{2}a(Y_3-Y_2)=\mu, \\ 2)\ a(d-a)+c(c-b)+\frac{1}{2}bY_1+\frac{\sqrt{2}}{2}dY_3=\mu, \\ 3)\ a(a-d)+c(b+c)-\frac{1}{2}bY_1-\frac{\sqrt{2}}{2}Y_2d=\mu, \\ 4)\ \frac{1}{4}c^2+\frac{1}{4}\left((b+2c)Y_3+\sqrt{2}aY_1-bY_2\right)=0, \\ 5)\ -\frac{5}{4}c^2+\frac{1}{4}\left((b-2c)Y_2-\sqrt{2}aY_1-bY_3\right)=0, \\ 6)\ ad-a^2-cb+\frac{3}{4}c^2+\frac{1}{2}\left(Y_1b+\frac{1}{\sqrt{2}}d(Y_2+Y_3)\right)=0. \end{cases} \tag{2.16}$$

Now, since $ac=0$, we consider the possible states $a=0$ and $c=0$ separately. Note that if $a=c=0$, then the meter $\mathfrak{g}_{IV}$ is flat.

**Case** $(a)$: By substituting $a=0\neq c$ in the above system, we obtain $\mu=-c^2$. Solving the fifth and the sixth equations gives

$$Y_2=-(\frac{5}{2}c+b),\quad Y_3=-(\frac{5}{4}b+\frac{1}{2}c).$$

Using other equations, one can conclude that $Y_1=\frac{3c^2+4bc}{8b}$. Also, we must have

$$d=-2\sqrt{2}c,\qquad 4b=3c.$$

**Case** $(b)$**:** Suppose $c=0\neq a$. From the equality of the second and third equations (2.16), one gets:

$$Y_1b+\frac{1}{\sqrt{2}}d(Y_2+Y_3)=2a(a-d). \tag{2.17}$$

By adding the fourth and fifth equations, we obtain: $2\sqrt{2}aY_1=0$ which gives $Y_1=0$.

By substituting $Y_1=0$, into the equation (2.17) we have: $Y_2+Y_3=\frac{2\sqrt{2}a(a-d)}{d}$ which gives $d\neq 0$. On the other hand, from the first equation (2.16), the value $\mu$ is obtained as $\mu=\frac{\sqrt{2}}{2}a(Y_3-Y_2)$. Comparing the second and third equations in (2.16), we conclude that $Y_2=Y_3$:

$$Y_2=Y_3=\frac{\sqrt{2}a(a-d)}{d}\Rightarrow\mu=0.$$

**Theorem 2.7.** *The 3D Lorentzian $\mathcal{LG}$ admits steady and expanding $b-\mathcal{RS}$ structure.*

## References


1. M. M. Akbar and E. Woolgar, *Ricci solitons and Einstein-scalar field theory*, Class. Quantum Grav. **26**(2009), 055015.
2. S. Azami, G. Fasihi-Ramandi and M. A. Choudhary, *Some characterization of $\alpha$-cosymplectic manifolds admitting hyperbolic Ricci solitons (HRS)*, J. Finsler Geom. Appl. **7**(1) (2026), 20-30.
3. S. Brendle, S. Hirsch, and F. John, *A generalization of Geroch's conjecture*, Commn. Pure. Appl. Math. **77**(2024), 441-456.

4. M. Brozos-Vazquez, G. Calvaruso, E. Garcia-rio, and S. Gavino-Fernandez, *Three-dimensional lorentzian homogeneous Ricci solitons*, arXiv:0911.1247.
5. M. Brozos-Vazquez, E. Garcia-Rio, P. Gilkey, S. Nikcevic, and R. Vazquez-Lorenzo, *The geometry of Walker manifolds*, Synthesis Lectures on Mathematics and Statistics 5, Morgan & Claypool Publ., 2009.
6. G. Calvaruso, *Homogeneous structures on three-dimensional Lorentzian manifolds*, J. Geom. Phys. **57**(2007), 1279-1291.
7. G. Calvaruso, *Einstein-like metrics on three-dimensional homogeneous Lorentzian manifolds*, Geom. Dedicata **127**(2007), 99-119.
8. G. Calvaruso and O. Kowalski, *On the Ricci operator of locally homogeneous Lorentzian 3-manifolds*, Central Eur. J. Math. **7**(2009), 124-139.
9. H.-D. Cao, *Recent progress on Ricci solitons*, arXiv:0908.2006.
10. J. S. Case, *Singularity theorems and the Lorentzian splitting theorem for the Bakry-Emery-Ricci tensor*, arXiv:0712.1321.
11. O. Chodosh, C. Li, P. Minter, and D. Stryker, *Stable minimal hypersurfaces*, Preprint, arXiv:2401.01492.
12. L. F. di Cerbo, *Generic properties of homogeneous Ricci solitons*, arXiv:0711.0465.
13. M. Chaichi, E. Garcia-Rio, and M. E. Vazquez-Abal, *Three-dimensional Lorentz manifolds admitting a parallel null vector field*, J. Phys. A. **38**(2005), 841-850.
14. L. A. Cordero and Ph. Parker, *Left-invariant Lorentzian metrics on 3-dimensional Lie groups*, Rend. Mat. VII. **17**(1997), 129-155.
15. D. Friedan, *Nonlinear models in* $2+\varepsilon$ *dimensions*, Ann. Phys. **163**(1985), 318-419.
16. E. Garcia-Rio, A. Haji-Badali, and R. Vazquez-Lorenzo, *Lorentzian 3-manifolds with special curvature operators*, Class. Quantum Grav. **25**(2008), 015003.
17. Ch. Guenther, J. Isenberg, and D. Knopf, *Linear stability of homogeneous Ricci solitons*, arXiv:math/0606793.
18. G. S. Hall, T. Morgan, and Z. Perjes, *Three-dimensional space-times*, Gen. Rel. Gravitation. **19**(1987), 1137-1147.
19. S. Hervik, *Ricci nilsoliton black holes*, J. Geom. Phys. **58**(2008), 1253-1264.
20. A. L. Kholodenko, *Towards physically motivated proofs of the Poincare and the geometrization conjectures*, J. Geom. Phys. **58**(2008), 259-290.
21. J. Lott, *On the long-time behavior of type-III Ricci flow solutions*, Math. Ann. **339**(2007), 627-666.
22. L. Mazet, *Stable minimal hypersurfaces*, Preprint, arXiv:2405.14676.
23. T. L. Payne, *The existence of soliton metrics for nilpotent Lie groups*, Geom. Dedicata, (to appear).
24. R. Pina and K. Tenenblat, *On solutions of the Ricci curvature and the Einstein equation*, Israel J. Math. **171**(2009), 61-76.
25. S. Rahmani, *Metriques de Lorentz sur les groupes de Lie unimodulaires de dimension trois*, J. Geom. Phys. **9**(1992), 295-302.
26. A. G. Walker, *Canonical form for a Riemannian space with a parallel field of null planes*, Quart. J. Math. Oxford. **1**(1950), 69-79.
27. M. Yar Ahmadi and S. Hedayatian, *Finite topological type of complete gradient shrinking GRF system solitons*, J. Finsler Geom. Appl. **7**(1) (2026), 123-129.

Received: 03.12.2025
Accepted: 16.04.2026