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Regularity of generalized lower-C^2 functions in Hadamard manifolds

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Abstract. In this paper we introduce and study the classes of lower-C^2 and upper-C^2 functions on Hadamard manifolds. As applications, we investigate their regularity properties and analyze stationary points of associated minimization problems. Our results show that the class of lower-C^2 functions is regular in this setting, while the class of upper-C^2 functions is not.

Keywords: Lower-C^2 function, regular function, stationary point, Hadamard manifold.

1. **Introduction**

The linear structure plays a crucial role in employing conventional tools of smooth and nonsmooth analysis. At the same time, many optimization problems arising in various applications cannot be posed in linear spaces and instead require a Riemannian manifold structure for their formalization and study. There exist functions that are non-Lipschitz and non-convex in linear spaces but become Lipschitz and convex on Riemannian manifolds when endowed with suitable Riemannian metrics, see [19, 33, 44] and references therein. Nevertheless, in recent years, the extension of concepts and techniques from linear spaces

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to Riemannian manifolds has received considerable attention; see, for example, [5, 10, 11, 17, 20, 21, 29, 35, 37, 42]. Convexity of functions plays a vital role in nonlinear analysis and optimization theory. Since convexity is often not enjoyed by the real problems, various approaches to the generalization have been proposed to relax the convexity assumption. The class of subsmooth functions, which includes lower-C^1 and lower-C^2 functions, represents a major generalization in this field. Lower-C^1 functions were first introduced by Spingarn in [41] who shows that these functions are semi-smooth and Clarke regular, and that are characterized by sub-monotonicity. Following Spingarn's work, the class of lower-C^2 functions on $\mathbb{R}^n$ was introduced by Rockafellar in [40]. In particular, he established that such functions can be locally expressed as a difference $g - \|.\|^2$, where g is finite, convex function. Bounkhel in [13] generalized this class to the Hilbert space setting and obtained the same expression as above with g finite, continuous and convex. Since then numerous articles have appeared in the literature reflecting further generalizations and applications in this category, see for example [1, 8, 12, 13, 14, 15, 23, 24, 25, 26, 41] and references therein. Note that the notions and results considered here in the context of Hadamard manifolds are more complicated in comparison to their counterparts on linear spaces and most of the known techniques in linear spaces do not work in this nonlinear setting. For instance, even for a convex subset S, the behavior (differentiability and convexity) of the distance function d_S in a Riemannian manifold M, under the effect of the curvature of M, has different properties to linear spaces, see [29, 32]. The class of lower-C^1 functions in the framework of Riemannian manifolds was introduced in [31], where several important properties were investigated. In this paper we deal with the study of lower-C^2 and upper-C^2 functions in setting of finite dimensional Hadamard manifolds. The organization of the paper is as follows: In Section 2 we state some basic notations and terminology used in this paper. In Section 3 we introduce the concept of lower-C^2 (upper-C^2) functions. Regularity properties and Stationary points of minimization programming problems as applications are investigated in Section 4.

2. **Preliminaries**

In this section, we state some notations, definitions and basic properties of Riemannian geometry which will be used throughout the paper. We refer [30, 43, 44] and references therein for the standard material of differential geometry. Throughout the paper, all manifolds are assumed to be of finite-dimensional. For a Riemannian manifold M, of dimension m, we denote by T_xM the tangent space at $x \in M$. The tangent bundle of a smooth manifold M is defined as the disjoint union

$$TM := \{(p, v) : p \in M,\ v \in T_pM\},$$

which is itself a smooth manifold of dimension $2m$. The Riemannian metric denoted by $\langle , \rangle$, and corresponding norm by $\| \ \|$. By minimizing the length functional defined by $l(\gamma) = \int_a^b \|\gamma^{'}(t)\| dt$ over the set of all piecewise C^1 curves γ joining x to y, we obtain the Riemannian distance $d(x,y)$. A vector field X on M is said to be parallel along a curve γ if $\nabla_{\gamma'} X = 0$, where ∇ is the Levi-Cività connection on M. If $\nabla_{\gamma'}\gamma' = 0$ then γ is said to be a geodesic. Recall that a geodesic from x to y in M is said to be minimal if its length equals $d(x,y)$. A Riemannian manifold is complete if for any $x \in M$ all geodesics emanating from x are defined on $\mathbb{R}$. By the Hopf-Rinow theorem, we know that if M is complete and connected and finite-dimensional, then any pair of points in M can be joined by a minimal geodesic. A subset S of a Riemannian manifold M is called convex if the unique minimal geodesic joining any two points x, $y \in S$ belongs entirely to S. For every $t \in [0,1]$, ∇ induces an isometry, relative to $\langle , \rangle$, $P^{x_2}_{x_1,\gamma} : T_{x_1}M \to T_{x_2}M$, the so-called parallel transport along the minimal geodesic γ from $\gamma(0) = x_1$ to $\gamma(1) = x_2$. Recall that a Hadamard manifold is a simply connected and complete Riemannian manifold, with non- positive sectional curvature. When M is a Hadamard manifold, $\gamma(t) = \exp_{x_1}(t \exp^{-1}_{x_1} x_2)$ so we denote that mapping by $P^{x_2}_{x_1}$. It is well known that, for any x_0 in a Hadamard manifold M, the map $x \mapsto d^2(x, x_0)$ belongs to $C^\infty(M)$ and $\text{grad} d^2(x, x_0) = -2\exp^{-1}_x x_0, \quad \text{for all } y \in M.$

Recall that a geodesic triangle $\triangle(x_1x_2x_3)$ consists of three points, called vertices and three minimal geodesics joining them. By [6, p. 918] and Proposition 4.5 in [43, p. 223] the following theorem holds.

Theorem 2.1. *Given a geodesic triangle $\triangle(x_1x_2x_3)$ in a Hadamard manifold M. Denote by γ_i the geodesic joining x_i to x_{i+1} for $i = 1,2,3(\text{mod}3)$. Then it holds that*

$$(i) \quad d^2(x_1,x_2) + d^2(x_2,x_3) - 2\langle \exp^{-1}_{x_2} x_1, \exp^{-1}_{x_2} x_3\rangle \le d^2(x_1,x_3),$$

$$(ii) \quad \langle \exp^{-1}_{x_2} x_1, \exp^{-1}_{x_2} x_3\rangle + \langle \exp^{-1}_{x_3} x_1, \exp^{-1}_{x_3} x_2\rangle \ge d^2(x_2,x_3).$$

A function $f : S \to \mathbb{R}$ is said to be convex if for every x, $y \in S$ we have

$$f(\gamma(t)) \le (1-t)f(x) + tf(y) \text{ for all } t \in [0,1],$$

where γ is the unique minimal geodesic joining x to y. A function $f : M \to \mathbb{R}$ is said to be locally Lipschitz on M, if for every $x \in M$ there exist $K \ge 0$ and a neighborhood U containing x such that

$$|f(z) - f(y)| \le Kd(z,y), \text{ for all } z, y \in U.$$

Definition 2.2. Let $f : M \to \mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative of f at $x \in M$ in the direction $v \in T_xM$, denoted by $f^\circ(x;v)$, is defined as

$$f^\circ(x;v) := \limsup_{y \to x, t \downarrow 0} \frac{f \text{o} \varphi^{-1}\big(\varphi(y) + td\varphi(x)v\big) - f(y)}{t},$$

that is, $f^\circ(x;v)=(fo\varphi^{-1})^\circ(\varphi(x);d\varphi(x)v)$, where (U,φ) is a chart at x. Note that this definition does not depend on charts. By considering the exponential chart at x we get $f^\circ(x;v)=(fo\varphi^{-1})^\circ(0_x;v)$. The generalized gradient (or Clarke subdifferential) of f at $x\in M$ is the subset $\partial_c f(x)$ of T_xM defined by

$$\partial_c f(x) := \{\zeta\in T_xM:\ f^\circ(x;v)\geq\langle\zeta,v\rangle_x, \text{for all } v\in T_xM\}.$$

We can see that $\partial_c f(x)=\partial_c(f\text{o}\exp_x)(0_x)$ (see [27]).

3. The lower-C^2(upper -C^2) functions

Lower-C^2 functions in context of Hadamard manifolds defined in this section.

Definition 3.1. Let M be a Hadamard manifold and $U\subseteq M$ be an open set. Then, the function $f:U\to\mathbb{R}$ is said to be a lower-C^2 function (upper-C^2 function) if for every $\bar{x}\in U$ there exist a neighborhood V of $\bar{x}$, some compact subset T of a topological space and a continuous function $F:V\times T\to\mathbb{R}$ such that

$$f(x)=\max_{t\in T}F(x,t)\big(f(x)=\min_{t\in T}F(x,t)\big),\ \text{for all } x\in V,$$

in which $x\mapsto F(x,t)$ is a C^2 function for all $t\in T$.

As a first example consider an open subset U of a Hadamard manifold M and pick $T=\{1,2,...,n\}$. Suppose that $f_t:U\to\mathbb{R}$, $t\in T$ are C^2 functions. Define the $F:U\times T\to\mathbb{R}$ as $F(x,t):=f_t(x)$. Now, $f:U\to\mathbb{R}$ defined by

$$f(x)=\max_{t\in T}f_t(x)\big(f(x)=\min_{t\in T}f_t(x)\big),$$

is a lower-C^2 (upper-C^2) function.

The following result states linear combination of lower-C^2 functions is a lower-C^2 function. The proof is similar to that in the linear space setting, as given in [40, p. 452].

Proposition 3.2. *Let M be a Hadamard manifold and U be an open subset of M. Suppose for every $i=1,\dots,m$ the function $f_i:U\to\mathbb{R}$ is lower-C^2 and $f(x):=\sum_{i=1}^m\lambda_i f_i$ for $\lambda_i\geq 0$. Then, f is lower-C^2 on U .*

The following theorem generalizes one side of theorem 10.33 in [39] in the context of Hadamard manifolds.

Theorem 3.3. *Let M be a Hadamard manifold and U be an open subset of M and $f:U\to\mathbb{R}$ be a function. Suppose that for every $\bar{x}\in U$ there is some open convex subset $V\subseteq U$ around $\bar{x}$ such that $f(x)=g(x)-h(x)$ for every $x\in V$, where $g:V\to\mathbb{R}$ is a finite convex function and $h:V\to\mathbb{R}$ is a C^2 function. Then f is a lower-C^2 function.*

Proof. Suppose that $\bar{x} \in U$ and f is expressed as $f(x) = g(x) - h(x), x \in V$, where g and h are convex and C^2 functions on V, respectively. Since g is convex so for every $x, y \in V$ and every $v \in \partial g(y)$ we have

$$g(x) \geq g(y) + \langle v, \exp_y^{-1} x \rangle. \tag{3.1}$$

Pick a compact set $T \subset V$ containing $\bar{x}$ and define

$$S := \big\{ s = (y, v) \in TM \bigm| y \in T,\ v \in \partial g(y) \big\},$$

where TM is the tangent bundle of M. For $s = (y, v) \in S$, set

$$a(s) := v, \qquad \alpha(s) := g(y).$$

Then for every $x \in T$,

$$\langle a(s), \exp_y^{-1} x \rangle + \alpha(s) = \langle v, \exp_y^{-1} x \rangle + g(y) \leq g(x).$$

Taking the supremum over $s \in S$ yields

$$\sup_{s \in S} \big\{ \langle a(s), \exp_{y(s)}^{-1} x \rangle + \alpha(s) \big\} \leq g(x).$$

Moreover choosing $y = x$ and any $v \in \partial g(x)$, the inequality (3.1) gives us

$$\langle v, \exp_x^{-1} x \rangle + g(x) = g(x),$$

and therefore equality holds so

$$g(x) = \sup_{s \in S} \big\{ \langle a(s), \exp_{y(s)}^{-1} x \rangle + \alpha(s) \big\}, \qquad \forall x \in T.$$

Since g is finite and convex on V, by Theorem 4.6 in [44, p. 74] $\partial g(y)$ is a convex and compact subset of $T_y M$. Because T is compact, it follows that $S \subset TM$ is compact. The mappings $s \mapsto a(s)$ and $s \mapsto \alpha(s)$ are continuous on S. Also for each $(y, v) \in S$, the function

$$x \mapsto \langle v, \exp_y^{-1} x \rangle + \alpha$$

is of class C^2 on T. Hence, g is a lower–C^2 function on T. On the other hand h is a C^2 function, so h is a lower-C^2 function. Therefore, by Proposition 3.2, $f = g - h$ is a lower-C^2 function. □

In view of the two results established above, we infer that any convex (respectively, concave) function defined on an open convex subset of a Hadamard manifold is lower–C^2 (respectively, upper–C^2). Furthermore, the example presented below demonstrates that the associated Moreau envelopes are themselves lower–C^2 functions.

Example 3.4. For a lower semicontinuous convex function $f : M \to \bar{\mathbb{R}}$ on Hadamard manifold M consider the Moreau envelope defined by

$$f_\lambda(x) := \inf_{y \in M} \Big\{ f(y) + \frac{1}{2\lambda} d^2(x, y) \Big\}.$$

By Lemma 3.2 in [3, p. 334] there are some $x_0 \in M$ and $c > 0$ such that $f(x) \geq -\frac{c}{2}(1 + d^2(x, x_0))$ for every $x \in M$. Fix $\lambda \in (0, \frac{1}{2c})$ and for every $x \in dom f$ define $h(x) := -f_\lambda(x)$. Then, it follows from proposition 2.1 in [3, p. 328] that there exists $\rho > 0$ such that $h(x) = \max_{y \in \bar{B}(x,\rho)} F(x, y)$ where

$$F(x, y) := -f(y) - \frac{1}{2\lambda} d^2(x, y).$$

We see that $F(x, y)$ is C^∞ in x and $T := \bar{B}(x, \rho)$ is compact, so h is lower-C^2 function, see also [40, p. 450].

The following example provides a lower-C^2 (upper-C^2) function within the framework of a Hadamard manifold.

Example 3.5. Consider the Poincar*é* upper half-plane model

$$\mathbb{H} = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_2 > 0\},$$

endowed with the Riemannian metric defined for every $(x_1, x_2) \in \mathbb{H}$ by

$$g_{ij}(x_1, x_2) := \frac{1}{x_2^2} \delta_{ij}, \quad \text{for } i, j = 1, 2.$$

Then $(\mathbb{H}, g)$ is a Hadamard manifold with sectional curvature -1, and the geodesics in $\mathbb{H}$ are the semi-lines and the semicircles orthogonal to the line $x_2 = 0$. The Riemannian distance between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{H}$ is given by

$$d(x, y) = \cosh^{-1}\left(1 + \frac{(y_1 - x_1)^2 + (y_2 - x_2)^2}{2x_2y_2}\right). \tag{3.2}$$

The unique minimal geodesic connecting two arbitrary points x, y is given in [44, p. 20] as

$$\gamma_{xy}(t) = \begin{cases} \left(x_1, e^{(1-t)\ln y_2 + t \ln x_2}\right), & x_1 = y_1, \\ \left(b - r \tanh t_{xy}, \frac{r}{\cosh t_{xy}}\right), x_1 \neq y_1, \end{cases}$$

where t_{xy}, b and r, respectively, are defined as follows

$$\begin{aligned} t_{xy} &= (1 - t)\Big(\frac{b - y_1}{r}\Big) + t\Big(\frac{b - x_1}{r}\Big), \\ b &= \frac{y_1^2 + y_2^2 - (x_1^2 + x_2^2)}{2(y_1 - x_1)}, \\ r &= \sqrt{(y_1 - b)^2 + y_2^2}. \end{aligned} \tag{3.3}$$

Moreover,

$$\exp_x^{-1} y = \begin{cases} \left(0, x_2 \ln \frac{y_2}{x_2}\right), & x_1 = y_1, \\ \frac{x_2}{r}\left(\tanh^{-1}\left(\frac{b - x_1}{r}\right) - \tanh^{-1}\left(\frac{b - y_1}{r}\right)\right)(x_2, b - x_1), x_1 \neq y_1. \end{cases} \tag{3.4}$$

Now, define the functions $F : U \times T \to \mathbb{R}$, as $F(x, t) = (x_1 - t)^2 + t^2 x_2^2$ and set $f(x) := \max_{t \in T} F(x, t)$ and $g(x) := \min_{t \in T} F(x, t)$, where $U := \{(x_1, x_2) \in$

$M|\ x_2 > \frac{1}{2}\}$ and $T := [-2, 2]$. It is easy to see that the functions f and g are lower-C^2 and upper-C^2 on U, respectively.

By a similar process to the one in Section 10.35 in [40] the following proposition can be proved.

Proposition 3.6. *Let M and N be two Hadamard manifolds and U be an open convex subset of M. Suppose that $g : U \to \mathbb{R}$ is a lower-C^2 function and $F : N \to M$ be a C^2 function. Then, the function $f = g \circ F$ is a lower-C^2 function on $F^{-1}(U)$.*

The following result, which is an improvement of proposition 10.30 in [39], provides a relationship between C^2, lower-C^2 and upper-C^2 functions.

Proposition 3.7. *Let M be a Hadamard manifold and U an open subset of M with $f : U \to \mathbb{R}$ a real valued function. The function f is a C^2 function if and only if it is simultaneously a lower-C^2 and upper-C^2 function.*

4. **Regularity Properties**

In this section, we extend and modify the corresponding notions of regularity for functions in $\mathbb{R}^n$ from [34, 36], using somewhat different terminology, to the Hadamard manifold setting. It is shown that the class of lower-C^2 functions is regular and the class of upper-C^2 functions is not regular in this context. We also investigate two types of stationary points for optimization problems. Recall that a locally Lipschitz function f is said to be Clarke-regular (or regular) at a point x provided that, for every $v \in T_xM$ its directional (Clarke-directional) derivative coincides with the ordinary convex directional derivative i.e., $f^\circ(x; v) = f'(x; v)$. As a first example, we note that every convex function is regular. Furthermore, every lower-C^2 function is regular, as shown in Theorem 4.1. Additionally, we investigate stationary points of optimizations problems of the the form

$$\min_{x \in S} f(x), \qquad (P)$$

where S is a nonempty convex and compact subset of an open subset U of a Hadamard manifold M and $f : U \to \mathbb{R}$ is an upper-C^2 function. The following theorem deals with some basic regularities properties of a lower-C^2 function.

Theorem 4.1. *Let M be a Hadamard manifold and U be an open convex subset of M. If $f(x) = \max_{t \in T} F(x, t)$ is a lower-C^2 function on U. Then the following assertions hold:*

(a) *f is locally Lipschitz and we have*

$$\partial_c f(x) = conv\{\mathrm{grad}_x F(x, t)|\ t \in I(x)\},\ \text{for all } x \in U.$$

(b) $f'(x;v) = f^\circ(x;v) = \max_{t\in I(x)} \langle \mathrm{grad}_x F(x,t), v\rangle$, for all $v \in T_xM$. *Hence f is Clarke-regular at x, where* $I(x) := \mathrm{argmax}_{t\in T} F(x,t)$.

Proof. The proof of both items (a) and (b) follows from the combining of proposition 3.6, proposition 3.4 in [18] and example 2.6 in [27]. □

In the next theorem, some properties of upper-C^2 functions are introduced.

Theorem 4.2. *Let M be a Hadamard manifold and U be an open subset of M. If $f(x) = \min_{t\in T} F(x,t)$ is an upper-C^2 function on U. Then the following assertions hold:*

(a) *f is locally Lipschitz and we have*

$$\partial_c f(x) = conv\{\mathrm{grad}_x F(x,t) |\ t \in I(x)\}, \text{ for all } x \in U.$$

(b) $f'(x;v) = \min_{t\in I(x)} \langle \mathrm{grad}_x F(x,t), v\rangle$, for all $v \in T_xM$, *where* $I(x) := \mathrm{argmin}_{t\in T} F(x,t)$.

Proof. (a) Pick $\varphi := -f$ and fix $v \in T_xM$. Since φ is a lower-C^2 function by part (a) of Theorem 4.1 it holds that $\varphi^\circ(x;v) = \varphi'(x;v)$. On the other hand

$$\begin{aligned} \varphi'(x;v) &= \lim_{t\downarrow 0} \frac{\varphi(\exp_x tv) - \varphi(x)}{t} \\ &= -\lim_{t\downarrow 0} \frac{-\varphi(\exp_x tv) - (-\varphi(x))}{t} \\ &= -\lim_{t\downarrow 0} \frac{f(\exp_x tv) - f(x)}{t} = -f'(x;v). \end{aligned} \tag{4.1}$$

So $f'(x;v) = -\varphi^\circ(x;v)$. Taking into account that φ is lower-C^2, $\partial_c f(x) = -\partial_c\varphi(x)$, $\varphi(x) = \max_{t\in T}\big(-F(x,t)\big)$ and using part (a) of Theorem 4.1 again, we get

$$\begin{aligned} \partial_c f(x) &= -\partial_c \varphi(x) \\ &= -conv\{-\mathrm{grad}_x F(x,t) |\ t \in I(x)\} \\ &= conv\{\mathrm{grad}_x F(x,t) |\ t \in I(x)\}. \end{aligned} \tag{4.2}$$

(b) Finally we see that

$$\begin{aligned} f'(x;v) &= -\varphi^\circ(x;v) \\ &= -\max_{-\theta\in\partial_c\varphi(x)} \langle -\theta, v\rangle = -\max_{\theta\in\partial_c f(x)} \langle -\theta, v\rangle = \min_{\theta\in\partial_c f(x)} \langle \theta, v\rangle. \end{aligned} \tag{4.3}$$

□

Observe from theorem 4.2 (b) that, unlike lower-C^2 functions the class of upper-C^2 functions may exhibit strict inequality (as we see in the next example), i.e., $f^\circ(x;v) > f'(x;v)$, so it is not regular.

Example 4.3. Let M be the Poincaré upper half-plane and define $U := \{(x_1, x_2) \in M|\ x_2 > \frac{1}{2}\}$ and

$$S := \Big\{(x_1, x_2) \in U | x_1 \geq 0, (x_1 - \sqrt{3})^2 + x_2^2 \geq 4, x_1^2 + x_2^2 \leq 4\Big\}.$$

Now, define the functions $f_i : U \to \mathbb{R}, i \in T := \{1, 2\}$ as

$$f_1(x) = (x_1 - 2)^2 + x_2^2, \qquad f_2(x) = (x_1 + 2)^2 + x_2^2,$$

and set $f(x) := \max_{i\in T} f_i(x)$ and $g(x) := \min_{i\in T} f_i(x)$. Hence f_1 and f_2 are lower-C^2 and upper-C^2 on U respectively. Pick $\bar{x} = (0, 1) \in S$, so $f(\bar{x}) = f_1(\bar{x}) = f_2(\bar{x}) = 4$. Simple computations deduce that

$$\text{grad} f_1(\bar{x}) = (-4, 2),\ \ \text{grad} f_2(\bar{x}) = (4, 2).$$

Set $y = (y_1, y_2) \in S$ and $v := \exp_{\bar{x}}^{-1} y = (v_1, v_2) \in T_{\bar{x}}M$. By easy calculations we have $v_1, v_2 \geq 0$. Now by using Theorem 4.1 we get

$$\begin{aligned}\partial_c f(\bar{x}) &= \text{conv}\{\text{grad} f_1(\bar{x}), \text{grad} f_2(\bar{x})\}\\ &= \{a(-4, 2) + (1 - a)(4, 2)|\ 0 \leq a \leq 1\}\\ &= \{(4 - 8a, 2)|\ 0 \leq a \leq 1\},\end{aligned}$$

and

$$\begin{aligned}&f'(\bar{x}; v) = f^\circ(\bar{x}; v) = \max_{\xi\in\partial_c f(\bar{x})} \langle \xi, (v_1, v_2)\rangle\\ &= \max_{0\leq a\leq 1} \langle (4 - 8a, 2), (v_1, v_2)\rangle = \max_{0\leq a\leq 1} \{(4 - 8a)v_1 + 2v_2\} = 4v_1 + 2v_2.\end{aligned}$$

While by applying Theorem 4.2 we get

$$g'(\bar{x}; v) = \min_{0\leq a\leq 1} \{(4 - 8a)v_1 + 2v_2\} = -4v_1 + 2v_2,$$

$$g^\circ(\bar{x}; v) = \max_{0\leq a\leq 1} \{(4 - 8a)v_1 + 2v_2\} = 4v_1 + 2v_2.$$

Pick $\bar{y} := (\frac{\sqrt{3}}{2}, \frac{\sqrt{13}}{2}) \in S$. Straight calculations by using formulas in example 3.5 gives us, $r = 2,\ b = \sqrt{3}$, so

$$v := \exp_{\bar{x}}^{-1} \bar{y} = \frac{1}{2}\big(\tanh^{-1}\big(\frac{\sqrt{3}}{2}\big) - \tanh^{-1}\big(\frac{\sqrt{3}}{4}\big)\big)\big(1, \sqrt{3}\big) \approx (0.85, 1.47).$$

Therefore, $g^\circ(\bar{x}; v) > 0 > g'(\bar{x}; v)$, hence g is not regular at point $\bar{x}$.

Motivated by definition 3.4 in [9] and [36] we consider two stationary concepts for problem (P) as follows:
- (Clarke stationary point) A point $\bar{x} \in S$ is said to be a C-stationary point for problem (P), if $0 \in \partial_c f(\bar{x}) + N_S(\bar{x})$, where

$$N_S(\bar{x}) := \{\theta \in T_{\bar{x}}M | \langle \theta, \exp_{\bar{x}}^{-1} x\rangle \leq 0 \text{ for all } x \in S\},$$

is the usual normal cone to S at $\bar{x}$.
- (Directional stationary point) A point $\bar{x} \in S$ is said to be a d-stationary point for problem (P) if $f'(\bar{x}; \exp_{\bar{x}}^{-1} x) \geq 0$ for all $x \in S$.

Note that both stationary concepts coincide when f is smooth at the point under consideration: in this case, $\partial_c f(\bar{x}) = \{\mathrm{grad}_x f(\bar{x})\}$.

Remark 4.4. It is known that if f attains a local minimum at $\bar{x}$, then $0 \in \partial_c f(\bar{x})$, so every local minimum point of problem (P) is stationary. However, the Clarke subdifferential may contain a stationary point that is not a local minima. For instance, consider the sets U and S defined in example 4.3 and define the function $f_1 : U \to \mathbb{R}$, $\bar{x} \in M$ as $f_1(x) := -d^2(x, \bar{x})$. Pick $\bar{x} := (0, 1)$. We see that $\partial_c f_1(\bar{x}) = \{0\}$, so $\bar{x}$ is a C-stationary point. A similar phenomena occurs for the function $f_2(x) := -d(x, \bar{x})$, where $\partial_c f_2(\bar{x}) = \mathbb{B}_{\bar{x}}$, with $\mathbb{B}_{\bar{x}}$ being the unite closed ball in $T_{\bar{x}}M$. Note that $f_2^{'}(\bar{x}; v) = -|v|$ for every $v \in T_{\bar{x}}S$, so $\bar{x}$ is not a d-stationary point.

Moreover, the point $\bar{x} \in S$ is a C-stationary point for problem (P) (with f and S defined in Example 4.3), because $f^{\circ}(\bar{x}; v) \geq 0$. However, $\bar{x}$ is not a d-stationary point.

Proposition 4.5. *Let M be a Hadamard manifold and U be an open subset of M. Suppose that S is a nonempty convex and compact subset of U. If $f(x) = \min_{t \in T} F(x, t)$ is an upper-C^2 function on U and $I(x) := \mathrm{argmax}_{t \in T} F(x, t)$. Then the following assertions hold:*

(a) If $\bar{x} \in S$ is a C-stationary point of problem (P) *then,*

$$\max_{x \in S} \langle \mathrm{grad}_x F(\bar{x}, \bar{t}), \exp_{\bar{x}}^{-1} x \rangle = 0, \text{ for at least one } \bar{\mathrm{t}} \in \mathrm{I}(\bar{\mathrm{x}}),$$

(b) If $\bar{x} \in S$ is a d-stationary point of problem (P) *then*

$$\min_{x \in S} \langle \mathrm{grad}_x F(\bar{x}, \bar{t}), \exp_{\bar{x}}^{-1} x \rangle = 0, \text{ for all } \bar{\mathrm{t}} \in \mathrm{I}(\bar{\mathrm{x}}).$$

Proof. (a) Taking into account that $\bar{x} \in S$ is a C-stationary point for problem (P) there exists a vector $\theta \in \partial_c f(\bar{x})$ such that $-\theta \in N_S(\bar{x})$. It follows that $\langle \theta, \exp_{\bar{x}}^{-1} x \rangle \geq 0$ for every $x \in S$. Therefore,

$$\min_{x \in S} \langle \theta, \exp_{\bar{x}}^{-1} x \rangle = 0, \text{ for at least one } \theta \in \partial_{\mathrm{c}} \mathrm{f}(\bar{\mathrm{x}}). \tag{4.4}$$

Thus, by employing the obtained representation for $\partial_c f(\bar{x})$ in Theorem 4.1 part (a) we deduce that

$$\min_{x \in S} \langle \mathrm{grad}_x F(\bar{x}, \bar{t}), \exp_{\bar{x}}^{-1} x \rangle = 0, \text{ for at least one } \bar{\mathrm{t}} \in \mathrm{I}(\bar{\mathrm{x}}), \tag{4.5}$$

which is required.

(b) Since $\bar{x} \in S$ is a d-stationary point for problem (P), $f'(\bar{x}; \exp_{\bar{x}}^{-1} x) \geq 0$ for all $x \in S$. So in view of part (a) in Theorem 4.2 we get

$$\min_{x \in S} \langle \theta, \exp_{\bar{x}}^{-1} x \rangle = 0, \text{ for all } \theta \in \partial_{\mathrm{c}} \mathrm{f}(\bar{\mathrm{x}}). \tag{4.6}$$

Therefore,

$$\min_{x \in S} \langle \mathrm{grad}_x F(\bar{x}, \bar{t}), \exp_{\bar{x}}^{-1} x \rangle = 0, \text{ for all } \bar{\mathrm{t}} \in \mathrm{I}(\bar{\mathrm{x}}). \tag{4.7}$$

□

Finally the next result shows that if a d-stationary point of (P) lies in the interior of S, then f is smooth at this point.

Proposition 4.6. *Let M be a Hadamard manifold and U be an open subset of M. Suppose that S is a nonempty convex and compact subset of U. If $\bar{x} \in \operatorname{int}(S)$ be a d-stationary point of* (P), *where the function f is lower$-C^2$, then $\partial_c f(\bar{x}) = \{0\}$.*

Proof. Pick $\theta \in \partial_c f(\bar{x})$. There exits $\varepsilon > 0$ such that for every

$$x := \exp_{\bar{x}}(-\varepsilon \frac{\theta}{||\theta||}) \in S.$$

Since $\bar{x}$ is a d-stationary point, by Proposition 4.5 we get

$$\langle \theta, \exp_{\bar{x}}^{-1} x \rangle \geq 0.$$

Thus

$$0 \leq \langle \theta, \exp_{\bar{x}}^{-1} x \rangle = \langle \theta, \exp_{\bar{x}}^{-1} \exp_{\bar{x}}(-\varepsilon \frac{\theta}{||\theta||}) \rangle = \langle \theta, -\varepsilon \frac{\theta}{||\theta||} \rangle = -\varepsilon ||\theta||,$$

hence $\theta = 0$ which is required. □

5. Conclusion

In this work, we introduced the classes of lower-C^2 and upper-C^2 functions in the setting of Hadamard manifolds. Within this framework, we analyzed their regularity properties and investigated the characterization of stationary points of minimization problems. Our results established that the class of lower-C^2 functions is regular, whereas the class of upper-C^2 functions is not.

Several natural directions remain open for future research. One avenue is to extend the results presented here to more general Riemannian manifolds, where additional geometric challenges may arise. Another is to design and analyze suitable algorithms for effectively computing stationary points of problem (P). Moreover, the complete characterization of lower-C^2 functions in this setting remains an open problem of particular importance.

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