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Study of η-RS on Lorentzian metric PS manifolds endowed GSM connection

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Abstract. The present study initially identifies the generalized symmetric connection of the type (α_1, α_2), which can be regarded as more generalized forms of quarter and semi-symmetric connections. The goal of this endeavor is to look at the η-Ricci Soliton(RS) on Lorentzian metric P-Sasakian(PS) manifold with Generalized Symmetric Metric(GSM) connection of the kind (α_1,α_2). Ricci and η-Ricci solitons with generalized symmetric metric connection of the type (α, α_2) have been discussed, satisfying the conditions $\bar{R}\cdot\bar{S}$, $\bar{S}\cdot\bar{R}$. Finally, we have constructed an example of LP-Sasakian manifold with generalized symmetric metric connection of the type (α_1, α_2) admitting η-Ricci solitons.

Keywords: Lorentzian metric PS manifold, GSM connection, η-RS, RS, Einstein manifold.

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1. **Introduction**

Hayden established ametric connection with non-zero torsion on a Riemannian manifold [16]. Golab [11] proposed quarter-symmetric connections on a differentiable manifold as a more generalized version of semi-symmetric connections. Meanwhile, Lorentzian para-contact manifolds were introduced by Matsumoto [19].

It is known that a (semi-)Riemannian manifold possesses a linear connection that can be classified as a generalized symmetric connection when its torsion tensor T is defined in the following manner:

$$T(X_1, X_2) = \alpha_1\{v(X_2)X_1 - v(X_1)X_2\} + \alpha_2\{v(X_2)\varphi X_1 - v(X_1)\varphi X_2\}, \quad (1.1)$$

for any manifold with vector fields X_1, X_2 and smooth functions α_1, α_2. v is a 1-form connected to a non-vanishing smooth non-null unit vector field ξ, and φ is a tensor of type $(1,1)$. Furthermore, if there is a Riemannian metric g in M such that $\overline{\nabla}g = 0$, the connection is said to be a GSM connection otherwise, it is non-metric. If $\alpha_1 = 0(\alpha_2 = 0)$ in equation (1.1), the generalized symmetric connection is referred to as α_2-quarter-symmetric (α_1-semi-symmetric connection), respectively. Moreover, by selecting $(\alpha_1,\ \alpha_2)$=$(1, 0)$ and $(\alpha_1, \alpha_2) = (0, 1)$, the generalized symmetric connection can be decomposed into a semi-symmetric connection and a quarter-symmetric connection. As a result, it is possible to think of a generalized symmetric connection as a generalization of a semi-symmetric and a quarter-symmetric connection. These two connections are significant for the investigation of geometry as well as physics applications. Several authors have extensively investigated the properties of Riemannian manifolds equipped with semi-symmetric(symmetric) and non-metric connections (see [15, 10, 28, 33, 2, 9]). An almost contact manifold's semi-symmetric metric connection was defined by Sharfuddin and Hussian in [30], by setting

$$T(X_1, X_2) = \eta(X_2)X_1 - \eta(X_1)X_2.$$

The authors of [14, 25] and [32] accomplished a study on the semi-symmetric non-metric connection in Kenmotsu and nearly trans-hyperbolic Sasakian manifolds.

In the present investigation, we introduce a new connection termed the GSM connection on a Lorentzian metric PS manifold. This connection serves as an extension of the semi-symmetric and quarter-symmetric metric connections. On the other hand, a RS is a natural generalization of an Einstein metric. R. S. Hamilton [12] said in 1982, that the RS moved under the Ricci flow simply by diffeomorphisms of the initial metric, that is, they are stationary points of the Ricci flow:

$$\frac{\partial g}{\partial t} = -2Ric(g) \quad (1.2)$$

Definition: A RS (g, V, λ) on a Riemannian manifold is defined by

$$Ł_V g + 2S + 2\lambda g = 0, \tag{1.3}$$

where S is the Ricci tensor, $Ł_V$ is the Lie derivative along the vector field V on M and λ is a real scalar. RS is said to be shrinking, steady or expending according as $\lambda<0$, $\lambda=0$ and $\lambda>0$ respectively.

The exploration of RS in the context of Riemannian geometry was initiated by R. Sharma [31]. Subsequently, the study of RSin almost contact metric manifolds has been extensively investigated by Tripathi [34], Nagaraja et al. [21], and other researchers such as C. S. Bagewadi et al. [5].The notion of η-RS was introduced in 2009 by J. T. Cho and M. Kimura, who additionally presented a categorization scheme for real hypersurfaces in non-flat complex space forms that admit η-RS. A. M. Blaga et al. [6] have examined η- RS in almost paracontact metric manifolds. A. M. Blaga and a number of other authors have also investigated η-RS on manifolds with different structures(see [15, 6, 27, 35, 23, 29]). With this new connection, it is natural and intriguing to analyze η-RS in almost contactmetric manifolds.

As a result of the aforementioned investigations, we shall examine the η-RS in a Lorentzian metric PS manifold with regard to a GSM connection in this work. We shall consider η-RS in the almost contact geometry, precisely, on a Lorentzian metric PS manifold with GSM connection which satisfies certain curvature properties $\bar{R}.\bar{S}$ and $\bar{S}.\bar{R}$ respectively.

2. **Preliminaries**

Let M denote a differentiable manifold of dimension n equipped with a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a 1-form η, and a Lorentzian metric g. This manifold satisfies the following properties:

$$\phi\xi = 0, \tag{2.1}$$

$$\eta(\phi X_1) = 0, \tag{2.2}$$

$$\eta(\xi) = -1, \tag{2.3}$$

$$\phi^2 X_1 = X_1 + \eta(X_1)\xi, \tag{2.4}$$

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) + \eta(X_1)\eta(X_2), \tag{2.5}$$

$$g(X_1, \xi) = \eta(X_1), \tag{2.6}$$

for all vector fields X_1, X_2 on M, where ∇ is the Levi-civita connection with respect to the Lorentzian metric g. Such manifold (M, ϕ, ξ, η, g) is called a Lorentzian metric PS (shortly, LPS) manifold ([17, 19]). The following are satisfied by an Lorentzian metric PS manifold:

$$(\nabla_{X_1}\phi)X_2 = g(X_1, X_2)\xi + \eta(X_2)X_1 + 2\eta(X_1)\eta(X_2)\xi, \tag{2.7}$$

$$\nabla_{X_1}\xi = \phi X_1. \tag{2.8}$$

If we write $g(\phi X_1, X_2)$=$\Phi(X_1, X_2)$, where X_1 and X_2 are vector fields on the manifold M, then the tensor field Φ is classified as a symmetric $(0,2)$ tensor field.

In Lorentzian metric PS manifold the following relations hold ([18], [20]):

$$(\nabla_{X_1}\eta)X_2 = g(\phi X_1, X_2), \tag{2.9}$$

$$g(R(X_1, X_2)X_3, \xi) = \eta(R(X_1, X_2)X_3) = g(X_2, X_3)\eta(X_1) - g(X_1, X_3)\eta(X_2), \tag{2.10}$$

$$R(\xi, X_1)X_2 = g(X_1, X_2)\xi - \eta(X_2)X_1, \tag{2.11}$$

$$R(X_1, X_2)\xi = \eta(X_2)X_1 - \eta(X_1)X_2, \tag{2.12}$$

$$R(\xi, X_1)\xi = X_1 + \eta(X_1)\xi, \tag{2.13}$$

$$S(X_1, \xi) = (n-1)\eta(X_1), \tag{2.14}$$

$$S(\phi X_1, \phi X_2) = S(X_1, X_2) + (n-1)\eta(X_1)\eta(X_2), \tag{2.15}$$

for all vector fields X_1, X_2 and X_3 on M, in which R and S can be viewed as the curvature tensor and the Ricci tensor of M, respectively.

An Lorentzian metric PS manifold M is said to be a generalized η-Einstein of the non-vanishing Ricci tensor S of M satisfies the relation

$$S(X_1, X_2) = ag(X_1, X_2) + b\eta(X_1)\eta(X_2) + cg(\phi X_1, X_2), \tag{2.16}$$

for every X_1, $X_2 \in \Gamma(TM)$ where a, b, and c are considered as scalar functions on the manifold M. If c=0, then then the manifold M is considered to be a η-Einstein manifold.

On contracting (2.16), we obtain

$$r = na - b \tag{2.17}$$

In a similar way, setting X_1=X_2=ξ in (2.16) and using (2.14), we get

$$-(n-1) = -a + b \tag{2.18}$$

In view of (2.17) and (2.18), we have

$$a = \frac{r-(n-1)}{n-1},\ b = -\left[\frac{n(n-1)-r}{n-1}\right] \tag{2.19}$$

Therefore, the Ricci tensor S of a generalized η-Einstein Lorentzian metric PS manifold M can be represented as.

$$S(X_1, X_2) = \frac{r-(n-1)}{n-1}g(X_1, X_2) - \left[\frac{n(n-1)-r}{n-1}\right] + cg(\phi X_1, X_2)$$

3. GSM Connection in a Lorentzian metric PS Manifold

Let $\overline{\nabla}$ denote a linear connection and ∇ denote a Levi-Civita Connection of a Lorentzian para-contact metric manifold M such that

$$\overline{\nabla}_{X_1}X_2 = \nabla_{X_1}X_2 + H(X_1, X_2), \tag{3.1}$$

for any vector field X_1 and X_2. The following is obtained so that $\overline{\nabla}$ is a generalized symmetric connection of ∇, in which H is viewed as a tensor of type $(1, 2)$;

$$H(X_1, X_2) = \frac{1}{2}\Big[T(X_1, X_2) + T^{'}(X_1, X_2) + T^{'}(X_2, X_1)\Big], \tag{3.2}$$

where T is the torsion tensor of $\overline{\nabla}$ and

$$g(T'(X_1, X_2), X_3) = g(T(X_3, X_1), X_2) \tag{3.3}$$

Derived from (1.1) and (3.3), we obtain the following

$$T'(X_1, X_2) = \alpha_1\{\eta(X_1)X_2 - g(X_1, X_2)\xi\} + \alpha_2\{\eta(X_1)\phi X_2 - g(\phi X_1, X_2)\xi\}. \tag{3.4}$$

Using (1.1), (3.2) and (3.4) we obtain

$$H(X1, X_2) = \alpha_1\{\eta(X_1)X_2 - g(X_1, X_2)\xi\} + \alpha_2\{\eta(X_1)\phi X_2 - g(\phi X_1, X_2)\xi\}. \tag{3.5}$$

Corollary 3.1. *For an Lorentzian metric PS manifold, GSM connection* ∇ *is given by*

$$\overline{\nabla}_{X_1}X_2 = \nabla_{X_1}X_2 + \alpha_1\{\eta(X_2)X_1 - g(X_1, X_2)\xi\} + \alpha_2\{\eta(X_2)\phi X_1 - g(\phi X_1, X_2)\xi\}. \tag{3.6}$$

If we select $(\alpha_1, \alpha_2)=(1, 0)$ and $(\alpha_1, \alpha_2)=(0, 1)$, a generalized metric connection is reduced to a semi-symmetric and quarter-symmetric connection as seen below:

$$\overline{\nabla}_{X_1}X_2 = \nabla_{X_1}X_2 + \eta(X_2)X_1 - g(X_1, X_2)\xi. \tag{3.7}$$

$$\overline{\nabla}_{X_1}X_2 = \nabla_{X_1}X_2 + \eta(X_2)\phi X_1 - g(\phi X_1, X_2)\xi. \tag{3.8}$$

From (3.6), we can derive the following proposition:

Proposition 3.2. *The subsequent relations are derived when* M *is an Lorentzian metric PS manifold with generalized metric connection:*

$$\begin{aligned}(\overline{\nabla}_{X_1}\phi)X_2 = {}&[(1-\alpha_2)g(X_1, X_2) + (2-2\alpha_2)\eta(X_1)\eta(X_2) - \alpha_1\Phi(X_1, X_2)]\xi + \\ &(1-\alpha_2)\eta(X_2)X_1 - \alpha_1(X_2)\phi X_1,\end{aligned} \tag{3.9}$$

$$\overline{\nabla}_{X_1}\xi = (1-\alpha_2)\phi X_1 - \alpha_1 X_1 - \alpha_1\eta(X_1)\xi, \tag{3.10}$$

$$(\overline{\nabla}_{X_1}\eta)X_2 = (1-\alpha_2)\Phi(X_1, X_2) - \alpha_1 g(\phi X_1, \phi X_2), \tag{3.11}$$

for any X_1, X_2, $X_3 \in \Gamma(TM)$.

Example 3.3. *A* 3*-dimensional manifold* $M = \{(x, y, z) \in \mathbb{R}^3\}$ *is considered, in which* (x, y, z) *are regarded as the standard coordinates in* $\mathbb{R}^3$*. Suppose that* e_1*,* e_2*,* e_3 *are linearly independent global frame on* M *as presented below:* $e_1 = e^z \frac{\partial}{\partial y}$*,* $e_2 = e^z(\frac{\partial}{\partial x} + \frac{\partial}{\partial y})$*,* $e_3 = \frac{\partial}{\partial z} = \xi$*. Assume that* g *is a Lorentzian metric defined as*

$$g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0,$$
$$g(e_1, e_1) = g(e_2, e_2) = 1, g(e_3, e_3) = -1.$$

When we consider that η *is a* 1*-form represented as* $\eta(X_1) = g(X_1, e_3)=g(X_1, \xi)$ *for every* $X_1 \in TM$ *and* ϕ *is the* $(1,1)$ *tensor field presented as* $\phi e_1=-e_1$*,* $\phi e_2 = -e_2$ *and* $\phi e_3=0$*, we thereby get*

$$\eta(e_3) = -1, \quad \phi^2 X_1 = X_1 + \eta(X_1)e_3, \quad g(\phi X_1, \phi X_2) = g(X_1, X_2) + \eta(X_1)\eta(X_2)$$

for all X_1*,* $X_2 \in TM$ *through use of linearity of* ϕ *and* g*. Therefore for* $e_3=\xi$*,* (ϕ, ξ, η, g) *describes a Lorentzian para-contact structure on* M*. Therefore* ∇ *is the Levi-Civita connection concerning the Riemannian metric* g*.*

The following are obtained:

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -e_1, \quad [e_2, e_3] = -e_2.$$

Using Koszul's formula, the following can be calculated in an easy way

$$\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -e_1,$$
$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_3 = -e_2,$$
$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0. \quad \nabla_{e_3} e_3 = 0.$$

The above relations show that (ϕ, ξ, η, g) *is an LP-Sasakian structure on* M [18].

We now perform similar calculations for the generalized symmetric metric connection. Using 3.6 *in above equations, we get*

$$\overline{\nabla}_{e_1} e_1 = (-1 - \alpha + \beta)e_3, \ \overline{\nabla}_{e_1} e_2 = 0, \ \overline{\nabla}_{e_1} e_3 = (-1 - \alpha + \beta)e_1,$$

$$\overline{\nabla}_{e_2} e_1 = 0, \ \overline{\nabla}_{e_2} e_2 = (-1 - \alpha + \beta)e_3, \ \overline{\nabla}_{e_2} e_3 = (-1 - \alpha + \beta)e_2,$$

$$\overline{\nabla}_{e_3} e_1 = 0, \ \overline{\nabla}_{e_3} e_2 = 0, \ \overline{\nabla}_{e_3} e_3 = 0. \tag{3.12}$$

*We can easily see that equation (*3.12*) holds the relation (*1.1*). Also we obtain* $\overline{\nabla} g = 0$*. Thus* $\overline{\nabla}$ *is a generalized symmetric metric connection on* M*.*

4. Curvature Tensor on Lorentzian metric PS manifold with GSM Connection

Consider that M is an n-dimensional Lorentzian metric PS manifold, then the following can define the curvature tensor $\overline{R}$ of the generalized metric connection $\overline{\nabla}$ on M

$$\overline{R}(X_1, X_2)X_3 = \overline{\nabla}_{X_1}\overline{\nabla}_{X_2}X_3 - \overline{\nabla}_{X_2}\overline{\nabla}_{X_1}X_3 - \overline{\nabla}_{[X_1,X_2]}X_3. \tag{4.1}$$

When proposition (1) is used, through (3.6) and (3.12), we obtain

$$\begin{aligned}\overline{R}(X_1, X_2)X_3 = R(X_1, X_2) + K_1(X_2, X_3)X_1 - K_1(X_1, X_3)X_2 + K_2(X_2, X_3)\phi X_1 \\ - K_2(X_1, X_3)\phi X_2 + \{K_3(X_1, X_2)X_3 - K_3(X_2, X_1)X_3\}\xi\end{aligned} \tag{4.2}$$

where

$$K_1(X_2, X_3) = (\alpha_1\alpha_2 - \alpha_1)\Phi(X_2, X_3) + \alpha_1^2 g(X_2, X_3) + (\alpha_1^2 + \alpha_2 - \alpha_2^2)\eta(X_2)\eta(X_3) \tag{4.3}$$

$$K_2(X_2, X_3) = (\alpha_2^2 - 2\alpha_2)\Phi(X_2, X_3) - \alpha_1(1 - \alpha_2)g(X_2, X_3) \tag{4.4}$$

$$K_3(X_1, X_2)X_3 = \{(\alpha_1^2 + \alpha_2)g(X_2, X_3) + \alpha_1\alpha_2\Phi(X_2, X_3)\}\eta(X_3) \tag{4.5}$$

From (2.3-2.8), (2.11), (2.12) and (4.1-4.4), We present the following proposition:

Proposition 4.1. *When M is a manifold with a Lorentzian metric of dimension n and a GSM connection, the following equations hold.:*

$$\begin{aligned}\overline{R}(X_1, X_2)\xi = (1 - \alpha_2 + \alpha_2^2)[\eta(\alpha_2)X_1 - \eta(X_1)X_2] + \alpha_1(1 - \alpha_2)[\eta(X_1)\phi X_2 \\ -\eta(X_2)\phi X_1)],\end{aligned} \tag{4.6}$$

$$\begin{aligned}\overline{R}(\xi, X_2)X_3 = \{-\alpha_1\Phi(X_2, X_3) + (1 - \alpha_2)g(X_2, X_3) - \alpha_2^2\eta(X_2)\eta(X_3)\}\xi - (1- \\ \alpha_2 + \alpha_2^2)\eta(X_3)X_2 + \alpha_1(1 - \alpha_2)\eta(X_3)\phi X_2,\end{aligned} \tag{4.7}$$

$$\overline{C}(\xi, X_2)\xi = (1 - \alpha_2 + \alpha_2^2)[\eta(X_2)\xi + X_2] + \alpha_1(\alpha_2 - 1)\phi X_2, \tag{4.8}$$

$$\begin{aligned}\eta(\overline{R}(X_1, X_2)X_3) = (1 - \alpha_2)\{g(X_2, X_3)\eta(X_1) - g(X_1, X_3)\eta(U_2)\} + \alpha_1\{\eta(X_2) \\ g(\phi X_1, X_3) - \eta(X_1)g(\phi X_2, X_3)\}\end{aligned} \tag{4.9}$$

for every X_1, X_2, $X_3 \in \Gamma(TM)$.

The subsequent appearance provides the Ricci tensor $\overline{S}$ and the scalar curvature $\overline{r}$ of a Lorentzian metric PS manifold, along with the GSM connection

$$\overline{S}(X_1, X_2) = \sum_{i=1}^{n} g(\overline{R}(e_i, X_1)X_2, e_i),$$

and

$$\overline{r} = \sum_{i=1}^{n} \epsilon_i \overline{S}(e_i, e_i),$$

respectively, let $\epsilon_i{=}g(e_i, e_i)$, where X_1 and X_2 are elements of $\Gamma(TM)$, and $\{e_1, e_2, e_3, ..., e_n\}$ is considered to be an orthonormal frame.. Then by using (2.7), (2.8) and (4.2) we obtain

$$\begin{aligned}\overline{S}(X_1, X_2) = S(X_1, X_2) &+ \sum_{i=1}^{n} k_1(X_1, X_2)g(e_i, e_i) - \sum_{i=1}^{n} k_1(e_i, X_2)g(X_1, e_i) \\ &+ \sum_{i=1}^{n} k_2(X_1, X_2)g(\phi e_i, e_i) - \sum_{i=1}^{n} k_2(e_i, X_2)g(\phi X_1, e_i) \\ &+ \sum_{i=1}^{n} \{K_3(e_i, X_1)X_2 - K_3(X_1, e_i)X_2\}\eta(e_i).\end{aligned} \tag{4.10}$$

Then by using (4.3), (4.4), (4.5) and (4.10) we obtain

$$\begin{aligned}&\overline{S}(X_1, X_2) = S(X_1, X_2) + \{-\alpha_1\alpha_2 + (n-2)(\alpha_1\alpha_2 - \alpha_1) + (\alpha_2^2 - 2\alpha_2)trac\Phi\} \\ &\Phi(X_1, X_2) + \{-2\alpha_1^2 + \alpha_2 - \alpha_2^2 + n\alpha_1^2 + (\alpha_1\alpha_2 - \alpha_1)trac\Phi\}g(X_1, X_2) + \{-2\alpha_1^2 \\ &\qquad + n(\alpha_1^2 + \alpha_2 - \alpha_2^2)\}\eta(X_1)\eta(X_2).\end{aligned} \tag{4.11}$$

where S is Ricci tensor with respect to Levi-Civita connection.

5. Ricci and η-RS on (M, ϕ, ξ, η, g)

An η-RS can be defined as a tuple (g, V, λ, μ), where V represents a vector field on M, λ and μ are real constants, and g denotes a Riemannian (or Pseudo-Riemannian) metric that satisfies the equation.

$$\text{Ł}_{\xi} g + 2\overline{S} + 2\lambda + 2\mu\eta \otimes \eta = 0, \tag{5.1}$$

where λ and μ are real constants, Ł_{ξ} is the Lie derivative in the direction of ξ and $\overline{S}$ is the Ricci tensor with regard to the GSM connection. Writing the GSM connection $\overline{\nabla}$ in terms of Ł_{ξ}, we get:

$$2\overline{S}(X_1, X_2) = -g(\overline{\nabla}_{X_1}\xi, X_2) - g(X_1, \overline{\nabla}_{X_2}\xi) - 2\lambda g(X_1, X_2) - 2\mu\eta(X_1)\eta(X_2), \tag{5.2}$$

for any $X_1,\ X_2 \in \Gamma(TM)$.

The data (g, ξ, λ, μ) that satisfies the equation (5.1) is said to be an η-RS on M [10]. In especially, if $\mu{=}0$, then (g, ξ, λ) is known as a RS [8] and is referred to as shrinking, steady or expanding depending on whether λ is negative, zero, or positive [8].

Here is an illustration of η-RS on Lorentzian metric PS manifold with GSM connection.

Example 5.1. *: Let $M(\phi,\xi,\eta,g)$ be the Lorentzian metric PS manifold considered in example* 3.3.
Using Koszul's formula, the following can be calculated in an easy way
$\nabla_{e_1}e_1=-e_3,\quad \nabla_{e_1}e_2=0,\quad \nabla_{e_1}e_3=-e_1,$
$\nabla_{e_2}e_1=0,\quad \nabla_{e_2}e_2=-e_3,\quad \nabla_{e_2}e_3=-e_2,$
$\nabla_{e_3}e_1=0,\quad \nabla_{e_3}e_2=0,\quad \nabla_{e_3}e_3=0.$
Let $\overline{\nabla}$ be a generalized symmetric metric connection, we obtain using the above relations, we can calculate the non-vanishing components of the curvature tensor as follows:

$$R(e_1,e_2)e_1=-e_2,\ R(e_1,e_2)e_2=e_1,\ R(e_1,e_3)e_1=-e_3,$$
$$R(e_1,e_3)e_3=-e_1,\ R(e_2,e_3)e_2=-e_3,\ R(e_2,e_3)e_3=-e_2 \tag{5.3}$$

Using the equation (5.3), we can conveniently compute the non-vanishing components of the Ricci tensor are as follows:

$$S(e_1,e_1)=2,\ S(e_2,e_2)=2,\ S(e_3,e_3)=-2, \tag{5.4}$$

Similar calculations for the generalized metric connection yield. Using 3.6 in the above equations, we obtain

$$\overline{\nabla}_{e_1}e_1=-(1+\alpha-\beta)e_3,\ \ \overline{\nabla}_{e_1}e_2=0,\ \ \overline{\nabla}_{e_1}e_3=-(1+\alpha-\beta)e_1,$$
$$\overline{\nabla}_{e_2}e_1=0,\ \ \overline{\nabla}_{e_2}e_2=-(1+\alpha-\beta)e_3,\ \ \overline{\nabla}_{e_2}e_3=-(1+\alpha-\beta)e_2,$$
$$\overline{\nabla}_{e_3}e_1=0,\ \ \overline{\nabla}_{e_3}e_2=0,\ \ \overline{\nabla}_{e_3}e_3=0. \tag{5.5}$$

Using 5.5, the components of the curvature tensor with respect to the generalized metric connection are as follows:

$$\overline{R}(e_1,e_2)e_1=-(1+\alpha-\beta)^2e_2,\ \overline{R}(e_1,e_2)e_2=(1+\alpha-\beta)^2e_1,$$
$$\overline{R}(e_1,e_3)e_1=-(1+\alpha-\beta)e_3,\ \overline{R}(e_1,e_3)e_3=-(1+\alpha-\beta)e_1,$$
$$\overline{R}(e_2,e_3)e_2=-(1+\alpha-\beta)e_3,\ \overline{R}(e_2,e_3)e_3=-(1+\alpha-\beta)e_2,$$
$$\overline{R}(e_3,e_2)e_1=0,\ \overline{R}(e_3,e_1)e_2=0, \tag{5.6}$$

From (5.6), the components of the Ricci tensor are as follows:

$$\overline{S}(e_1,e_1)=(1+\alpha-\beta)(2+\alpha-\beta),\ \overline{S}(e_2,e_2)=(1+\alpha-\beta)(2+\alpha-\beta),$$
$$\overline{S}(e_3,e_3)=-2(1+\alpha-\beta). \tag{5.7}$$

From (5.2) and (5.5) we get

$$-2(1+\alpha-\beta)[g(e_i,e_i)+\eta(e_i)\eta(e_i)]+2\overline{S}(e_i,e_i)+2\lambda g(e_i,e_i)+2\mu\eta(e_i)\eta(e_i)=0 \tag{5.8}$$

for all $i\in\{1,2,3\}$, and we have $\lambda{=}-(1+\alpha-\beta)^2$ (i.e. $\lambda<0$) and $\mu=1-(\alpha-\beta)^2$, the data (g,ξ,λ,μ) is an η- Ricci soliton on (M,ϕ,ξ,η,g).

6. Second order parallel symmetric tensors and η-RS in Lorentzian metric PS manifold

A well-known geometrical object in the study of RS is a symmetric $(0,2)$-tensor field that is parallel to the GSM connection.
Let h be a symmetric tensor field of type $(0,2)$, which we assume is parallel with respect to the GSM connection $\overline{\nabla}$, so $\overline{\nabla}h$=0.
Using the Ricci identity [8]

$$\overline{\nabla}^2 h(X_1,X_2;X_3,X_4)+\overline{\nabla}^2 h(X_1,X_2;X_4,X_3)=0, \tag{6.1}$$

we obtain the relation

$$h(\overline{R}(X_1,X_2)X_3,X_4)+h(X_3,\overline{R}(X_1,X_2)X_4)=0. \tag{6.2}$$

Replacing X_3=X_4=ξ in (6.2) and using (4.6) and it follows from the symmetry of h $h(\overline{R}(X_1,X_2)\xi,\xi)=0$ for any X_1, $X_2\in\Gamma(TM)$ and

$$\begin{aligned}&(1-\alpha_2+\alpha_2^2)\eta(X_2)h(X_1,\xi)-(1-\alpha_2+\alpha_2^2)\eta(X_1)h(X_2,\xi)+\alpha_1(1-\alpha_2)\eta(X_1)h\\&(\phi X_2,\xi)-\alpha_1(1-\alpha_2)\eta(X_2)h(\phi X_1,\xi)+(1-\alpha_2+\alpha_2^2)\eta(X_2)h(\xi,X_1)-(1-\alpha_2\\&+\alpha_2^2)\eta(X_1)h(\xi,X_2)+\alpha_1(1-\alpha_2)\eta(X_1)h(\xi,\phi X_2)-\alpha_1(1-\alpha_2)\eta(X_2)h(\xi,\phi X_1)=0.\end{aligned} \tag{6.3}$$

Putting X_1=ξ in (6.3) and by the virtue of (2.4), we obtain

$$2(1-\alpha_2+\alpha_2^2)[\eta(X_2)h(\xi,\xi)+h(\xi,X_2)]-2\alpha_1(1-\alpha_2)h(\xi,\phi X_2)=0 \tag{6.4}$$

or

$$2(1-\alpha_2+\alpha_2^2)[h(\xi,X_2)+\eta(X_2)h(\xi,\xi)]-2\alpha_1(1-\alpha_2)h(\phi X_2,\xi)=0. \tag{6.5}$$

Suppose $(1-\alpha_2+\alpha_2^2)\neq 0$, α_1=0 it results

$$h(X_2,\xi)+\eta(X_2)h(\xi,\xi)=0, \tag{6.6}$$

for any $X_2\in\Gamma(TM)$, equivalent to

$$h(X_2,\xi)+g(X_2,\xi)h(\xi,\xi)=0, \tag{6.7}$$

for any $X_2\in\Gamma(TM)$. Differentiating the equation 6.7 covariantly with respect to the vector field $X_1\in\Gamma(TM)$, we obtain

$$\begin{aligned}h(\nabla_{X_1}X_2,\xi)+h(X_2,\nabla_{X_1}\xi)=&-h(\xi,\xi)[g(\nabla_{X_1}X_2,\xi)+g(X_2,\nabla_{X_1}\xi)]\\&-2g(X_2,\xi)h(\nabla_{X_1}\xi,\xi).\end{aligned} \tag{6.8}$$

Using (4.6) in (6.8), we obtain

$$h(X_1,X_2)=g(X_1,X_2)h(\xi,\xi), \tag{6.9}$$

for any X_1, $X_2\in\Gamma(TM)$.

Proposition 6.1. *Assume that* (M, ψ, ξ, η, g) *is a Lorentzian metric PS manifold with GSM connection as well non-vanishing* ξ*-sectional curvature and provided with a symmetric and* ϕ*-skew symmetric tensor field of type* $(0, 2)$*. If* h *is parallel to* $\overline{\nabla}$*, then the metric tensor* g *is a constant multiple of* h*.*

On a Lorentzian metric PS manifold with GSM connection using (3.10) and $Ł_\xi g{=}2(g - \eta \otimes \eta)$, the equation (5.2) becomes:

$$\overline{S}(X_1, X_2) = -(1 - \alpha_2)g(X_1, \phi X_2) + (\alpha_1 - \lambda)g(X_1, X_2) + (\alpha_1 - \mu)\eta(X_1)\eta(X_2) \tag{6.10}$$

In particular, $X_1{=}\xi$, we obtain

$$\overline{S}(X_1, \xi) = -(\lambda - \mu)\eta(X_1). \tag{6.11}$$

The Ricci operator $\overline{Q}$ is used in this scenario is defined by $g(\overline{Q}X_1, X_2){=}\overline{S}(X_1, X_2)$ has the expression

$$\overline{Q}X_1 = -(1 - \alpha_2)\phi X_1 + (\alpha_1 - \lambda)X_1 + (\alpha_1 - \mu)\eta(X_1)\xi. \tag{6.12}$$

We will now apply the earlier findings to η-RS.

Proposition 6.2. *Let* (M, ϕ, ξ, η, g) *be a Lorentzian metric PS manifold with GSM connection. Let us suppose that the symmetric* $(0, 2)$*-tensor field* $h{=}Ł_\xi g + 2\overline{S} + 2\mu\eta \otimes \eta$ *is equivalent to the GSM connection associated with* g*. Then* $(g, \xi, 2 + \frac{1}{2}h(\xi, \xi), \mu)$ *yields an* η*-RS.*

Proof. . Now, we evaluate

$$h(\xi, \xi) = Ł_\xi g(\xi, \xi) + 2\overline{S}(\xi, \xi) + 2\mu\eta(\xi)\eta(\xi) = -4 + 2\lambda, \tag{6.13}$$

so $\lambda{=}2 + \frac{1}{2}h(\xi, \xi)$. From (6.9) we conclude that $h(X_1, X_2) = 2(\lambda - 2)g(X_1, X_2)$, for any X_1, $X_2 \in \Gamma(TM)$. Therefore $Ł_\xi g + 2\overline{S} + 2\mu\eta \otimes \eta{=}2(\lambda - 2)g$. □

The equation $Ł_\xi g + 2\overline{S} - h(\xi, \xi)g = 0$ holds when $\mu = 0$. Based on the given information, it can be inferred that we deduce.

Corollary 6.3. *In the context of a Lorentzian metric PS manifold* (M, ϕ, ξ, η, g) *equipped with a GSM connection, it is observed that the symmetric* $(0, 2)$*-tensor field* h *can be expressed as* $L_\xi g + 2\overline{S}$ *and is parallel with respect to the GSM connection associated with* g*. In this scenario, the equation (5.1) defines an RS when* μ *is equal to zero.*

Conversely, we shall study the consequences of the existence of η-RS on a Lorentzian metric PS manifold with GSM connection. From 6.10, we give the conclusion:

Proposition 6.4. *If equation (5.1) defines an* η*-Ricci soliton on a Lorentzian metric PS manifold* (M, ϕ, ξ, η, g) *with GSM, then* (M, g) *is quasi-Einstein.*

Remember that the manifold is known as quasi-Einstein [7] if the Ricci curvature tensor field S is a linear combination (with real scalars λ, μ) respectively, with $\mu{\neq}0$ of g and the tensor product of a non-zero 1-form η satisfying $\eta{=}g(X_1, \xi)$, for ξ a unit vector field and respectivey, Einstein [7] if S is collinear with g.

Remark 6.5. *Specifically, the existence of an η- RS on a Lorentzian metric PS manifold with GSM connection which is Ricci symmetric (i.e. $\overline{\nabla} S = 0$) implies that M is Einstein manifold. The class of Ricci symmetric manifold represents an extension of class of Einstein manifold to which the locally symmetric manifold also belong (i.e. $\overline{\nabla} R = 0$). The condition $\overline{\nabla} S = 0$ implies $\overline{R}.\overline{S} = 0$ and the manifolds satisfying this condition are called Ricci semi-symmetric [8].*

In what follows we shall consider η-RS requiring for the curvature to satisfy

$$\overline{R}(\xi, X_1).\overline{S} = 0, \quad \text{and} \quad \overline{S}.\overline{R}(\xi, X_1) = 0.$$

7. η-RS on a Lorentzian metric PS manifold with GSM connection satisfying $\overline{R}(\xi, X_1).\overline{S} = 0$

We are now considering a Lorentzian metric PS manifold with a GSM connection $\overline{\nabla}$ fulfilling the criteria

$$\overline{S}(\overline{R}(\xi, X_1)X_2, X_3) + \overline{S}(X_2, \overline{R}(\xi, X_1)X_3) = 0,$$

for any X_1, $X_2 \in \Gamma(TM)$. Replacing the expression of $\overline{S}$ from (6.10) and from the symmetries of $\overline{R}$ we obtain

$$\begin{aligned}
&(1-\alpha_2)(1-\alpha_2+\alpha_2^2)[\eta(X_1)g(X_1, \phi X_3) + \eta(X_3)g(X_2, \phi X_1)] - \alpha_1(1-\alpha_2^2)[\eta(X_2) \\
&g(X_1, X_3) + \eta(X_3)g(X_2, X_1)] - 2\alpha_1(1-\alpha_2)^2\eta(X_1)\eta(X_2)\eta(X_3) - \alpha_1(\alpha_1-\lambda)[g(\phi X_1, \\
&X_2)\eta(X_3) + g(\phi X_1, X_3)\eta(X_2)] + (\alpha_1-\lambda)(1-\alpha_2)[g(X_1, X_2)\eta(X_3) + g(X_1, X_3)\eta(X_2)] \\
&- 2\alpha_2^2(\alpha_1-\lambda)\eta(X_1)\eta(X_2)\eta(X_3) - (\alpha_1-\lambda)(1-\alpha_2+\alpha_2^2)[\eta(X_2)g(X_2, X_3) + \eta(X_3) \\
&g(X_2, X_1)] + \alpha_1(\alpha_1-\lambda)(1-\alpha_2)[\eta(X_2)g(\phi X_1, X_3) + \eta(X_3)g(X_2, \phi X_1)] + \alpha_1(\alpha_1-\mu) \\
&[\eta(X_3)g(\phi X_1, X_2) + \eta(X_2)g(\phi X_1, X_3)] - (\alpha_1-\mu)(1-\alpha_2)[\eta(X_3)g(X_1, X_2) + \eta(X_2) \\
&g(X_1, X_3) - 2(\alpha_1-\mu)(1-\alpha_2)\eta(X_1)\eta(X_2)\eta(X_3) = 0.
\end{aligned}$$

for any X_1, X_2, $X_3 \in \Gamma(TM)$.

For $X_3 = \xi$ we have

$$\begin{aligned}
&\{(\alpha_1-\lambda)\alpha_2^2 + (\alpha_1-\mu)(1-\alpha_2) + \alpha_1(1-\alpha_2)^2\}g(X_1, X_2) + \{\alpha_1(1-\alpha_2)^2 + \alpha_2^2(\alpha_1-\lambda) \\
&+ (\alpha_1-\mu)(1-\alpha_2)\}\eta(X_1)\eta(X_2) + \{-(1-\alpha_2)(1-\alpha_2+\alpha_2^2) + \alpha_1(\alpha_1-\lambda)(1-\alpha_2) + \alpha_1 \\
&(\alpha_1-\mu)\}g(X_2, \phi X_1) = 0
\end{aligned}$$

for any X_1, $X_2 \in \Gamma(TM)$. Therefore, we can assert the subsequent theorem.

Proposition 7.1. *If a Lorentzian metric PS manifold with a GSM connection* $\overline{\nabla}$, (g,ξ,λ,μ) *is an* η*-RS on* M *and it satisfies* $\overline{R}(\xi,X_1)\overline{S}=0$, *a generalized* η*-Einstein manifold is the result.*

For $\mu=0$, We conclude:

Corollary 7.2. *On a Lorentzian metric PS manifold with a GSM connection satisfying* $\overline{R}(\xi,X_1).\overline{S}=0$, *With the potential vector field* ξ, *there is no RS.*

8. η-RS on Lorentzian metric PS manifold with GSM connection satisfying $\overline{S}.\overline{R}(\xi,X_1)=0$

In this part, we examined the Lorentzian metric PS manifold along with a GSM connection $\overline{S}$ fulfilling the criteria

$$\begin{aligned}&\overline{S}(X_1,\overline{R}(X_2,X_3)X_4)\xi-\overline{S}(\xi,\overline{R}(X_2,X_3)X_4)X_1+\overline{S}(X_1,X_2)\overline{R}(\xi,X_3)X_4-\overline{S}(\xi,X_2)\\&\overline{R}(X_1,X_3)X+\overline{S}(X_1,X_3)\overline{R}(X_2,\xi)X_4-\overline{S}(\xi,X_3)\overline{R}(X_2,X_1)X_4+\overline{S}(X_1,X_4)\\&\overline{R}(X_2,X_3)\xi-\overline{D}(\xi,X_4)\overline{R}(X_2,X_3)X_1=0,\end{aligned}\tag{8.1}$$

for any X_1, X_2, X_3, $X_4\in\Gamma(TM)$.
When the inner product with ξ is taken into account, equation (8.1) becomes

$$\begin{aligned}&\overline{S}(X_4,\overline{R}(X_2,X_3)X_4)\eta(\xi)-\overline{S}(\xi,\overline{R}(X_2,X_3)X_4)\eta(X_1)+\overline{S}(X_1,X_2)\eta(\overline{R}(\xi,X_3)X_4)\\&-\overline{S}(\xi,X_2)\eta(\overline{R}(X_1,X_3)X_4)+\overline{S}(X_1,X_3)\eta(\overline{R}(X_2,\xi)X_4)-\overline{S}(\xi,X_3)\eta(\overline{R}(X_2,X_1)X_4)\\&+\overline{S}(X_1,X_4)\eta(\overline{R}(X_2,X_3)\xi)-\overline{S}(\xi,X_4)\eta(\overline{R}(X_2,X_3)X_1=0\end{aligned}\tag{8.2}$$

for any X_1, X_2, X_3, $X_4\in\Gamma(TM)$.
For X_4=ξ, using the (4.6), (4.7), (4.9) and (6.10) in (8.2), we get

$$\begin{aligned}&\{(1-\alpha_2)(1-\alpha_2+\alpha_2^2)+\alpha_1(\alpha_1-\lambda)(1-\alpha_2)-(\lambda-\mu)\alpha_1\}[g(X_1,\phi X_3)\eta(X_2)\\&-g(X_1,\phi X_2)\eta(X_3)]\{\alpha_1(1-\alpha_2)^2+(\alpha_1-\lambda)(1-\alpha_2+\alpha_2^2)\\&-(\lambda-\mu)(1-\alpha_2)\}[\eta(X_3)g(X_1,X_2)-\eta(X_2)g(X_1,X_3)]=0\end{aligned}$$

for any X_1, X_2, X_3, $X_4\in\Gamma(TM)$. Therefore, we can assert the subsequent theorem.

Proposition 8.1. *If* (M,ϕ,ξ,η,g) *is a Lorentzian metric PS manifold with a GSM connection,* (g,ξ,λ,μ) *is an* ζ*-RS on* M *and it satisfies* $\overline{S}.\overline{R}(\xi,X_1)=0$. *Then*

$$\{(1-\alpha_2)(1-\alpha_2+\alpha_2^2)+\alpha_1(\alpha_1-\lambda)(1-\alpha_2)-(\lambda-\mu)\alpha_1\}=0,\tag{8.3}$$

and

$$\{\alpha_1(1-\alpha_2)^2+(\alpha_1-\lambda)(1-\alpha_2+\alpha_2^2)-(\lambda-\mu)(1-\alpha_2)\}=0\tag{8.4}$$

Setting μ=0 yields

$$\{(1-\alpha_2)(1-\alpha_2+\alpha_2^2)+\alpha_1(\alpha_1-\lambda)(1-\alpha_2)-\lambda\alpha_1\}=0,$$
$$\{\alpha_1(1-\alpha_2)^2+(\alpha_1-\lambda)(1-\alpha_2+\alpha_2^2)-\lambda(1-\alpha_2)\}=0,$$

Consequently, we are able to determine the following consequence:

Corollary 8.2. *On a Lorentzian metric PS manifold with a GSM connection satisfying* $\overline{S}.\overline{R}(\xi, X_1)$=0*, the RS defined by (5.1),* μ=0*, we obtain the following conditions:*
(i). $\{(1-\alpha_2)(1-\alpha_2+\alpha_2^2)+\alpha_1(\alpha_1-\lambda)(1-\alpha_2)-\lambda\alpha_1\}=0$
and $\{\alpha_1(1-\alpha_2)^2+(\alpha_1-\lambda)(1-\alpha_2+\alpha_2^2)-\lambda(1-\alpha_2)\}=0,$
(ii). If $\alpha_1=\lambda$ *which is either shrinking or expanding and steady.*

9. Conclusion

We investigate some interesting results on ζ-RS on Lorentzian metric PS manifold endowed GSM connection and give two examples on such manifold. We also discuss Ricci and η-RS with GSM connection of type (α_1, α_2). Certain curvature conditions on such manifolds are derived.

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