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η-Ricci solitons on contact pseudo-metric manifolds

Eftekhar Asgharzadeh[a] and Morteza Faghfouri[a*]

[a]Department of Pure Mathematics,
Faculty of Mathematical Sciences,
University of Tabriz, Tabriz, Iran.

E-mail: ef.asgharzadeh@gmail.com
E-mail: faghfouri@tabrizu.ac.ir

Abstract. In this paper, we investigate the geometry of contact pseudo-metric manifolds admitting an $\eta-$Ricci soliton. We establish that a Sasakian pseudo-metric manifold admitting an η-Ricci soliton is necessarily an η-Einstein manifold. Furthermore, if the potential vector field of the soliton is not Killing, then the manifold is $\mathcal{D}$-homothetically fixed, and the vector field preserves the structure tensor field. We also prove that a K-contact pseudo-metric manifold endowed with a gradient η-Ricci soliton metric is η-Einstein. In addition, we examine contact pseudo-metric manifolds admitting an η-Ricci soliton whose potential vector field is pointwise colinear with the Reeb vector field. Finally, we analyze gradient η-Ricci solitons on (κ, μ)-contact pseudo-metric manifolds, providing new insights into their structure and curvature properties.

Keywords: Ricci soliton, η-Ricci soliton, (κ, μ)-contact pseudo-metric manifolds, gradient η-Ricci solitons.

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*Corresponding Author
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1. **Introduction**

A Ricci soliton is a natural generalization of an Einstein metric, which was introduced by Hamilton [19] as the fixed point of the Hamilton's Ricci flow $\frac{\partial}{\partial t}g = -2\,\mathrm{Ric}$. The Ricci flow is a nonlinear diffusion equation analogue of the heat equation for metrics. A Ricci soliton (g, V, λ) on the pseudo-Riemannian manifold (M, g) is defined by the following equation

$$\pounds_V g + 2\,\mathrm{Ric} + 2\lambda g = 0,$$

where $\pounds_V$ is the Lie derivative along the potential vector field V, and λ is a constant real number. The Ricci soliton is called shrinking, steady, expanding if $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$, respectively. If $V = Df$, where Df is the gradient of the smooth function f, then the Ricci soliton is called a gradient Ricci soliton. The Ricci solitons have been studied in many different contexts (see [2–4, 11, 12, 15, 18, 24, 28]). They are also of interest to physicists because of their relations to string theory [1, 22], and physicists refer to Ricci solitons as quasi-Einstein metrics [14].

The η-Ricci soliton notion, as a generalization of a Ricci soliton, was introduced by Cho and Kimura [10]. An η-Ricci soliton on a manifold M is a tuple (g, V, λ, μ), where g is a pseudo-Riemannian metric, V is the potential vector field, and λ, μ are constant real numbers satisfying

$$\pounds_V g + 2\,\mathrm{Ric} + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.1}$$

where η is a 1-form on M. Notice that when $\mu = 0$ then an η-Ricci soliton reduces to a Ricci soliton. Moreover, if $V = Df$, the η-Ricci soliton is called a gradient η-Ricci soliton and Eq.(1.1) becomes

$$\mathrm{Hess}\, f + \mathrm{Ric} + \lambda g + \mu\eta \otimes \eta = 0. \tag{1.2}$$

The η-Ricci solitons have been studied in many different settings, Blaga studied η-Ricci solitons on para-Kenmotsu [5] and Lorentzian para-Sasakian manifolds [6]. Devaraja and Venkatesha studied η-Ricci solitons on para-Sasakian manifolds [25], etc.

Contact geometry is an odd-dimensional analogue of the symplectic geometry and has been studied in many different contexts (particularly) those related to physics. It has been used as a proper framework for classical thermodynamics [7,27], and as a geometrical approach to magnetic field [8]. Also, it was studied in relation with the Yang-Mills theory [21], quantum mechanics [20], gravitational waves [23], etc. Studying contact structures with pseudo-Riemannian metrics was started by Takahashi in [29], but he just studied the Sasakian case. Recently, Calvaruso and Perrone [9] have studied a contact pseudo-metric manifold in the general case. Ghaffarzadeh and second author studied nullity conditions on the contact pseudo-metric manifolds and have introduced the "(κ, μ)-contact pseudo-metric manifold" notion [16, 17]. The relevance for the general relativity of contact pseudo-metric manifolds was studied in [13]. In view of the above applications and the lack, to the best of the authors' knowledge, of a comprehensive study concerning Ricci solitons with arbitrary potential vector fields on contact pseudo-metric manifolds, this work is devoted to the investigation of Ricci solitons in the contact pseudo-metric setting. Moreover, as Ricci solitons arise as particular cases of η-Ricci solitons, we naturally extend our study to the more general framework of η-Ricci solitons on contact pseudo-metric manifolds.

The present paper has been organized as follows. In Section 2, we recalled the contact pseudo-metric manifold notion and proved some lemmas that are used in the next sections. In Section 3, we studied η-Ricci solitons on Sasakian pseudo-metric manifolds and showed that a Sasakian pseudo-metric manifold, which admits an η-Ricci soliton, is an η-Einstein manifold and if the potential vector field of the η-Ricci soliton is not a Killing vector field, then the manifold is $\mathcal{D}$-homothetically fixed, and presented an example for it. Moreover, we showed a K-contact pseudo-metric manifold which admits a gradient η-Ricci soliton is an η-Einstein manifold. Also, we studied an η-Ricci soliton that has a potential vector field colinear to the Reeb vector field on a contact pseudo-metric manifold and showed that the manifold is K-contact. In the last section, we studied gradient η-Ricci solitons on a (κ, μ)-contact pseudo-metric manifold and obtained some conditions on the curvature tensor of the manifold.

2. Preliminaries

In this section, we recall some definitions and results needed in the rest of the paper.

A $(2n+1)$-dimensional manifold M is called an almost contact pseudo-metric manifold, if there exists an almost contact pseudo-metric structure (φ, ξ, η, g) on M, where φ, ξ, η, g are a $(1,1)$-tensor field, a vector field, a 1-form and a compatible pseudo-Riemannian metric, respectively, which satisfy the following equations

$$\eta(\xi) = 1, \quad \varphi^2(X) = -X + \eta(X)\xi, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.2}$$

where $\epsilon = \pm 1$, and X, Y are arbitrary vector fields. Using the above equations, we have

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0,$$
$$\eta(X) = \epsilon g(\xi, X), \quad g(\varphi X, Y) = -g(X, \varphi Y)$$

and especially $g(\xi, \xi) = \epsilon$.

Note that the signature of the metric g depends on whether the vector field ξ is spacelike or timelike. If ξ is spacelike, that is, $g(\xi, \xi) > 0$, then the signature of g is $(2p+1,\, 2n-2p)$. If ξ is timelike, meaning that $g(\xi,\xi)<0$, then the signature of g is $(2p,\, 2n-2p+1)$.

The fundamental 2-form Φ of an almost contact pseudo-metric manifold (M,φ,ξ,η,g) is defined as $\Phi(X,Y)=g(X,\varphi Y)$, where $X,Y\in\Gamma(M)$. If

$$g(X,\varphi Y)=d\eta(X,Y), \tag{2.3}$$

then η is a contact form, (φ,ξ,η,g) is a contact pseudo-metric structure and M is called a contact pseudo-metric manifold.

Throughout this paper, we use $R(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}$, where $X,Y\in\Gamma(M)$, as the Riemannian curvature tensor definition. In a contact pseudo-metric manifold (M,φ,ξ,η,g) the $(1,1)$-tensor field ℓ and h are defined by

$$\ell X=R(X,\xi)\xi,\quad hX=\frac{1}{2}(\pounds_\xi\varphi)X.$$

Also, notice the ℓ and h are self-adjoint operators. In the contact pseudo-metric manifold (M,φ,ξ,η,g), we have the following equations [9, 26]

$$\operatorname{trace}(h)=\operatorname{trace}(h\varphi)=0, \tag{2.4}$$

$$\eta\circ h=0,\quad \ell\xi=0, \tag{2.5}$$

$$h\varphi=-\varphi h,\quad h\xi=0, \tag{2.6}$$

$$\nabla_\xi\varphi=0, \tag{2.7}$$

$$\nabla_X\xi=-\epsilon\varphi X-\varphi hX, \tag{2.8}$$

$$\operatorname{Ric}(\xi,\xi)=2n-trh^2, \tag{2.9}$$

where X is an arbitrary vector field.

A contact pseudo-metric manifold (M,φ,ξ,η,g) is a K-contact pseudo-metric manifold if ξ is a Killing vector field or equivalently $h=0$. So, we have the following equations

$$Q\xi=2n\epsilon\xi, \tag{2.10}$$

$$\nabla_X\xi=-\epsilon\varphi X, \tag{2.11}$$

where Q is the Ricci operator of the metric g and $X\in\Gamma(M)$.

Lemma 2.1. *Let (M,φ,ξ,η,g) be a $(2n+1)$-dimensional K-contact pseudo-metric manifold, then*

$$(\nabla_X Q)\xi=-2n\varphi X+\epsilon Q\varphi X, \tag{2.12}$$

$$(\nabla_\xi Q)X=\epsilon(Q\varphi-\varphi Q)X, \tag{2.13}$$

where X is an arbitrary vector field.

Proof. First, differentiating (2.10) along an arbitrary vector field X and using (2.11), we obtain (2.12). Next, taking the Lie derivative of the Ricci tensor Ric along ξ, we have

$$(\mathcal{L}_\xi\operatorname{Ric})(X,Y)=g\big((\nabla_\xi Q)X+Q(\nabla_X\xi),\,Y\big)+g\big(QX,\,\nabla_Y\xi\big),$$

for all $X, Y \in \Gamma(M)$. Since ξ is a Killing vector field, it follows that $\mathcal{L}_\xi \mathrm{Ric} = 0$. Using this fact together with (2.11) in the above expression yields (2.13), which completes the proof. □

An almost contact pseudo-metric structure (φ, ξ, η, g) is called normal if $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$. A normal contact pseudo-metric manifold is a Sasakian pseudo-metric manifold. A Sasakian pseudo-metric manifold is a K-contact pseudo-metric manifold, satisfying

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \epsilon\eta(Y)X, \tag{2.14}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.15}$$

where $X, Y \in \Gamma(M)$.

Lemma 2.2. *Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian pseudo-metric manifold then $Q\varphi = \varphi Q$.*

Proof. First, using equation (2.14) to compute the curvature tensor, we obtain

$$\begin{aligned} R(X,Y,\varphi Z,W) + R(X,Y,Z,\varphi W) = \\ &\epsilon g(Z,\varphi Y)g(X,W) - \epsilon g(Z,\varphi X)g(Y,W) \\ &- \epsilon g(X,Z)g(\varphi Y,W) + \epsilon g(Y,Z)g(\varphi X,W), \end{aligned}$$

where $X, Y, Z, W \in \Gamma(M)$ and

$$R(X,Y,Z,W) = g(R(X,Y,Z),W).$$

Now, let X, Y, Z, W be orthogonal to ξ. Then, from the above relation, we obtain

$$R(X,Y,Z,W) = R(\varphi X,\varphi Y,\varphi Z,\varphi W),$$

This identity implies

$$\mathrm{Ric}(X,\varphi Y) + \mathrm{Ric}(\varphi X,Y) = 0,$$

for all vector fields X, Y orthogonal to ξ. Using this last equation, we conclude that $Q\varphi = \varphi Q$, which completes the proof. □

A contact pseudo-metric manifold $(M, \varphi, \xi, \eta, g)$ is called an η-Einstein manifold if the Ricci curvature is of the form $\mathrm{Ric} = ag + b\eta \otimes \eta$, where a, b are smooth functions on the manifold M. If the manifold M is a K-contact pseudo-metric manifold with dimension greater than three, then a, b are constants.

Let $(M, \varphi, \xi, \eta, g)$ be a contact pseudo-metric manifold. For any nonzero real constant $t \neq 0$, is defined a new contact pseudo-metric manifold $(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$, where $\tilde{\eta} = t\eta$, $\tilde{\xi} = \frac{1}{t}\xi$, $\tilde{\varphi} = \varphi$ and $\tilde{g} = tg + \epsilon t(t-1)\eta \otimes \eta$ [9].

This transition is called a $\mathcal{D}$-homothetic deformation. It preserves several fundamental properties of the structure, in particular the K-contact and

Sasakian conditions. Furthermore if (M,φ,ξ,η,g) be a K-contact pseudo-metric manifold, under a $\mathcal{D}$-homothetic deformation we have

$$\tilde{\mathrm{Ric}} = \mathrm{Ric} - 2\epsilon(t-1)g + 2(t-1)(nt+n+1)\eta\otimes\eta, \tag{2.16}$$

where Ric and $\tilde{Ric}$ are Ricci tensors of (M,φ,ξ,η,g) and $(M,\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ respectivly [9].

Proposition 2.3. *Let (M,φ,ξ,η,g) be an η-Einstein K-contact pseudo-metric manifold of dimension $2n+1$ such that $\mathrm{Ric} = ag + b\eta\otimes\eta$, and let t be a nonzero real number. Under a $\mathcal{D}$-homothetic deformation, the Ricci tensor of the deformed structure satisfies*

$$\tilde{\mathrm{Ric}} = \tilde{a}\,\tilde{g} + (2n - \epsilon\tilde{a})\tilde{\eta}\otimes\tilde{\eta},$$

where $\tilde{a} = \left(\frac{a-2\epsilon t+2\epsilon}{t}\right)$.

Proof. First using $\tilde{\eta} = t\eta$ in $\tilde{g} = tg + \epsilon t(t-1)\eta\otimes\eta$, we have

$$g = \frac{1}{t}\tilde{g} - \frac{1}{t^2}\epsilon(t-1)\tilde{\eta}\otimes\tilde{\eta}. \tag{2.17}$$

Now putting $\tilde{\eta} = t\eta$ and (2.17) in (2.16) gives

$$\tilde{\mathrm{Ric}} = \left(\frac{a-2\epsilon t+2\epsilon}{t}\right)\tilde{g} + \left(2n + \frac{2t-2-a\epsilon}{t}\right)\tilde{\eta}\otimes\tilde{\eta},$$

and it proves the result. □

Notice in the previous proposition, when $a = -2\epsilon$, the form of the Ricci tensor remains unchanged under the $\mathcal{D}$-homothetic deformation. Thus, we have the following definition.

Definition. An η-Einstein K-contact pseudo-metric manifold with $a = -2\epsilon$ is said to be $\mathcal{D}$-homothetically fixed.

3. η-Ricci solitons on Sasakian pseudo-metric manifolds

In this section, we have studied η-Ricci solitons on Sasakian pseudo-metric manifolds.

Theorem 3.1. *Let (M,φ,ξ,η,g) be a $(2n+1)$-dimensional Sasakian pseudo-metric manifold. If (g,V,λ,μ) be an η-Ricci soliton on the manifold M, then M is an η-Einstein manifold. Moreover, the Ricci tensor and the scalar curvature of g are given by*

$$\mathrm{Ric} = (n\epsilon + \frac{\mu\epsilon-\lambda}{2})g + (\frac{n}{2}(\epsilon+1) + \frac{\lambda}{4}(\epsilon+1) + \frac{(\epsilon-3)}{4}\mu)\eta\otimes\eta, \tag{3.1}$$

$$r = \frac{1}{4}\epsilon\left(\lambda - \mu + 8n^2 + (4\mu+6)n\right) + \frac{1}{4}(-\lambda + \mu - 4\lambda n + 2n), \tag{3.2}$$

where Ric and r are the Ricci tensor and the scalar curvature of the metric g, respectively.

Proof. Applying equation (1.1) to the following formula given in [30, p. 23],

$$(\pounds_V\nabla_x g - \nabla_X\pounds_V g - \nabla_{[V,X]}g)(Y,Z) = -g((\pounds_V\nabla)(X,Y),Z) - g((\pounds_V\nabla)(X,Z),Y),$$

where X, Y and Z are arbitrary vector fields, we find

$$\begin{aligned} g((\pounds_V\nabla)(X,Y),Z) =& (\nabla_Z(\mathrm{Ric}+\mu\eta\otimes\eta))(X,Y) \\ &-(\nabla_X(\mathrm{Ric}+\mu\eta\otimes\eta))(Y,Z) \\ &-(\nabla_Y(\mathrm{Ric}+\mu\eta\otimes\eta))(Z,X). \end{aligned} \tag{3.3}$$

Using Lemma 2.1 and Lemma 2.2, we obtain

$$\nabla_\xi Q = 0. \tag{3.4}$$

Substituting $Y = \xi$ in (3.3) and using the above relation as well as Lemma 2.1, we obtain

$$(\pounds_V\nabla)(X,\xi) = (4n+2\mu)\varphi X - 2\epsilon Q\varphi X. \tag{3.5}$$

Differentiating (3.5) along an arbitrary vector field Y and using (2.14) yield

$$\begin{aligned} (\nabla_Y\pounds\nabla)(X,\xi) - \epsilon(\pounds_V\nabla)(X,\varphi Y) = & \, 2\mu g(X,Y)\xi - (4n+2\mu)\epsilon\eta(X)Y \\ & - 2\epsilon(\nabla_Y Q)(\varphi X) + 2\eta(X)QY. \end{aligned} \tag{3.6}$$

Using (3.6) in the following commutation formula (see [30]),

$$(\pounds_V R)(X,Y)Z = (\nabla_X\pounds_V\nabla)(Y,Z) - (\nabla_Y\pounds_V\nabla)(X,Z), \tag{3.7}$$

where $X, Y, Z \in \Gamma(M)$ are arbitrary vector fields, we obtain:

$$\begin{aligned} (\pounds_V R)(X,Y) = & \, 2\epsilon(\nabla_Y Q)\varphi X - 2\epsilon(\nabla_X Q)\varphi Y \\ & + 2\eta(Y)QX - 2\eta(X)QY \\ & + (4n+2\mu)\epsilon\eta(X)Y - (4n+2\mu)\epsilon\eta(Y)X \\ & + \epsilon(\pounds_V\nabla)(Y,\varphi X) - \epsilon(\pounds_V\nabla)(X,\varphi Y). \end{aligned} \tag{3.8}$$

Substituting ξ for Y in (3.8) and using (3.5), we obtain

$$(\pounds_V R)(X,\xi)\xi = 4QX - 4\epsilon(2n+\mu)X + 4\mu\epsilon\eta(X)\xi, \qquad X \in \Gamma(M). \tag{3.9}$$

Using (1.1), we have

$$(\pounds_V g)(X,\xi) + (4n + 2\lambda\epsilon + 2\mu)\eta(X) = 0, \qquad X \in \Gamma(M),$$

and this equation yields

$$\epsilon(\pounds_V\eta)(X) - g(X,\pounds_V\xi) + 2(2n+\lambda\epsilon+\mu)\eta(X) = 0, \tag{3.10}$$

$$\eta(\pounds_V\xi) = (2n\epsilon + \mu\epsilon + \lambda), \tag{3.11}$$

where X is an arbitrary vector field. Next Lie-differentiating the formula

$$R(X,\xi)\xi = X - \eta(X)\xi$$

along the vector field V and using (3.10), (3.11) and (2.15), we have

$$QX = \frac{(\epsilon-1)}{4}\left((\pounds_V\eta)X\right)\xi + \left(n\epsilon + \frac{\mu\epsilon-\lambda}{2}\right)X + \left(n + \frac{\lambda\epsilon+(1-2\epsilon)\mu}{2}\right)\eta(X)\xi, \tag{3.12}$$

where $X \in \Gamma(M)$. Now using the foregoing equation and symmetry of the Ricci tensor, we deduce

$$\frac{\epsilon(\epsilon-1)}{4}(\pounds_V\eta)X\eta(Y) = \frac{\epsilon(\epsilon-1)}{4}(\pounds_V\eta)Y\eta(X) \qquad X, Y \in \Gamma(X).$$

Using the above equation and (3.11), we find (3.1), and in turn it yields (3.2), completing the proof. □

Theorem 3.1 imposes a strong condition on the potential vector field of an η-Ricci soliton on a Sasakian pseudo-metric manifold. The following lemma is needed for further study.

Lemma 3.1. *Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional contact pseudo-metric manifold. If $\mathrm{Ric} = ag + b\eta\otimes\eta$, where $a, b \in \mathbf{R}$, then*

$$\mathrm{Ric}^{ij}\,\mathrm{Ric}_{ij} + \lambda r + \mu(\epsilon a + b) = 0, \tag{3.13}$$

where r is the scalar curvature of the metric g.

Proof. Using equation (1.1) in the following formula (see [30])

$$\pounds_V\Gamma^h_{ij} = \frac{1}{2}g^{ht}\left(\nabla_j(\pounds_V g_{it}) + \nabla_i(\pounds_V g_{jt}) - \nabla_t(\pounds_V g_{ij})\right),$$

where Γ^h_{ij} denote the Christoffel symbols of the metric g, we obtain

$$\pounds_V\Gamma^h_{ij} = \nabla^h(\mathrm{Ric}_{ij} + \mu\eta_i\eta_j) - \nabla_i(\mathrm{Ric}^h_j + \mu\eta_j\eta^h) - \nabla_j(\mathrm{Ric}^h_i + \mu\eta_i\eta^h).$$

Next, substituting this expression into the following identity (see [30])

$$\pounds_V R^h_{kji} = \nabla_k(\pounds_V\Gamma^h_{ij}) - \nabla_j(\pounds_V\Gamma^h_{ki}),$$

we obtain

$$\begin{aligned}\pounds_V R^h_{kji} = {} & \nabla_k\nabla^h(\mathrm{Ric}_{ij} + \mu\eta_i\eta_j) - \nabla_k\nabla_i(\mathrm{Ric}^h_j + \mu\eta_j\eta^h) \\ & - \nabla_k\nabla_j(\mathrm{Ric}^h_i + \mu\eta_i\eta^h) - \nabla_j\nabla^h(\mathrm{Ric}_{ki} + \mu\eta_k\eta_i) \\ & + \nabla_j\nabla_k(\mathrm{Ric}^h_i + \mu\eta_i\eta^h) + \nabla_j\nabla_i(\mathrm{Ric}^h_k + \mu\eta_k\eta^h).\end{aligned}$$

The above equation, together with the assumption of the lemma, yields

$$\pounds_V\,\mathrm{Ric}_{ji} = \nabla_h\nabla^h(\mathrm{Ric}_{ij} + \mu\eta_i\eta_j) - \nabla_h\nabla_i(\mathrm{Ric}^h_i + \mu\eta_j\eta^h) - \nabla_h\nabla_j(\mathrm{Ric}^h_i + \mu\eta_i\eta^h).$$

From equation (1.1), we have

$$\pounds_V g^{ij} = 2\,\mathrm{Ric}^{ij} + 2\lambda g^{ij} + 2\mu\eta^i\eta^j.$$

Using this relation, together with (1.1) and the previous equation, we obtain (3.13). This completes the proof. □

Theorem 3.2. *Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian pseudo-metric manifold and let (g, V, λ, μ) be an η-Ricci soliton on M.*

(a) *If ξ is a timelike vector field then, V is a Killing vector field.*
(b) *If ξ is a spacelike vector field and V is not a Killing vector field, then M is $\mathcal{D}$-homothetically fixed , $\lambda - \mu = 2n + 4$ and $\pounds_V \varphi = 0$.*

Proof. In the case of (a), using (2.10), we find $\lambda - \mu = 2n$, this, (3.1) and (1.1) yield V is Killing.

In the case of (b), using Lemma 3.1, we obtain $(-\lambda+\mu+2n+4)(\lambda+\mu+2n) = 0$. According to the theorem's assumption V is not a Killing vector field, so $\lambda - \mu = 2n + 4$, using this in (3.1), we deduce

$$\mathrm{Ric} = -2g + 2(n+1)\eta \otimes \eta, \tag{3.14}$$

so M is $\mathcal{D}$-homothetically fixed. Using the foregoing equation and (3.3), we obtain

$$(\pounds_V \nabla)(Y, Z) = 2(2n + 2 + \mu)(\eta(Z)\varphi Y + \eta(Y)\varphi Z), \qquad Y, Z \in \Gamma(M).$$

Differentiating the above equation along an arbitrary vector field X, using (3.7), and then contracting with respect to X, we obtain

$$(\pounds_V \mathrm{Ric})(Y, Z) = 2(2n + 2 + \mu)(2g(Y, Z) - (4n + 2)\eta(Y)\eta(Z)), \tag{3.15}$$

where Y, Z are arbitrary vector fields.

Substituting (3.14) into (1.1), we obtain

$$(\pounds_V g)(Y, Z) = -(2n + \lambda + \mu)(g + \eta \otimes \eta)(Y, Z), \qquad Y, Z \in \Gamma(M). \tag{3.16}$$

Lie-differentiating (3.14) along the vector field V gives us

$$\begin{aligned}(\pounds_V \mathrm{Ric})(Y, Z) =& 2(2n + \lambda + \mu)\{g(Y, Z) + \eta(Y)\eta(Z)\}+ \\ & 2(n + 1)\{\eta(Z)(\pounds_V \eta)Y + \eta(Y)(\pounds_V \eta)Z\},\end{aligned} \tag{3.17}$$

where $Z, Y \in \Gamma(M)$. Substituting ξ for Y in (3.17) and (3.15), using (3.11), we obtain

$$(\pounds_V \eta)Y = -2(2 + 2n + \mu)\eta(Y), \qquad Y \in \Gamma(M). \tag{3.18}$$

Operating the above equation by d and noticing the fact that d commutes with Lie-derivative we deduce

$$(\pounds_V d\eta)(X, Y) = -2(2 + 2n + \mu)g(X, \varphi Y), \qquad X, Y \in \Gamma(M).$$

Lie-differentiating (2.3) along the vector field V and using the above equation, yield $\pounds_V \varphi = 0$, and it completes the proof. □

Corollary 3.3. *Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian pseudo-metric manifold with the spacelike vector field ξ and let $(g, V, \lambda, 0)$ be a Ricci soliton on M. If the vector field V is not a Killing vector field, then the Ricci soliton is an expanding soliton on M with $\lambda = 2n + 4$.*

Example 3.4. *Consider* $\mathbf{R}^3$ *with the standard coordinate system* (x, y, z)*. Let* $\xi = 2\frac{\partial}{\partial z}$, $\eta = \frac{1}{2}(-ydx + dz)$, $\varphi(\frac{\partial}{\partial x}) = -\frac{\partial}{\partial y}$, $\varphi(\frac{\partial}{\partial y}) = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$ *and* $\varphi(\frac{\partial}{\partial z}) = 0$*. If* $g = \epsilon\eta \otimes \eta + \frac{1}{4}(dx^2 + dy^2)$*, then* $(M, \varphi, \xi, \eta, g)$ *is a Sasakian pseudo-metric manifold. By direct calculation, we have* $\mathrm{Ric} = -2\epsilon g + 4\eta \otimes \eta$*. Now, let* V *be a vector field defined by*

$$V = ((2 - 6\epsilon + (\epsilon - 1)\lambda + (1 - 2\epsilon)\mu)x\frac{\partial}{\partial x} + (2\epsilon - \lambda)y\frac{\partial}{\partial y} - (2 + \epsilon\lambda + \mu)z\frac{\partial}{\partial z}.$$

If ξ *be a spacelike vector field and* $\lambda - \mu = 6$ *then* (g, V, λ, μ) *is an* η*-Ricci soliton on* M*,* $\pounds_V\varphi = 0$ *and* V *is not a Killing vector field. But if* ξ *is a timelike vector field then* (g, V, λ, μ) *is an* η*-Ricci soliton on* M *iff* V *is a Killing vector field, and this condition is satisfied if* $\lambda = -2$ *and* $\mu = -4$*.*

Proposition 3.5. *Let* $(M, \varphi, \xi, \eta, g)$ *be a* K*-contact pseudo-metric manifold. If* (g, V, λ, μ) *is a gradient* η*-Ricci soliton on* M *then* M *is an* η*-Einstein manifold and* $\mathrm{Ric} = -\lambda g - \mu\eta \otimes \eta$*, where* $-\epsilon\lambda - \mu = 2n$*.*

Proof. First (1.2) gives

$$\nabla_X Df + QX + \lambda X + \epsilon\mu\eta(X)\xi = 0, \qquad X \in \Gamma(M). \tag{3.19}$$

Calculating $R(X, Y)Df$ by the above equation, we deduce

$$\begin{aligned} R(X, Y)Df = {} & \epsilon\mu(\nabla_Y\eta)X\xi + \epsilon\mu\eta(X)\nabla_Y\xi + (\nabla_Y Q)X \\ & - \epsilon\mu(\nabla_X\eta)Y\xi - \epsilon\mu\eta(Y)\nabla_X\xi - (\nabla_X Q)Y. \end{aligned} \tag{3.20}$$

Substituting ξ for Y in the last equation and using Lemma 2.1, we find

$$R(X, \xi)Df = (\mu + 2n)\varphi X - \epsilon\varphi QX, \qquad X \in \Gamma(M).$$

Scalar product of the above equation with ξ gives $df = (\xi(f))\eta$, operating d on this equation, we obtain $d\eta \wedge (\xi(f)) + \eta \wedge d(\xi(f)) = 0$, taking exterior product of the last equation with η and using $\eta \wedge d\eta \neq 0$, we have $\xi(f) = 0$, so f is a constant function. Next using this consequence in (3.19), we find $\mathrm{Ric} = -\lambda g - \mu\eta \otimes \eta$ and this gives $-\epsilon\lambda - \mu = 2n$, completing the proof. □

One may ask, what will happen if the potential vector field of an η-Ricci soliton on a contact pseudo-metric manifold $(M, \varphi, \xi, \eta, g)$ is ξ, we have answered this question in the following theorem.

Theorem 3.6. *Let* $(M, \varphi, \xi, \eta, g)$ *be a contact pseudo-metric manifold, and let* $(g, \varphi, \lambda, \mu)$ *be an* η*-Ricci soliton on the manifold* M*. If* V *is colinear with* ξ *and* $Q\varphi = \varphi Q$*, then* M *is an* η*-Einstein* K*-contact pseudo-metric manifold and* $\mathrm{Ric} = -\lambda g - \mu\eta \otimes \eta$*, where* $-\epsilon\lambda - \mu = 2n$*.*

Proof. Let $V = f\xi$, where f is a non-zero smooth function on the manifold M. Using this in (1.1), we have

$$\begin{aligned} & \epsilon X(f)\eta(Y) + \epsilon Y(F)\eta(X) - 2fg(\varphi hX, Y) + 2\,\mathrm{Ric}(X, Y) \\ & \qquad + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0, \end{aligned} \tag{3.21}$$

for $X, Y \in \Gamma(M)$. Substituting ξ for Y in (3.21), we deduce

$$\epsilon Df + 2Q\xi + (\xi(f) + 2\lambda + 2\epsilon\mu)\xi = 0. \tag{3.22}$$

By assumption $Q\varphi = \varphi Q$, so $\varphi Q\xi = 0$, using this and (2.9), we have $Q\xi = \epsilon(2n - trh^2)\xi$. Substituting this consequence in (3.22), we find

$$\epsilon Df + (2\epsilon(2n - trh^2) + \xi(f) + 2\lambda + 2\mu\epsilon)\xi = 0. \tag{3.23}$$

Next, substituting ξ for X, Y in (3.21), we obtain

$$2n - trh^2 = -\epsilon(\xi(f)) - \lambda\epsilon - \mu.$$

The above equation and (3.23) give $Df = \epsilon(\xi(f))\xi$, differentiating this equation along an arbitrary vector field X and using (2.8), we find

$$g(\nabla_X(Df), Y) = X(\xi(f))\eta(Y) - \epsilon\xi(f)\{g(\epsilon\varphi X, Y) + g(\varphi hX, Y)\}, \quad X, Y \in \Gamma(M).$$

Using the above equation, (2.3) and the known formula $g(\nabla_X(Df), Y) = g(\nabla_Y(Df), X)$, where $X, Y \in \Gamma(M)$, we deduce

$$X(\xi(f))\eta(Y) - Y(\xi(f))\eta(X) = -2\xi(f)d\eta(X, Y), \qquad X, Y \in \Gamma(M).$$

Considering X, Y as arbitrary orthogonal vector fields to ξ in the above equation and noticing that $d\eta \neq 0$, we deduce $X(f) = 0$, so f is a constant function on the manifold M. Using this consequence in (3.21) gives

$$-f\varphi hX + QX + \lambda X + \epsilon\mu\eta(X)\xi = 0, \qquad X \in \Gamma(M). \tag{3.24}$$

Substituting φX for X in the above equation, we find

$$-f\varphi h\varphi X + QX + \lambda X = 0, \qquad X \in \Gamma(M). \tag{3.25}$$

Operating φ on (3.24) and using $\varphi h = -h\varphi$, we have

$$f\varphi h\varphi X + QX + \lambda X = 0, \qquad X \in \Gamma(M). \tag{3.26}$$

Using the above equation, (3.25), (2.1) and $Q\xi = (-\lambda - \mu\epsilon)\xi$, we obtain:

$$\mathrm{Ric} = -\mu\eta \otimes \eta - \lambda g.$$

Using the above equation in (1.1) gives $\pounds_\xi g = 0$, so M is a K-contact pseudo-metric manifold and $-\epsilon\lambda - \mu = 2n$, completing the proof. □

4. η-Ricci solitons on (κ, μ)-contact pseudo-metric manifolds

Studying nullity conditions on manifolds is one of the interesting topics in differential geometry, specially in the context of contact pseudo-metric manifolds. In [16], Ghaffarzadeh and second author introduced the notion of a (κ, μ)-contact pseudo-metric manifold. According to them a contact pseudo-metric manifold (M, φ, ξ, η) is called a (κ, μ)-contact pseudo-metric manifold if it satisfies

$$R(X, Y)\xi = \epsilon\kappa(\eta(Y)X - \eta(X)Y) + \epsilon\mu(\eta(Y)hX - \eta(X)hY), \tag{4.1}$$

where R is the Riemannian curvature tensor of M, κ, μ are constant real numbers, and X, Y are arbitrary vector fields. For a $(\kappa, \mu)-$*contact* pseudo-metric manifold we have the following formulas [16]

$$h^2 = (\epsilon\kappa - 1)\varphi^2, \tag{4.2}$$

$$Q\xi = 2n\kappa\xi, \tag{4.3}$$

$$(\nabla_\xi h) = -\epsilon\mu\varphi h, \tag{4.4}$$

furthermore if $\epsilon\kappa < 1$ then we have [16]

$$\begin{aligned} QX &= \epsilon[2(n-1) - n\mu]X + (2(n-1) + \mu)hX \\ &\quad + [2(1-n)\epsilon + 2n\kappa + n\epsilon\mu]\eta(X)\xi, \end{aligned} \tag{4.5}$$

$$r = 2n(\kappa - 2\epsilon) + 2n^2\epsilon(2 - \mu), \tag{4.6}$$

where X and r are, an arbitrary vector field and the scalar curvature of the manifold, respectively.

Lemma 4.1. *Let $(M, \varphi, \xi, \eta, g)$ be a (κ, μ)-contact pseudo-metric manifold, and let $\epsilon\kappa < 1$. If (g, V, λ, τ) is a gradient η-Ricci soliton on the manifold M then*

$$\epsilon\kappa(-2 + \mu) = n\mu + \mu + \tau. \tag{4.7}$$

Proof. Differentiating (4.3) along an arbitrary vector field X and using (2.8), we deduce

$$(\nabla_X Q)\xi = Q(\epsilon\varphi + \varphi h)X - 2n\kappa(\epsilon\varphi + \varphi h)X, \qquad X \in \Gamma(M). \tag{4.8}$$

Taking scalar product of (3.20) and ξ, and using (4.8), we have

$$\begin{aligned} g(R(X,Y)Df, \xi) &= \epsilon g((Q\varphi + \varphi Q)Y, X) + g((Q\varphi h + h\varphi Q)Y, X) \\ &\quad + (-4n\kappa\epsilon - 2\tau)g(\varphi Y, X), \quad X, Y \in \Gamma(M). \end{aligned} \tag{4.9}$$

Substituting φX for X and φY for Y in (4.1) give $R(\varphi X, \varphi Y)\xi = 0$, using this, (2.1) and the above equation, we obtain

$$\epsilon(\varphi Q + Q\varphi)X - (\varphi Qh + hQ\varphi)X + (-4n\kappa\epsilon - 2\tau)\varphi X = 0, \tag{4.10}$$

where X is an arbitrary vector field. Now, substituting φX for X in (4.5), we have

$$Q\varphi X = \epsilon[2(n-1) - n\mu]\varphi X + (2(n-1) + \mu)h\varphi X, \qquad X \in \Gamma(M). \tag{4.11}$$

Next, operating φ on (4.5), we obtain

$$\varphi Qx = \epsilon[2(n-1) - n\mu]\varphi X + (2(n-1) + \mu)\varphi hX, \qquad X \in \Gamma(M). \tag{4.12}$$

Substituting hX for X in (4.12), using (4.2) and (2.1) give

$$\varphi QhX = \epsilon[2(n-1) - n\mu]\varphi hX - (\epsilon\kappa - 1)(2(n-1) + \mu)\varphi X, \qquad X \in \Gamma(M). \tag{4.13}$$

Operating h on (4.11), using (4.2) and (2.1), we have

$$hQ\varphi X = \epsilon[2(n-1) - n\mu]h\varphi X - (\epsilon\kappa - 1)(2(n-1) + \mu)\varphi X, \qquad X \in \Gamma(M). \tag{4.14}$$

Using the last four equations in (4.10) and $h\varphi = -\varphi h$ give (4.7), completing the proof. ☐

Theorem 4.2. *Let $(M, \varphi, \xi, \eta, g)$ be a (κ, μ)-contact pseudo-metric manifold and let $\epsilon\kappa < 1$. If (g, V, λ, τ) is a gradient η-Ricci soliton on M, then $\mu = 0, \tau = -2\epsilon\kappa$, or $\mathrm{Ric} = -\lambda g - \tau\eta \otimes \eta$ and $\mu = 2 - 2n, \tau = 2n(-\frac{1}{n} + n - \epsilon\kappa)$.*

Proof. First, substituting ξ for X in (4.9), using (4.1) and (4.3), we have

$$\kappa(\xi(f)\xi - \epsilon Df) - \epsilon\mu hDf = 0.$$

Differentiating the above equation along vector field ξ and using (4.4), we have

$$\kappa\xi(\xi(f))\xi + \epsilon\kappa(2n\kappa + \lambda + \tau\epsilon)\xi + (\epsilon\mu)^2\varphi hDf = 0.$$

Now, operating φ on the last equation we find $\mu^2 hDf = 0$, taking h from this and using (4.2), we obtain

$$\mu^2(\epsilon\kappa - 1)(-Df + \eta(Df)\xi) = 0.$$

Examining the above equation we have either, i)$\mu = 0$ or ii)$\mu \neq 0$.

In the case i, using (4.7), we obtain $\tau = -2\epsilon\kappa$. In the case ii, we have $Df = \eta(Df)\xi$, differentiating this along arbitrary vector field X and using (2.8), we have

$$g(\nabla_X Df, Y) = X(\xi(f))\eta(Y) - \xi(f)g(\varphi X, Y) - \epsilon\xi(f)f(\varphi hX, Y),$$

where X, Y are arbitrary vector fields. Using the above equation and

$$g(\nabla_X Df, Y) = g(\nabla_Y Df, X),$$

we find

$$X(\xi(f))\eta(Y) - Y(\xi(f))\eta(X) + 2\xi(f)d\eta(X, Y) = 0, \qquad X, Y \in \Gamma(M).$$

Substituting φX for X and φY for Y, and noticing the fact that $d\eta \neq 0$, we have $\xi(f) = 0$. So f is a constant function and $\mathrm{Ric} = -\lambda g - \tau\eta \otimes \eta$. Using this gives $r = (2n+1)(-\lambda) - \epsilon\tau$, comparing the last consequence and (4.6), we have

$$n\mu = -2 + 2n - 2n\kappa\epsilon - \tau.$$

Now using the above equation and (4.7), we obtain, $\mu = 2 - 2n$ and $\tau = 2n(-\frac{1}{n} + n - \epsilon\kappa)$, completing the proof. ☐

Corollary 4.3. *Let $(M, \varphi, \xi, \eta, g)$ be a (κ, μ)-contact pseudo-metric manifold and let $\epsilon\kappa < 1$. If $(g, V, \lambda, 0)$ is a gradient η-Ricci soliton (in fact a gradient Ricci soliton) on M, then $R(X, Y)\xi = 0$, where X, Y are arbitrary vector fields.*

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