Journal of Finsler Geometry and its Applications
Vol. 7, No. 2 (2026), pp 97-110
https://doi.org/10.22098/jfga.2026.17967.1172

Static fluid space time with three dimensional para contact metric manifolds as special component

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Abstract. This paper presents a study on static equations within three-dimensional almost paracontact metric manifolds. We prove that para- cosymplectic manifolds satisfying the static equation are either Ricci flat or gradient of function is collinear with ξ. In three-dimensional para-Kenmotsu manifolds, a non-trivial solution implies that either the scalar curvature is constant, and the manifold is Einstein with sectional curvature -1 or f is constant. Additionally, for trans-para-Sasakian manifolds, such a solution indicates that either the scalar curvature is constant and the manifold is Einstein, or it becomes a $\beta-$para-Kenmotsu manifold.

Keywords: Para-cosymplectic, para-Kenmotsu manifold, trans-para-Sasakian manifold, static equation.

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AMS 2020 Mathematics Subject Classification: 53C25, 53C15, 53D15

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1. Introduction

Static space-times represent an important class of solutions to Einstein's field equations in general relativity, offering valuable insight into the relationship between matter distributions and the geometry of space-time. These solutions are particularly significant in contexts where the gravitational field is time-independent and free of rotational effects, providing simplified yet physically meaningful models for various astrophysical and cosmological scenarios.
A static space-time is typically described by a metric of the form $\hat{g} = -f^2\, dt^2 + g$ where f is a smooth, positive function on a Riemannian manifold, and g is a Riemannian metric on the special hypersurface. This form of the metric ensures that the space-time admits a global hypersurface-orthogonal, timelike Killing vector field, reflecting its static nature.

The Einstein field equations governing such a space-time are given by

$$\mathrm{S}_{\hat{g}} - \frac{1}{2} r_{\hat{g}}\, \hat{g} = T, \tag{1.1}$$

where $S_{\hat{g}}$ and $r_{\hat{g}}$ denote the Ricci tensor and scalar curvature associated with the metric $\hat{g}$, respectively. The tensor T represents the energy-momentum stress tensor of a perfect fluid, expressed as $T = -\mu \frac{d^2 f}{dt^2} - \rho g$ with μ and ρ denoting the smooth energy density and pressure functions of the fluid. This formulation captures the dynamic interplay between matter and geometry, making static space-times a valuable setting for investigating fundamental aspects of gravitational theory. According to Einstein's field equations, the Friedmann–Lemaître–Robertson–Walker (FLRW) models describe a homogeneous universe permeated by a perfect fluid and undergoing accelerated expansion. In this context, static perfect fluid space-times serve as generalizations of static vacuum solutions and hold considerable importance for both physicists and mathematicians.

We now recall the definition of static perfect fluid space-time (see [2, 4, 5]).

Definition 1.1. *A Riemannian manifold (M^n, g) is referred to as the special component of a static perfect fluid space-time if there exist smooth functions $f, \rho : M^n \to R$ with $f > 0$ such that*

$$f\, S^{\circ} = \nabla^{\circ 2} f. \tag{1.2}$$

$$\Delta f = \left(\frac{n-2}{2(n-1)} r + \frac{n}{n-1} \rho \right) f. \tag{1.3}$$

Here $S^{\circ} = S - \frac{r}{n}\, g$, $\quad \nabla^{\circ 2} f = \nabla^2 f - \frac{\Delta f}{n}\, g$, $\quad S^{\circ}$ is the traceless part of the Ricci tensor and $\nabla^{\circ} f$ is the traceless Hessian of f. Additionally, if the manifold has a boundary, it is required that the level set where f coincides with the boundary $f^{-1}(0) = \partial M$.

When $\rho = r = 0$ in equations (1.2) and (1.3), the system reduces to the well-known static vacuum Einstein equations $f\,\mathrm{S} = \nabla f$, $\Delta f = 0$. These equations

characterize static vacuum Einstein spaces, which emerge as special cases of static perfect fluid space-time.

The study of paracontact geometry, which can be viewed as the counterpart of contact geometry in pseudo-Riemannian settings, has seen significant development since the notion of an almost paracontact manifold was first introduced by Sato [7]. Subsequently, Kaneyuki and Williams [6] associated a pseudo-Riemannian metric with an almost paracontact manifold, extending the framework set by Takahashi [9] for contact manifolds equipped with a pseudo-Riemannian structure, particularly in Sasakian geometry. Notably, Zamkovoy [12] demonstrated that any almost paracontact structure admits a compatible pseudo-Riemannian metric of signature $(n+1, n)$.

In recent years, particular attention has been given to the classification and geometric properties of special classes of three-dimensional paracontact manifolds. Three-dimensional cases are especially significant due to their relative computational tractability and their ability to model various physical and geometrical phenomena in pseudo-Riemannian geometry.

The concept of para-cosymplectic manifolds arises naturally within the broader framework of paracontact geometry, which was initiated by Sato in 1976 [7], when he introduced the idea of an almost paracontact manifold.

The concept of para-Kenmotsu manifolds was introduced by Zamkovoy [12] in the early 2000s as part of his systematic study of paracontact geometry. In his work, Zamkovoy extended the classical notion of contact structures to the pseudo-Riemannian setting, thereby formulating para-Kenmotsu manifold as a distinguished subclass of paracontact metric manifolds. These structures can be regarded as the pseudo-Riemannian analogues of the Kenmotsu manifolds, initially established by Kenmotsu in the context of Riemannian geometry.

Zamkovoy [11] introduced a trans-para-Sasakian manifold as a analogue of trans-Sasakian manifold and showed that a trans-para-Sasakian manifold is trans-para-Sasakian structure of type (α, β), where α and β are smooth functions. The trans-para-Sasakian structure is the para-Sasakian, para-Kenmotsu or para-cosymplectic structure in the case of $\alpha = 1$, $\beta = 1$ or $\alpha = \beta = 0$ respectively.

Inspired by the preceding studies, we have undertaken an investigation of static equations on para-cosymplectic manifolds. It was found that in such manifolds, the Ricci tensor vanishes, and a gradient of function is collinear with ξ. Additionally, in para-Kenmotsu manifolds, the scalar curvature is constant and the sectional curvature is equal to -1. In the case of trans-para-Sasakian manifolds, it is established that either the scalar curvature remains constant, or the manifold reduces to a β-para-Kenmotsu manifold. These results contribute to the broader understanding of the interplay between static equations and the geometric structures of paracontact metric manifolds.

2. **Preliminaries**

An almost paracontact structure on a $(2n+1)$-dimensional smooth manifold M is given by a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η satisfying the following conditions:

(i) $\eta(\xi) = 1, \qquad \phi^2 = I - \eta \otimes \xi,$
(ii) the eigen distributions D^+ and D^- of ϕ corresponding to the eigenvalues 1 and -1, respectively, have equal dimension n.

It follows that $\phi \circ \xi = 0$, $\eta \circ \phi = 0$, and $\text{rank}(\phi) = 2n$.If an almost paracontact manifold admits a semi-Riemannian (or compatible) metric g satisfying

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.1}$$

for any vector fields X and Y on M, then the quadruple (ϕ, ξ, η, g) defines an almost paracontact metric structure.

The associated metric g must have signature $(n+1, n)$ and satisfies $\eta = g(\,\cdot\,, \xi)$ and $g(\,\cdot\,, \phi\,\cdot\,) = -g(\phi\,\cdot\,, \cdot\,)$. The fundamental 2-form Φ of an almost paracontact metric structure (ϕ, ξ, η, g) is given by $\Phi(X, Y) = g(X, \phi Y)$, for all vector fields X and Y on M. If $\Phi(X, Y) = d\eta(X, Y)$, then manifold M with this structure (ϕ, ξ, η, g) is called paracontact metric manifold. An almost paracontact metric manifold is said to be normal [13] if

$$N(X, Y) - 2d\eta(X, Y) = 0,$$

where

$$N(X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$$

is the Nijenhuis torsion tensor of ϕ.

In this paper, we investigate para-cosymplectic, para-Kenmotsu, and trans-para-Sasakian structures as the spatial components of static fluid space-times. Although all three are three-dimensional paracontact metric manifolds, each exhibits distinct geometric properties that justify a comparative study. Para-cosymplectic manifolds are characterized by vanishing structure tensors, which lead to zero curvature in certain directions and thus provide a flat geometric model. Para-Kenmotsu manifolds, on the other hand, possess negative constant ϕ-sectional curvature and non-vanishing torsion, giving rise to expanding or contracting flow-like geometries. Trans-para-Sasakian manifolds include both of these as special cases by introducing two functions, α and β, which regulate the deformation of curvature and structural tensors. The detailed formulations of these three structures are presented in Sections 3, 4, and 5.

3. Static fluid space time with three-dimensional para-cosymplectic manifold as special component

A $(2n+1)-$dimensional almost paracontact metric manifold $(M,\, \phi,\, \xi,\, \eta,\, g)$ is called an almost α-paracosymplectic manifold when it fulfills the characteristic conditions associated with almost α-paracosymplectic structures.

$$d\eta = 0, \qquad d\Phi = 2\alpha\, \eta \wedge \Phi, \tag{3.1}$$

where α may be a constant or function on M.

For the specific case when the function $\alpha = 0$ the resulting manifolds are termed almost paracosymplectic manifolds. If, in addition, the normality condition is satisfied, these manifolds are referred to as paracosymplectic manifolds. We also recall that any paracosymplectic manifold satisfies

$$R(X,Y)\xi = 0, \quad (\nabla_X \varphi)Y = 0, \quad \nabla_X \xi = 0, \quad S(X,\xi) = 0, \tag{3.2}$$

where R is the Riemannian curvature tensor and S is Ricci tensor defined by $S(X,Y) = g(QX,Y)$, where Q is Ricci operator on M.

We recall that the curvature tensor of a 3-dimensional semi-Riemannian manifold satisfies

$$\begin{aligned} R(X,Y)Z =& g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y \\ & - \frac{r}{2}\left(g(Y,Z)X - g(X,Z)Y\right). \end{aligned} \tag{3.3}$$

For a three-dimensional paracosymplectic manifold, using first and third terms of (3.2) in (3.3), we have

$$QX = \frac{r}{2}\left(X - \eta(X)\,\xi\right) \tag{3.4}$$

$$S(X,Y) = \frac{r}{2}\left(g(X,Y) - \eta(X)\,\eta(Y)\right), \tag{3.5}$$

for any vector fields X, Y on M.

Using (3.4) in (3.3), we have

$$\begin{aligned} R(X,Y)Z =& \frac{r}{2}\big(g(Y,Z)(X - \eta(X)\xi) - g(X,Z)(Y - \eta(Y)\xi) + \eta(X)\eta(Z)Y \\ & - \eta(Y)\eta(Z)X\big). \end{aligned} \tag{3.6}$$

Taking covariant derivative along Y in (3.4), we obtain

$$(\nabla_Y Q)(X) = \frac{(Yr)}{2}(Y - \eta(Y)\xi). \tag{3.7}$$

Lemma 3.1. *If a smooth, nontrivial function f on a Riemannian manifold (M^n, g) satisfies equation (1.2), then the curvature tensor R can be written as*

$$\begin{aligned} R(X,Y)\,Df =& (Xf)\,QY - (Yf)\,QX + f\left[(\nabla_X Q)Y - (\nabla_Y Q)X\right] \\ & + (X\beta)\,Y - (Y\beta)\,X, \end{aligned} \tag{3.8}$$

where $\beta = -\frac{(r-2\rho)}{2(n-1)}f$.

Proof. Let (M^n, g) be a Riemannian manifold admitting a smooth, nontrivial function f which satisfies (1.2) and (1.3). Then we get

$$\nabla_X Df = f\,QX + \beta\,X. \tag{3.9}$$

Taking covariant derivative along Y in (3.9), we get

$$\nabla_Y \nabla_X Df = (Yf)\,QX + f\,(\nabla_Y Q)X + f\,Q(\nabla_Y X) + (Y\beta)\,X + \beta\,(\nabla_Y X). \tag{3.10}$$

Using repeatedly (3.10) in $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, we get (3.8). □

Theorem 3.2. *If a para-cosymplectic manifold satisfies (1.2) and (1.3), then either manifold is Ricci flat or gradient of function is collinear with ξ.*

Proof. Making use of (3.7) in (3.8) and taking inner product with W, we get

$$\begin{aligned} g(R(X,Y)Df,W) =& f\bigg(\frac{(Xr)}{2}(g(Y,W) - \eta(Y)\eta(W)) - \frac{(Yr)}{2}(g(X,W) \\ & - \eta(X)\eta(W))\bigg) + (Xf)g(QY,W) - (Yf)g(QX,W) \\ & + (X\beta)g(Y,W) - (Y\beta)g(X,W). \end{aligned} \tag{3.11}$$

Taking $Z = Df$ in (3.6) and taking inner products with W, we get

$$\begin{aligned} g(R(X,Y)Df,W) =& \frac{r}{2}((Yf)(g(X,W) - \eta(X)\eta(W)) - (Xf)(g(Y,W) \\ & - \eta(Y)\eta(W)) + g(Y,W)\eta(X)(\xi f) - g(X,W)\eta(Y)(\xi f)). \end{aligned} \tag{3.12}$$

On contracting last equation, we have

$$S(Y,Df) = \frac{r}{2}((Yf) - (\xi f)\eta(Y)). \tag{3.13}$$

On comparing (3.11) and (3.12), we get

$$\begin{aligned} & \frac{r}{2}((Yf)(g(X,W) - \eta(X)\eta(W)) - (Xf)(g(Y,W) - \eta(Y)\eta(W)) \\ & + g(Y,W)\eta(X)(\xi f) - g(X,W)\eta(Y)(\xi f)) \\ & = f\left(\frac{(Xr)}{2}(g(Y,W) - \eta(Y)\eta(W)) - \frac{(Yr)}{2}(g(X,W) - \eta(X)\eta(W))\right) \\ & + (Xf)g(QY,W) - (Yf)g(QX,W) + (X\beta)g(Y,W) - (Y\beta)g(X,W). \end{aligned} \tag{3.14}$$

On contracting foregoing equation with respect to $X = W = e_i$, where e_i, $i = 1$ to 3 is orthonormal frame field , we get

$$\frac{r}{2}((Yf) - (\xi f)\eta(Y)) = \frac{f}{2}((Yr) - (\xi r)\eta(Y)) + g(QY,Df) - (Yf)r - 2(Y\beta). \tag{3.15}$$

Using (3.13) in (3.15), we obtain

$$f(Yr) + r(Yf) + 2(Y\beta) = 0. \tag{3.16}$$

We know that if the scalar curvature r is constant, then last equation becomes

$$(Y\beta) = -\frac{r}{2}(Yf). \tag{3.17}$$

Make use of (3.17) in (3.14) and taking r as a constant, for $X = \xi$ in obtained equation, we get

$$\frac{r}{2}((Yf) - (\xi f)\eta(Y)) = 0. \tag{3.18}$$

There arises two cases either $r = 0$ or $r \neq 0$.

Case(i): If $r = 0$ then from (3.4), we obtain $QX = 0$ i.e., manifold is Ricci flat.

Case(ii): If $r \neq 0$ then (3.18), becomes $Df = (\xi f)\xi$, for all vector fields Y.
Taking covariant derivative aling X in foregoing equation and utilizing third term of (3.2), we obtain

$$\nabla_X Df = X((\xi f))\xi. \tag{3.19}$$

Making use of (3.9) in (3.19), we get

$$fQX + \beta X = X((\xi f))\xi. \tag{3.20}$$

Taking inner product with Y in last equation and contracting obtained equation, we get

$$fr + 3\beta = \xi(\xi f). \tag{3.21}$$

Setting $X = \xi$ in (3.20) and taking inner product with ξ in obtained equation, we get

$$\beta = \xi(\xi f). \tag{3.22}$$

By comparing the last two equations, we obtain

$$f = -\frac{2}{r}\beta.$$

Taking $X = \xi$ in (3.9) and (3.19), we get

$$\nabla_\xi Df = \xi(\xi f)\xi. \tag{3.23}$$

Using (3.22) in (3.21), we obtain

$$\xi(\xi f) = -\frac{fr}{2}. \tag{3.24}$$

Making use of (3.24) in (3.23), we get

$$\nabla_\xi Df = -\frac{fr}{2}\xi.$$

This shows gradient function is collinear with ξ. □

4. Static fluid space time with three-dimensional para-Kenmotsu manifold as special component

If, in equation (3.1), the function $\alpha = 1$, then the manifold is known as an almost para-Kenmotsu manifold. Moreover, if this manifold satisfies the normality condition, it is referred to as a para-Kenmotsu manifold. It is known that in any para-Kenmotsu manifold the following relations hold: [12]

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\,\xi - \eta(Y)\,\varphi X, \quad \nabla_X \xi = X - \eta(X)\,\xi, \tag{4.1}$$

$$R(X,Y)\,\xi = \eta(X)\,Y - \eta(Y)\,X, \quad S(X,\xi) = -(n-1)\,\eta(X). \tag{4.2}$$

If we replace $Y = Z = \xi$ in (3.3) and use (4.2) in the resulting equation, we get

$$QX = \left(\frac{r+2}{2}\right) X - \left(\frac{r+6}{2}\right) \eta(X)\,\xi. \tag{4.3}$$

Making use of (4.3) in (3.3), we get

$$\begin{aligned} R(X,Y)Z = &\left(\frac{r+4}{2}\right)(g(Y,Z)X - g(X,Z)Y) - \left(\frac{r+6}{2}\right)(g(Y,Z)\eta(X)\xi \\ &- g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y). \end{aligned} \tag{4.4}$$

Taking covariant derivative along Y (4.3), we have

$$\begin{aligned} (\nabla_Y Q)(X) = &\frac{(Yr)}{2}X - \frac{(Yr)}{2}\eta(X)\xi - \big(\frac{r+6}{2}big\big)\big(- g(X,Y)\xi \\ &+ 2\eta(X)\eta(Y)\xi - \eta(X)Y\big). \end{aligned} \tag{4.5}$$

Lemma 4.1. [12] *A 3−dimensional para-Kenmotsu manifold is a manifold of constant negative sectional curvature if and only if the scalar curvature $r = -6$.*

Theorem 4.2. *Let (g, f) be a non-trivial solution of static equation on a three-dimensional para-Kenmotsu manifold. Then either scalar curvature r is constant, and manifold is Einstein with sectional curvature -1 or f is constant.*

Proof. Taking inner product with Z and utilizing (4.5) in (3.8), we obtain

$$\begin{aligned} g(R(X,Y)Df, Z) = &\left(\frac{r+2}{2}\right)(Xf)g(Y,Z) - \left(\frac{r+6}{2}\right)(Xf)\eta(Y)\eta(Z) \\ &- \left(\frac{r+2}{2}\right)(Yf)g(X,Z) + \left(\frac{r+6}{2}\right)(Yf)\eta(X)\eta(Z) \\ &+ (X\beta)g(Y,Z) - (Y\beta)g(X,Z) + f(\frac{(Xr)}{2}(g(Y,Z) \\ &- \eta(Y)\eta(Z)) - \frac{(Yr)}{2}(g(X,Z) - \eta(X)\eta(Z)) \\ &+ \frac{r+6}{2}(g(X,Z)\eta(Y) - g(Y,Z)\eta(X))). \end{aligned} \tag{4.6}$$

On contracting last equation, we get

$$-r(Yf) - 2(Y\beta) + \frac{f}{2}(Yr) = 0. \tag{4.7}$$

Setting $Z = Df$ and taking scalar product with Z, we get

$$\begin{aligned} g(R(X,Y)Df,Z) = &\left(\frac{r+4}{2}\right)((Yf)g(X,Z) - (Xf)g(Y,Z)) \\ &- \left(\frac{r+6}{2}\right)((Yf)\eta(X)\eta(Z) - (Xf)\eta(Y)\eta(Z) \\ &+ (\xi f)g(X,Z)\eta(Y) - (\xi f)g(Y,Z)\eta(X)). \end{aligned} \tag{4.8}$$

Taking $X = Z = \xi$ in (4.6) and (4.8), we have

$$g(R(\xi,Y)Df,\xi) = -2(\xi f)\eta(Y) + 2(Yf) + (\xi\beta)\eta(Y) - (Y\beta) \tag{4.9}$$

$$\text{and } g(R(\xi,Y)Df,\xi) = (\xi f)\eta(Y) - (Yf)\eta(X). \tag{4.10}$$

On comparing last two equations, we get

$$-3(\xi f)\eta(Y) - 3(Yf) = (\xi\beta)\eta(Y) - (Y\beta), \tag{4.11}$$

for all vector fields Y. The last equation becomes

$$-3(\xi f)\eta - 3df = (\xi\beta)\eta - d\beta. \tag{4.12}$$

Operating d on both sides of (4.12), and using Poincare Lemma i.e. $(d^2 = 0)$, we have

$$((\xi\beta) + 3(\xi f))\eta \wedge d\eta = 0. \tag{4.13}$$

Since $\eta \wedge d\eta \neq 0$, we have $(\xi\beta) = -3(\xi f)$. Using this in (4.11), we get

$$(Y\beta) = 3(Yf). \tag{4.14}$$

Utilizing last equation in (4.6), we get

$$\begin{aligned} g(R(X,Y)Df,Z) = &\left(\frac{r+8}{2}\right)(Xf)g(Y,Z) - \left(\frac{r+6}{2}\right)(Xf)\eta(Y)\eta(Z) \\ &- \left(\frac{r+8}{2}\right)(Yf)g(X,Z) + \left(\frac{r+6}{2}\right)(Yf)\eta(X)\eta(Z) \\ &+ f(\frac{(Xr)}{2}(g(Y,Z) - \eta(Y)\eta(Z)) - \frac{(Yr)}{2}(g(X,Z) \\ &- \eta(X)\eta(Z)) + (\frac{r+6}{2})(g(X,Z)\eta(Y) - g(Y,Z)\eta(X))). \end{aligned} \tag{4.15}$$

On contracting (4.15), we get

$$S(Y,Df) = -\left(\frac{r+10}{2}\right)(Yf) - \left(\frac{r+6}{2}\right)(\xi f)\eta(Y) - \frac{f}{2}(Yr). \tag{4.16}$$

On comparing (4.16) and (4.3), we have

$$(r+6)(Yf) = -\frac{f}{2}(Yr). \tag{4.17}$$

Using (4.14) in (4.7), we have

$$(r+6)(Yf) = \frac{f}{2}(Yr). \tag{4.18}$$

On comparing (4.17) and (4.18), we get

$$(r+6)(Yf) = 0. \tag{4.19}$$

There arises two cases either $r = -6$ or $(Yf) = 0$.

Case(i) : If $r = -6$ in (4.4) and (4.3), we get

$$R(X,Y)Z = -(g(Y,Z)X - g(X,Z)Y),$$

from which we have sectional curvature$= -1$ and $QX = -2X$.

Case(ii) : If $(Yf) = 0$, then $f =$ constant. Using this in (3.9), we have $f\,QX = \beta\,X$. □

Example 4.3. *We consider the 3-dimensional manifold* $M = \{(x,y,z) \in R^3 \ / z \neq 0\}$, (x,y,z) *are the standard co-ordinates in* R^3. *Let* e_1, e_2 *and* e_3 *be the vector fields on* M *given by*

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},$$

where e_1, e_2 *and* e_3 *are linearly independent at each point of* M.

Let g *be a semi-Riemannian metric defined by*

$$g(e_1,e_1) = g(e_3,e_3) = 1, \quad g(e_2,e_2) = -1, \quad g(e_i,e_j) = 0, \quad i \neq j \in \{1,2,3\}.$$

Let η *be a 1-form on* M *defined by* $\eta(X) = g(X,e_3)$, *for all* X *on* M *and* $e_3 = \xi$. *Let* ϕ *be a* $(1,1)$ *tensor field on* M *defined by*

$$\phi e_2 = e_1, \quad \phi e_1 = e_2, \quad \phi e_3 = 0.$$

Using linearity property of ϕ *and* g, *we have*

$$\phi^2 X = X - \eta(X)\xi, \quad g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y).$$

Thus the structure (ϕ,ξ,η,g) *defines an almost paracontact structure on* M. *Let* ∇ *be the Levi-Civita-connection with respect to the metric* g. *Then we have*

$$[e_1,e_2] = 0, \ [e_1,e_3] = e_1, \ [e_2,e_3] = e_2.$$

The Riemannian connection ∇ *with respect to the metric* g *is given by*

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z) \\ & -g([Y,Z],X) + g([Z,X],Y). \end{aligned}$$

By above Koszul's formula we can easily calculate

$$\nabla_{e_1}e_1 = -e_3, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_3 = e_1,$$
$$\nabla_{e_2}e_1 = 0, \quad \nabla_{e_2}e_2 = e_3, \quad \nabla_{e_2}e_3 = e_2,$$
$$\nabla_{e_3}e_1 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_3 = 0.$$

The equation shows that $\nabla_X\xi$ *and* $(\nabla_X\phi)(Y)$ *satisfies equations of (4.1). Then* M *is a para-Kenmotsu manifold. With the help of above we can easily calculate the following*

$$R(e_1,e_2)e_1 = e_2, \quad R(e_1,e_2)e_2 = e_1, \quad R(e_1,e_2)e_3 = 0,$$
$$R(e_1,e_3)e_1 = e_3, \quad R(e_1,e_3)e_2 = 0, \quad R(e_1,e_3)e_3 = -e_1,$$
$$R(e_2,e_3)e_1 = 0, \quad R(e_2,e_3)e_2 = -e_3, \quad R(e_2,e_3)e_3 = -e_2.$$

Let

$$Df = (e_1f)e_1 - (e_2f)e_2 + (e_3f)e_3.$$

From this we obtain the following:

$$\nabla_{e_1}Df = (e_1(e_1f) + (e_3f))e_1 + (e_1(e_3f) - (e_3f))e_3 - e_1(e_2f)e_2.,$$
$$\nabla_{e_2}Df = e_2(e_1f)e_1 - (e_2(e_2f) - (e_3f))e_2 + (e_2(e_3f) - (e_2f))e_3,$$
$$\nabla_{e_2}Df = e_3(e_1f)e_1 - e_3(e_2f)e_2 + e_3(e_3f)e_3.$$

Consider

$$\nabla_X Df = fQX + \beta X.$$

Using (4.3) in last equation, we get

$$\nabla_X Df = f\left(\frac{r+2}{2}X - \frac{r+6}{2}\eta(X)\xi\right) + \beta X.$$

$\nabla_{e_1}Df = \left(f(\frac{r+2}{2}) + \beta\right)e_1$, $\nabla_{e_2}Df = \left(f(\frac{r+2}{2}) + \beta\right)e_2$, $\nabla_{e_3}Df = (\beta - 2f)\,e_3$.

From these equations, we arrive at

$$e_1(e_1f) + (e_3f) = f(\tfrac{r+2}{2}) + \beta,\ e_2(e_2f) + (e_3f) = -f(\tfrac{r+2}{2}) - \beta,$$
$$e_3(e_3f) = \beta - 2f.$$

Adding all the above equations, we get

$$rf = -3\beta, \qquad \beta = 2f.$$

Using this in last equation, we get $r = -6$*, which verifies the main result.*

5. Static perfect fluid space time on three-dimensional trans-para-Sasakian manifold as special component

A $(2n+1)-$dimensional trans-para-Sasakian manifold is trans-para-Sasakian if and only if [11]

$$(\nabla_X\phi)(Y) = \alpha(-g(X,Y)\xi + \eta(Y)X) + \beta(g(X,\phi Y)\xi + \eta(Y)X), \tag{5.1}$$

where α and β are constants. From (5.1), we have

$$\nabla_X\xi = -\alpha\phi X - \beta(X - \eta(X)\xi), \tag{5.2}$$

for all vector fields X and Y on M.

In a $(2n+1)-$dimensional trans-para-Sasakian manifold, the following holds for constants α and β [8] :

$$(\nabla_X \eta)(Y) = \alpha g(X, \phi Y) - \beta(g(X, Y) - \eta(X)\eta(Y)) \tag{5.3}$$

$$R(\xi, X)Y = -(\alpha^2 + \beta^2)(g(X, Y)\xi - \eta(Y)X) \tag{5.4}$$

$$Q\xi = -2n(\alpha^2 + \beta^2)\xi. \tag{5.5}$$

If we replace $Y = Z = \xi$ in (3.3) and use (5.4) and(5.5), we get

$$QX = \left(\frac{r}{2} + (\alpha^2 + \beta^2)\right) X - \left(\frac{r}{2} + 3(\alpha^2 + \beta^2)\right) \eta(X)\xi. \tag{5.6}$$

Utilizing (5.6) in (3.3), we obtain

$$\begin{aligned} R(X, Y)Z = &\left(\frac{r}{2} + 2(\alpha^2 + \beta^2)\right)(g(Y, Z)X - g(X, Z)Y) \\ &- \left(\frac{r}{2} + 3(\alpha^2 + \beta^2)\right)(g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi) \\ &+ \left(\frac{r}{2} + 3(\alpha^2 + \beta^2)\right)(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X). \end{aligned} \tag{5.7}$$

Taking covariant derivative along Y in (5.6), we get

$$\begin{aligned} (\nabla_Y Q)(X) = &\frac{(Yr)}{2}(X - \eta(X)\xi) - \left(\frac{r}{2} + 3(\alpha^2 + \beta^2)\right)\Big(\alpha g(\phi X, Y)\xi - \beta g(X, Y)\xi \\ &+ 2\beta\eta(X)\eta(Y)\xi - \alpha\eta(X)\phi Y - \beta\eta(X)Y\Big). \end{aligned} \tag{5.8}$$

Theorem 5.1. *If a three-dimensional trans-para-Sasakian manifold admits a non-trivial static solution (g, f), then either it is Einstein with constant scalar curvature, or it reduces to a $\beta-$para-Kenmotsu manifold.*

Proof. Taking inner product of (3.8) with Z and using (5.8) in (3.8), we obtain

$$\begin{aligned} g(R(X, Y)Df, Z) = &(Xf)\{\left(\frac{r}{2} + (\alpha^2 + \beta^2)\right) g(Y, Z) - \left(\frac{r}{2} - 3(\alpha^2 + \beta^2)\right) \\ &\eta(Y)\eta(Z)\} - (Yf)\{\left(\frac{r}{2} + (\alpha^2 + \beta^2)\right) g(X, Z) \\ &- \left(\frac{r}{2} + 3(\alpha^2 + \beta^2)\right)\eta(X)\eta(Z)\} + f\{\frac{(Xr)}{2}(g(Y, Z) \\ &- \eta(Y)\eta(Z)) - \left(\frac{r}{2} + 3(\alpha^2 + \beta^2)\right)(\alpha g(\phi Y, X)\eta(Z) \\ &- \beta g(X, Y)\eta(Z) + 2\beta\eta(X)\eta(Y)\eta(Z) - \alpha g(\phi X, Z)\eta(Y) \\ &- \beta g(X, Z)\eta(Y)) - \frac{(Yr)}{2}(g(X, Z) - \eta(X)\eta(Z)) \\ &- \left(\frac{r}{2} + 3(\alpha^2 + \beta^2)\right)(\alpha g(\phi X, Y)\eta(Z) - \beta g(X, Y)\eta(Z) \\ &+ 2\beta\eta(X)\eta(Y)\eta(Z) - \alpha g(\phi Y, Z)\eta(X) - \beta g(Y, Z)\eta(X))\} \\ &+ (X\beta)g(Y, Z) - (Y\beta)g(X, Z). \end{aligned} \tag{5.9}$$

Setting $X = \xi$ in (5.9), we get

$$g(R(\xi,Y)Df,Z) = (\xi f)\{\Big(\frac{r}{2}+(\alpha^2+\beta^2)\Big)g(Y,Z) - \Big(\frac{r}{2}+3(\alpha^2+\beta^2)\Big)\eta(Y)\eta(Z)\}$$
$$+ (\xi\beta)g(Y,Z) - (Y\beta)\eta(Z) + f\{\frac{(\xi r)}{2}(g(Y,Z)-\eta(Y)\eta(Z))$$
$$- \Big(\frac{r}{2}+3(\alpha^2+\beta^2)\Big)(\beta\eta(Y)\eta(Z) - \alpha g(\phi Y,Z) - \beta g(Y,Z))\}. \tag{5.10}$$

From (5.4), we have

$$g(R(\xi,Y)Df,Z) = (\alpha^2+\beta^2)((\xi f)g(Y.Z) - (Yf)\eta(Z)). \tag{5.11}$$

On comparing (5.10) and (5.11), we have

$$(\xi f)\{\Big(\frac{r}{2}+(\alpha^2+\beta^2)\Big)g(Y,Z) - \Big(\frac{r}{2}+3(\alpha^2+\beta^2)\Big)\eta(Y)\eta(Z)\} + (\xi\beta)g(Y,Z)$$
$$- (Y\beta)\eta(Z) + f\{\frac{(\xi r)}{2}(g(Y,Z)-\eta(Y)\eta(Z)) - \Big(\frac{r}{2}+3(\alpha^2+\beta^2)\Big)(\beta\eta(Y)\eta(Z)$$
$$- \alpha g(\phi Y,Z) - \beta g(Y,Z))\} = (\alpha^2+\beta^2)((\xi f)g(Y.Z) - (Yf)\eta(Z)). \tag{5.12}$$

Interchanging Y and Z in last equation, and subtracting the obtained equation with foregoing equation, we obtain

$$(\alpha^2+\beta^2)((Zf)\eta(Y) - (Yf)\eta(Z)) + (Y\beta)\eta(Z) - (Z\beta)\eta(Y)$$
$$- 2f\alpha\Big(\frac{r}{2}+3(\alpha^2+\beta^2)\Big)g(\phi Z,Y) = 0. \tag{5.13}$$

Taking $Y = \xi$ in (5.13), we get

$$(\alpha^2+\beta^2)((Zf) - (\xi f)\eta(Z)) + (\xi\beta)\eta(Z) - (Z\beta) = 0, \tag{5.14}$$

for all vector fields Z. The last equation becomes

$$(\alpha^2+\beta^2)(df - (\xi f)\eta) + (\xi\beta)\eta - d\beta = 0. \tag{5.15}$$

Taking d on both sides of last equation, taking wedge product and using Poincare lemma, we obtain

$$\Big((\xi\beta) - (\alpha^2+\beta^2)(\xi f)\Big)\eta\wedge d\eta = 0.$$

Since $\eta\wedge d\eta \neq 0$, we get

$$(\xi\beta) = (\alpha^2+\beta^2)(\xi f). \tag{5.16}$$

Using (5.16) in (5.15), we have

$$(Z\beta) = (\alpha^2+\beta^2)(Zf). \tag{5.17}$$

Utilizing (5.16) and (5.15) in (5.13), we obtain

$$f\alpha\Big(\frac{r}{2}+3(\alpha^2+\beta^2)\Big) = 0.$$

Since $f \neq 0$, we have either $\alpha = 0$ or $\frac{r}{2} + 3(\alpha^2 + \beta^2) = 0$. If $\alpha = 0$ then from (5.2), the manifold becomes β-para-Kenmotsu manifold. In other case, from (5.7), we get $QX = -2(\alpha^2 + \beta^2)X$. □

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Received: 03.08.2025
Accepted: 11.04.2026