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Finslerian convolution metrics and their special classes

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Abstract. In this paper, we introduce a concept of convolution metric in Finslerian Geometry. We study some basic properties of Finslerian convolution metrics. Then, we characterize Riemannian, Minkowskian as well as Randers type convolution metrics among Finslerian convolution metrics. Furthermore, we provide several examples of Finslerian convolution metrics.

Keywords: Finslerian metric; Riemannian metric; Randers metric; Minkowskian metric; convolution metric.

1. Introduction

Convolution metrics play an important role in differential geometry and have applications in General Relativity, as they can be viewed as natural extensions of warped product metrics. As a concrete example of application, many basic solutions of the Einstein field equations on Riemannian manifolds have solutions when the Riemannian metrics are warped products at most; see, for example, [4].

Finsler metrics have attracted considerable interest due to their rich structure including mainly Riemann, Randers, Minkowski and Berwald type metrics. For details, the study of these special Finsler metrics and some important examples can be found in [1, 6] and in [5].

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In Riemannian Geometry, the notion of convolution metrics was introduced by Bang-Yen Chen to establish and characterize the Euclidean version of Segre embedding [2]. The following Proposition shows that the notion of convolution metrics arises very naturally from tensor product immersions.

Proposition 1.1. [3] *Let M_1 and M_2 be two C^∞ manifolds of dimensions n_1 and n_2 respectively. Consider g_1 and g_2 two Riemannian metrics on M_1 and M_2 respectively. If $\varphi_1 : (M_1, g_1) \longrightarrow \mathbb{R}^{n_1}\backslash\{0\}$ and $\varphi_2 : (M_2, g_2) \longrightarrow \mathbb{R}^{n_2}\backslash\{0\}$ are isometric immersions then the mapping*

$$\varphi = \varphi_1 \otimes \varphi_2 : M_1 \times M_2 \longrightarrow \mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2} = \mathbb{R}^{n_1 n_2}$$

defined by $\varphi(x_1, x_2) = \varphi_1(x_1)\varphi_2(x_2)$ gives rise to a convolution manifold $M_1 \bigstar M_2$ endowed with the convolution metric

$${}_{\rho_2}g_1 \star_{\rho_1} g_2 = \rho_2^2 g_1 + \rho_1^2 g_2 + 2\rho_1\rho_2 d\rho_1 \otimes d\rho_2$$

whenever ${}_{\rho_2}g_1 \star_{\rho_1} g_2$ is non-degenerate where $\rho_1 = \|\varphi_1\|$ and $\rho_2 = \|\varphi_2\|$.

In this paper, we construct an extension of the Riemannian convolution metric on a product manifold (See the Definition 2.5). More precisely, we define a new notion of convolution metric by taking account of, not only the Riemannian metrics but also the Minkowski, the Randers and the Berwald metrics. We have the following proposition.

Proposition 1.2. *Let (M_1, F_1) and (M_2, F_2) be two Finslerian manifolds. Consider $f_k : M_k \longrightarrow (0, \infty)$, for $k \in \{1, 2\}$, some smooth functions. If ${}_{f_2}F_1 \star_{f_1} F_2$ is a convolution Finslerian metric on the produit manifold $M_1 \times M_2$ then one of the following assertions holds:*

1. *the gradient of one of the functions f_k is orthogonal to any tangent vector on $M_1 \times M_2$.*
2. *one of the f_k is a nonzero constant. In this last case, when one of the $f_k \equiv 1$, the convolution Finslerian metric ${}_{f_2}F_1 \star_{f_1} F_2$ reduces to a Finslerian warped product metric.*

The main results concern the characterization of convolution metrics of Riemannian, Minkowskian and Randers type among Finslerian metrics. For example, we prove the following result.

Theorem 1.3. *Let (M_1, F_1) and (M_2, F_2) be two Finslerian manifolds of Randers type that is, for $k \in \{1, 2\}$, $F_k := \alpha_k + \beta_k$ where α_k is a Riemannian metric and β_k is a 1-form on M_k with $||\beta_k||_{\alpha_k} < 1$. If one of the $f_k : M_k \longrightarrow (0, \infty)$ is a nonzero constant function, then the Finslerian convolution metric ${}_{f_2}F_1 \star_{f_1} F_2$ on $M_1 \times M_2$ is a Randers metric if and only if*

$$\frac{\alpha_1}{\alpha_2} = \frac{\beta_1}{\beta_2}. \tag{1.1}$$

Finally, we give one of the examples of special Finslerian convolution.

Example 1.4. *Consider* $M = \{x = (x^1, x^2, x^3, x^4) \in \mathbb{R}^4 : x_1 > 0, x_3 > 0\}$ *and* $T_xM = \{(y^1, y^2, y^3, y^4) \in \mathbb{R}^4 : y_1 > 0, y_3 > 0\}$. *For* $\lambda \in [2, 4]$ *and for a positive integer* k, *define* $F : TM \longrightarrow (0, \infty)$ *by*

$$\begin{aligned} F(x^1, x^2, x^3, x^4, y^1, y^2, y^3, y^4) &= \Big\{ x_3^2 \left(y_1^4 + \lambda y_1^2 y_2^2 + y_2^4\right)^{\frac{1}{2}} + 8 x_1^3 x_3^3 y_1 y_3 \\ &\quad + x_3^2 \left[y_3^2 + y_4^2 + \lambda \left(y_3^{2k} + y_4^{2k}\right)^{\frac{1}{k}} \right] \Big\}^{\frac{1}{2}}. \quad (1.2) \end{aligned}$$

Then the function F *defined in (1.2) is a Finslerian convolution metric.*

The outline of this paper is organized as follows. In Section 2, we establish some basic notions on Finslerian convolution manifolds. Section 3 is devoted to the classification of special convolution metrics via the Finslerian fundamental function F conditions. In the Section 4, we provide examples of Finslerian convolution metrics.

Notation. In the rest of this paper, M is an $n-$dimensional product of C^2-manifolds M_1 and M_2 of dimensions n_1 and n_2 respectively. We denote by T_xM the tangent space at $x = (x_1, x_2) \in M$ and by $TM := \bigcup_{x \in M} T_xM$ the tangent bundle of M. We set $\mathring{T}M = \{(x_1, x_2, y_1, y_2) \in TM : y_1 \neq 0, y_2 \neq 0$ where 0 is a zero vector $\in T_{x_k}M_k$ for $k \in \{1, 2\}\}$ and $\pi : TM \to M$, $\pi(x, y) = x$ the natural projection. Let $(x^1, ..., x^n) = (x_1^1, ..., x_1^{n_1}, x_2^{n_1+1}, ..., x_2^n)$ be a local coordinate on an open subset U of M and $(x^1, ..., x^n, y^1, ..., y^n)$ be the local coordinate on $\pi^{-1}(U) \subset TM$. The local coordinate system $(x^i)_{i=1,...,n}$ produces the coordinate bases $\{\frac{\partial}{\partial x^i}\}_{i=1,...,n}$ and $\{dx^i\}_{i=1,...,n}$ respectively, for TM and cotangent bundle T^*M.

Convention. We use Einstein summation convention: repeated upper and lower indices will automatically be summed, from 1 to n, unless otherwise will be noted. For example $g_{ij}v^iv^j = \sum_{i,j=1}^n g_{ij}v^iv^j$.

2. Finslerian Convolution Manifold

We present basic definitions and observations.

Definition 2.1. *Let* M *be an* n*-dimensional* C^∞ *manifold. A scalar function* $F : TM \to [0, \infty)$ *is a Finslerian metric on* M *if the following conditions are fulfilled:*

(i) F *is* C^∞ *on the entire slit tangent bundle* $\mathring{T}M$,
(ii) F *is positively* 1*-homogeneous on the fibers of* TM, *that is for all* $c > 0$, $F(x, cy) = cF(x, y)$,
(iii) *the Hessian matrix* $(g_{ij}(x, y))_{1 \leq i,j \leq n}$ *with elements*

$$g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} \quad (2.1)$$

is positive-definite at every point (x,y) of $\mathring{T}M$.

Remark 2.2. *The Hessian matrix g whose elements are defined in (2.1) is a natural Riemannian metric on the pulled-back tangent bundle π^*TM over the manifold $\mathring{T}M$. The space π^*TM is a vector bundle and its fiber over a point $(x,y)\in\mathring{T}M$ is*

$$\pi^*TM|_{(x,y)} := \Big\{(x,y;v) : v\in T_xM\Big\}|_{(x,y)} \cong T_xM. \tag{2.2}$$

Definition 2.3. *Let (M,F) be an n-dimensional Finslerian manifold and g the Hessian matrix associated with F. The gradient of a smooth function u on M is, the section of the vector bundle π^*TM denoted by ∇u, given by*

$$g_{(x,y)}(\nabla u_{(x,y)}, \xi_{(x,y)}) = du_{(x,y)}(\xi_{(x,y)}) \tag{2.3}$$

for any $\xi\in\Gamma(\pi^*TM)$ and for every $(x,y)\in\mathring{T}M$. Locally,

$$\nabla u_{(x,y)} = g^{ij}(x,y)\frac{\partial u}{\partial x^i}\frac{\partial}{\partial x^j}. \tag{2.4}$$

Remark 2.4. *Let M_1 and M_2 be two C^∞ manifolds. For every $(x_1,x_2)\in M_1\times M_2$, we have the following properties derived from M_1 and M_2.*

(1) *The projections*

$$\begin{aligned}\sigma_1 &: M_1\times M_2 \longrightarrow M_1 \text{ such that } \sigma_1(x_1,x_2)=x_1\\ \sigma_2 &: M_1\times M_2 \longrightarrow M_1 \text{ such that } \sigma_2(x_1,x_2)=x_2\end{aligned}$$

are C^∞ submersions.

(2) *$dim(M_1\times M_2) = dimM_1 + dimM_2$.*

Definition 2.5. *Let (M_1,F_1) and (M_2,F_2) be two Finslerian manifolds. Consider $f_k : M_k\longrightarrow(0,\infty)$, for $k\in\{1;2\}$, some smooth functions. A convolution of (M_1,F_1) and (M_2,F_2), via f_1 and f_2, is the product manifold $M_1\times M_2$ endowed with a function $F_{f_1,f_2} : T(M_1\times M_2)\longrightarrow(0,\infty)$ defined by*

$$F_{f_1,f_2} := \sqrt{h_2^2(\sigma_1^*F_1)^2 + h_1^2(\sigma_2^*F_2)^2 + \frac{h_1h_2}{2}\frac{\partial(\sigma_1^*F_1)^2}{\partial y^i}\frac{\partial(\sigma_2^*F_2)^2}{\partial y^j}\big(\nabla h_1\big)^i\big(\nabla h_2\big)^j} \tag{2.5}$$

where $h_k := f_k\circ\sigma_k$. The function F_{f_1,f_2} is called a Finslerian convolution of F_1 and F_2, via f_1 and f_2, and is denoted by ${}_{f_2}F_1\star_{f_1}F_2$. Given a Finslerian convolution function ${}_{f_2}F_1\star_{f_1}F_2$ on $M_1\times M_2$, the pair $(M_1\times M_2, {}_{f_2}F_1\star_{f_1}F_2)$ is called a Finslerian convolution manifold and is denoted by ${}_{f_2}M_1\bigstar_{f_1}M_2$.

Remark 2.6. *For any $x\equiv(x_1,x_2)\in M\equiv M_1\times M_2$ and $y\equiv(y_1,y_2)\in T_xM$,*

(1) *$F_{f_1,f_2}(x,y)$ exists for some positive-real functions f_1 and f_2 on M_1 and M_2 respectively.*

(2) F_{f_1,f_2} *is not* C^∞ *on the tangent vectors of the form* $(y_1,0)$ *nor* $(0,y_2)$ *at a point* $(x_1,x_2)\in M_1\times M_2$ *but on* (y_1,y_2) *for* $y_1\neq 0$ *and* $y_2\neq 0$.

(3) ${}_{f_2}F_1\star_{f_1}F_2$ *is positively homogeneous of degree one with respect to* (y_1,y_2).

(4) *from the relation (2.5), we have*

$$\begin{aligned} F^2_{f_1,f_2}(x,y) &= 2f_1(x_1)f_2(x_2)F_1(x_1,y_1)F_2(x_2,y_2)\\ &\quad\times\frac{\partial F_1(x_1,y_1)}{\partial y^i}\frac{\partial F_2(x_2,y_2)}{\partial y^j}\big(\nabla f_1(x_1)\big)^i\big(\nabla f_2(x_2)\big)^j\\ &\quad+f_2^2(x_2)F_1^2(x_1,y_1)+f_1(x_1)^2F_2^2(x_2,y_2). \end{aligned}\tag{2.6}$$

Since (x_1,x_2) *is arbitrary a point of* M, *the expression (2.6) can be written as*

$$F^2_{f_1,f_2}=f_2^2F_1^2+f_1^2F_2^2+2f_1f_2F_1F_2\frac{\partial F_1}{\partial y^i}\frac{\partial F_2}{\partial y^j}\big(\nabla f_1\big)^i\big(\nabla f_2\big)^j.$$

The following lemma is needed throughout the rest of this work.

Lemma 2.7. *Let* M_1 *and* M_2 *be two* C^∞ *manifolds of dimensions* n_1 *and* n_2, *respectively, with* $n_1+n_2=n$. *If* $F={}_{f_2}F_1\star_{f_1}F_2$ *is a Finslerian convolution function on the manifold* $M_1\times M_2$ *then the Hessian matrix associated with* F *is given by*

$$\Big(g_{ij}(x_1,x_2,y_1,y_2)\Big)=\begin{pmatrix} f_2^2(x_2)\Big(g_{1\alpha\beta}(x_1,y_1)\Big) & 2f_1(x_1)f_2(x_2)\Big(\frac{\partial f_1(x_1)}{\partial x_1^\alpha}\frac{\partial f_2(x_2)}{\partial x_2^\nu}\Big)\\ \Big(0\Big) & f_1^2(x_1)\Big(g_{2\mu\nu}(x_2,y_2)\Big)\end{pmatrix}$$

at every point (x_1,x_2,y_1,y_2) *of* $\overset{\circ}{T}M_1\times\overset{\circ}{T}M_2$ *where* $i,j\in[1,...,n],\alpha,\beta\in[1,...,n_1],\mu,\nu\in[n_1+1,...,n]$ *and,* g_1 *and* g_2 *are the Hessian matrices of* F_1 *and* F_2 *respectively.*

Proof. The proof follows from a straightforward calculation. □

Remark 2.8. *In general, the components* g_{ij} *of the convolution matrix* g *are not symmetric in all indices.*

Because of the Remark 2.8, we have the Proposition 2.9.

Proposition 2.9. *Let* M_1 *and* M_2 *be two* C^∞ *manifolds of dimensions* n_1 *and* n_2, *respectively, with* $n_1+n_2=n$. *If* ${}_{f_2}F_1\star_{f_1}F_2$ *is a Finslerian convolution metric on a manifold* $M_1\times M_2$ *and if* $v\neq 0$ *is a tangent vector on* $M_1\times M_2$ *then the quadratic form* $({}_{f_2}g_1\star_{f_1}g_2)(v,v)$ *related with* F *is positive-definite if*

$$\frac{1}{f_1^2}g_{1\alpha\beta}v^\alpha v^\beta+\frac{1}{f_2^2}g_2v^\mu v^\nu>-\frac{2}{f_1f_2}\frac{\partial f_1}{\partial x_1^\alpha}\frac{\partial f_2}{\partial x_2^\nu}v^\alpha v^\nu,\tag{2.7}$$

where $\alpha,\beta\in[1,...,n_1],\mu,\nu\in[n_1+1,...,n]$ *and,* g_1 *and* g_2 *are the Hessian matrices of* F_1 *and* F_2 *respectively.*

Proof. From the Lemma 2.7, for any vector $v \neq 0$ on $M_1 \times M_2$, we have

$$\begin{aligned} g(v,v) &= \sum_{i,j=1}^{n} g_{ij} v^i v^j \\ &= \sum_{\alpha,\beta=1}^{n_1} g_{\alpha\beta} v^\alpha v^\beta + \sum_{\alpha=1}^{n_1} \sum_{\mu=n_1+1}^{n} g_{\alpha\mu} v^\alpha v^\mu \\ &\quad + \sum_{\mu=n_1+1}^{n} \sum_{\alpha=1}^{n_1} g_{\mu\alpha} v^\mu v^\alpha + \sum_{\mu,\nu=n_1+1}^{n} g_{\mu\nu} v^\mu v^\nu. \end{aligned}$$

Since $g \equiv_{f_2} g_1 \star_{f_1} g_2$, it follows that

$$\begin{aligned} g_{ij}(v,v) &= \sum_{\alpha,\beta=1}^{n_1} f_2^2 g_{1\alpha\beta} v^\alpha v^\beta + \sum_{\alpha=1}^{n_1} \sum_{\mu=n_1+1}^{n} 2 f_1 f_2 \frac{\partial f_1}{\partial x_1^\alpha} \frac{\partial f_2}{\partial x_2^\mu} v^\alpha v^\mu \\ &\quad + \sum_{\mu,\nu=n_1+1}^{n} f_1^2 g_{\mu\nu} v^\mu v^\nu \end{aligned} \tag{2.8}$$

with $g_{\alpha\mu} := 2 f_1 f_2 \frac{\partial f_1}{\partial x_1^\alpha} \frac{\partial f_2}{\partial x_2^\mu}$.

By the fact that F_1 and F_2 are Finslerian metrics, the first and the last terms in the right hand-side of (2.8) are positives. Hence $g(v,v) > 0$ if the equation (2.7) holds. □

When the matrix given in the Lemma 2.7 is positive-definite symmetric on $\mathring{T}(M_1 \times M_2)$, it defines a Finslerian metric on $M_1 \times M_2$. In this case, ${}_{f_2}F_1 \star_{f_1} F_2$ is called a convolution Finslerian metric and the Finslerian convolution manifold ${}_{f_2}M_1 \bigstar_{f_1} M_2$ is called a convolution Finslerian manifold.

Proof. The proof is obtained by the Lemma 2.7 together with the Proposition 2.9. □

3. Characterization of special Finslerian convolution metrics

Now, we study and characterize Riemannian, Randers and Minkowski convolution metrics among Finslerian convolution metrics by means of intrinsic conditions on the Finslerian fundamental function.

For the Riemannian case, we have the following definition and result.

Definition 3.1. *A Riemannian metric g on a manifold M is a family $\{g_x\}_{x\in M}$ where g_x is an inner product on the tangent space T_xM, such that the elements $g_{ij}(x) := g_x\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ are C^∞ in local coordinates. Since g_x is an inner product, the matrix $(g_{ij}(x))$ is positive-definite at every point $x \in M$. Thus every*

Riemannian metric g is a Finslerian metric which arises in the following manner

$$F(x,y) := \sqrt{g_{ij}(x)y^i y^j}.$$

In that case, $\frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}$ is simply $g_{ij}(x)$ which is independent of y.

Theorem 3.2. *A convolution Finslerian metric ${}_{f_2}F_1 \star_{f_1} F_2$ is Riemannian if and only if F_1 as well as F_2 are Riemannian metrics.*

Proof. Assume that $F =_{f_2} F_1 \star_{f_1} F_2$ is a Riemannian metric. By the Definition 3.1 the components g_{ij} associated with F are independent of (y_1, y_2). Hence, from the Lemma 2.7, we have

$$g_{1\alpha\beta}(x_1,y_1) = g_{1\alpha\beta}(x_1) \text{ and } g_{2\mu\nu}(x_2,y_2) = g_{2\mu\nu}(x_2).$$

That is $F_1(x_1,y_1)$ and $F_2(x_2,y_2)$ are Riemannian metrics. The converse follows immediately. □

In the Minkowskian realm, we have the following definition and result.

Definition 3.3. *A Finslerian metric F on a manifold M is locally Minkowski metric if, for every $x \in M$, there exists an neighborhood V of x such that the components g_{ij} of the Hessian matrix, given in (2.1), satisfy*

$$g_{ij}(x,y) = g_{ij}(y). \tag{3.1}$$

Theorem 3.4. *A convolution Finslerian metric ${}_{f_2}F_1 \star_{f_1} F_2$ is locally Minkowskian if and only if F_1 as well as F_2 are locally Minkowskian metrics and the functions $f_k, k \in \{1,2\}$ are both nonzero constants.*

Proof. The Theorem 3.4 is obtained by a straightforward calculation of the locally Minkowskian metric from the Lemma 2.7 by using the expression (3.1). □

Finally, the Randers convolution metrics are characterized as follows.

Definition 3.5. *A Randers metric F on an n-dimensional manifold M is a Finsler metric which has the following form*

$$F := \alpha + \beta$$

where, for every $(x,y) \in \mathring{T}M$, $\alpha_{(x,y)} := \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta_{(x,y)} := b_i(x)y^i$ is a 1-form on M with $||\beta||_\alpha := \sqrt{a^{ij}b_i b_j} < 1$.

Proof of Theorem 1.3: Suppose that f_1 is a nonzero constant function. Then, since F_1 and F_2 are Randers metrics, the third term of the right-hand side of the (2.7) vanishes. Hence, the convolution metric becomes

$${}_{f_2}F_1 \star_{f_1} F_2 = f_2^2(\alpha_1^2 + 2\alpha_1\beta_1 + \beta_1^2) + f_1^2(\alpha_2^2 + 2\alpha_2\beta_2 + \beta_2^2).$$

Since f_1 and f_2 are positive functions, denote $f_2\alpha_1, f_1\alpha_2, f_2\beta_1$ and $f_1\beta_2$ by $\alpha_1^*, \alpha_2^*, \beta_1^*$ and β_2^* respectively. We obtain

$$ {}_{f_2}F_1 \star_{f_1} F_2 = \alpha_1^{*2} + \alpha_2^{*2} + 2(\alpha_1^*\beta_1^* + \alpha_2^*\beta_2^*) + \beta_1^{*2} + \beta_2^{*2}. \tag{3.2}$$

By setting $\alpha = \sqrt{\alpha_1^{*2} + \alpha_2^{*2}}$ and $\beta = \sqrt{\beta_1^{*2} + \beta_2^{*2}}$, by using the Definition 3.5 and the relation (3.2), we get $\alpha\beta = \alpha_1^*\beta_1^* + \alpha_2^*\beta_2^*$. Hence the relation (1.1) follows from the fact that $\alpha_1^* = f_2\alpha_1, \beta_1^* = f_2\beta_1, \alpha_2^* = f_1\alpha_2$ and $\beta_2^* = f_1\beta_2$. □

As a consequence of Theorem 1.3, we have:

Corollary 3.6. *A convolution Finslerian metric ${}_{f_2}F_1\star_{f_1}F_2$ is Euclidean if and only if F_1 as well as F_2 are Euclidean metrics and the functions $f_k, k \in \{1,2\}$ are both nonzero constants.*

4. Some Examples of Finslerian Convolution Metrics

Now, we explicit non-trivial examples of convolution metrics.

Example 4.1. *(Finslerian convolution metric of Riemannian type) Denote by $\|.\|$ and $\langle .,.\rangle$ the standard Euclidean norm and the inner product in $\mathbb{R}^n$ respectively. Let $\mathbb{B}_1 = \{x_1 \in \mathbb{R}^3 : \|x_1\| < 1\}$ and $\mathbb{B}_2 = \{x_2 \in \mathbb{R}^2 : \|x_2\| < 1\}$ be two Euclidean balls. Define the functions*

$$\begin{aligned} F_k : T\mathbb{B}_k &\longrightarrow (0,\infty) \\ (x_k, y_k) &\to \frac{\sqrt{\|y_k\|^2 - (\|x_k\|^2\|y_k\|^2 - \langle x_k, y_k\rangle^2)}}{1 - \|x_k\|^2} \end{aligned}$$

for $k \in \{1,2\}$. Then the functions F_1 and F_2 are Riemannian metrics on $\mathbb{B}_1$ and $\mathbb{B}_2$ respectively known as Klein metrics.

Consider two smooth functions $\rho_k : \mathbb{B}_k \longrightarrow \mathbb{R}$. Then the function

$$\begin{aligned} F_{f_1,f_2}(x_1,x_2,y_1,y_2) = &\left\{ \frac{e^{2\rho_2}}{\left(1 - \sum_{r=1}^3 (x_1^r)^2\right)^2} \left[\sum_{r=1}^3 (y_1^r)^2 \left(1 - \sum_{r=1}^3 (x_1^r)^2\right) + \left(\sum_{r=1}^3 x_1^r y_1^r\right)^2 \right] \right. \\ &+ \frac{e^{2\rho_1}}{\left(1 - \sum_{s=4}^5 (x_2^s)^2\right)^2} \left[\sum_{s=4}^5 (y_2^s)^2 \left(1 - \sum_{s=4}^5 (x_2^s)^2\right) + \left(\sum_{s=4}^5 x_2^s y_2^s\right)^2 \right] \\ &\left. + 2e^{2\rho_1}e^{2\rho_2} \sum_{r=1}^3 \sum_{s=4}^5 \frac{\partial\rho_1}{\partial x_1^r}\frac{\partial\rho_2}{\partial x_2^s} y_1^r y_2^s \right\}^{\frac{1}{2}}. \end{aligned} \tag{4.1}$$

is the Riemannian convolution of F_1 and F_2, via $f_1 := e^{\rho_1}$ and $f_2 := e^{\rho_2}$.

When one the ρ_k is a constant function or when $\nabla\rho_k \perp v$ for each tangent vector v on $\mathbb{B}_k$, the Riemannian convolution function (4.1) becomes a convolution Riemannian metric on $\mathbb{B}_1 \times \mathbb{B}_2$.

Example 4.2. *(Minkowskian type Finslerian convolution metric) Let consider the Finslerian function F defined by the relation (1.2) in the Example 1.4.*

It can be shown that F has the form ${}_{f_2}F_1 \star_{f_1} F_2$, with $f_k : U \subset \mathbb{R}^2 \longrightarrow (0,\infty)$ such that $f_k(x_1,x_2) = c_k$. That is F is a Minkowskian convolution function on $M_1 \times M_2$ where $M_1 = \{(y^1,y^2) \in \mathbb{R}^2\}$, $M_2 = \{(y^3,y^4) \in \mathbb{R}^2\}$, $F_1 = \left(y_1^4 + \lambda y_1^2 y_2^2 + y_2^4\right)^{\frac{1}{4}}$ and $F_2 = \left[y_3^2 + y_4^2 + \lambda\left(y_3^{2k} + y_4^{2k}\right)^{\frac{1}{k}}\right]^{\frac{1}{2}}$ via c_1 and c_2.

Example 4.3. *(Finslerian convolution metric of Randers type) Denote by $\|.\|_V$ the norm on a 3-vector space V, by $\|.\|$ the standard Euclidean norm and by $\langle .,. \rangle$ the inner product in $\mathbb{R}^{n-3}$. Let $\mathbb{B} = \{x_2 \in \mathbb{R}^{n-3} : \|x_2\| < 1\}$ be an Euclidean ball. Then, for any $\epsilon \in [0,1)$, the following function $F : T\big(V \times \mathbb{B}\big) \longrightarrow (0,\infty)$ defined by*

$$\begin{aligned} F(x_1,x_2,y_1,y_2) &= F(x_1^1,x_1^2,x_1^3,x_2^4,...,x_2^n,y_1^1,y_1^2,y_1^3,y_2^4,...,y_2^n) \\ &= \Bigg\{ \|x_2\|^2 \left(\|y_1\|_V^2 + \epsilon y_1^3\right)^2 + 2\sum_{\alpha=1}^{3}\sum_{\nu=4}^{n} x_1^\alpha x_2^\nu y_1^\alpha y_2^\nu \\ &+ \|x_1\|_V^2 \left[\frac{\sqrt{\|y_2\|^2 - (\|x_2\|^2\|y_2\|^2 - \langle x_2,y_2\rangle^2)} + \langle x_2,y_2\rangle}{1-\|x_2\|^2}\right]^2 \Bigg\}^{\frac{1}{2}} \end{aligned} \tag{4.2}$$

for any $x_1 = (x_1^1,x_1^2,x_1^3) \in V$, $x_2 = (x_2^4,...,x_2^n) \in \mathbb{B}$, $y_1 = (y_1^1,y_1^2,y_1^3) \in T_{x_1}V \cong V$ and $.y_2 = (y_2^4,...,y_2^n) \in T_{x_2}\mathbb{B} \cong \mathbb{R}^{n-3}$ such that

$$\|x_2\|^2 \left(\|y_1\|_V^2 + \epsilon y_1^3\right)^2 + \|x_1\|_V^2 \left[\frac{\sqrt{\|y_2\|^2 - (\|x_2\|^2\|y_2\|^2 - \langle x_2,y_2\rangle^2)} + \langle x_2,y_2\rangle}{1-\|x_2\|^2}\right]^2 > -2\sum_{\alpha=1}^{3}\sum_{\nu=4}^{n} x_1^\alpha x_2^\nu y_1^\alpha y_2^\nu.$$

Then the function F defined in (4.2) is Randers convolution metric of $\|y_1\|_V^2 + \epsilon y_1^3$ and $\frac{\sqrt{\|y_2\|^2 - (\|x_2\|^2\|y_2\|^2 - \langle x_2,y_2\rangle^2)} + \langle x_2,y_2\rangle}{1-\|x_2\|^2}$, via $\|x_1\|$ and $\|x_2\|$.

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