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Ruled surfaces in a strict Walker 3-manifold

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Abstract. In this paper, we define and construct the ruled surfaces in a three-dimensional strict Walker manifold. We study the geometric properties of these families of surfaces. We give an example to illustrate our main results.

Keywords: Ruled surfaces, curves, mean curvature, Gauss curvature, Walker manifolds.

1. **Introduction**

Ruled surfaces in a three-dimensional manifold have important applications in physics and engineering and are essential to differential geometry. Rules surfaces are a basic type of surfaces in differential geometry that are defined by the constraint that at least one straight line, called a ruling, lies entirely on the surface through each point on the surface. These surfaces are useful in physics,

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engineering, and architecture because of their simple geometric structure and ease of construction. Many people studied ruled surfaces, for example, in [7], the author classifies the ruled surfaces in Euclidean and Minkowski spaces.

Ruled surfaces in Walker 3-manifold offer unique geometric properties when compared to their Euclidean counterparts, particularly when classified by spacelike, timelike, or lightlike rulings.

The study of ruled surfaces of a given ambient space is a natural and interesting problem. A surface Σ in M is said to be ruled if every point of Σ is on (a open geodesic segment) in M that lies in Σ (see [8]). Locally, a ruled surface is made by a one-parameter family of geodesic segments [3]. Several authors have studied problems on ruled surfaces (see [7, 11]).

The paper is organised as follows: in Section 2, we recall some preliminary results for the Walker manifold (M, g_f^ϵ). In Section 3, we give some basic formulas for immersed surfaces in (M, g_f^ϵ), and we construct the two families of ruled surfaces in (M, g_f^ϵ) which are used in the main result. In the last Section, we give a classification of ruled surfaces in a strict Walker 3-manifold.

2. Three Dimensional Walker Manifolds

A pseudo-Riemannian n-manifold is classified as a Walker manifold if it supports a null parallel r-plane field, where $r \leq \frac{n}{2}$. Following A. G. Walker's seminal work on metric canonical forms [10], specific coordinates can be adapted to this parallel plane field. In the three-dimensional case, the metric of a Walker manifold (M, g_f^ϵ) relative to coordinates (x, y, z) is defined by:

$$g_f^\epsilon = dx \circ dz + \epsilon dy^2 + f(x, y, z) dz^2. \tag{2.1}$$

The matrix representation of this metric and its inverse are given by:

$$g_f^\epsilon = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \epsilon & 0 \\ 1 & 0 & f \end{pmatrix}, \quad (g_f^\epsilon)^{-1} = \begin{pmatrix} -f & 0 & 1 \\ 0 & \epsilon & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

where $f(x, y, z)$ is a smooth function and $\epsilon = \pm 1$. The parallel degenerate line field is defined by $D = \text{Span}\{\partial_x\}$. More information about Walker manifolds can be found in [1, 4, 5].

2.1. Connection and Curvature.

Direct computation shows that the components of the Levi-Civita connection for the metric (2.1) are:

$$\begin{aligned} \nabla_{\partial_x} \partial_z &= \frac{1}{2} f_x \partial_x, \quad \nabla_{\partial_y} \partial_z = \frac{1}{2} f_y \partial_x, \\ \nabla_{\partial_z} \partial_z &= \frac{1}{2}(f f_x + f_z)\partial_x + \frac{1}{2} f_y \partial_y - \frac{1}{2} f_x \partial_z \end{aligned} \tag{2.2}$$

where $\{\partial_x, \partial_y, \partial_z\}$ are the standard coordinate vector fields. For a strict Walker manifold (where $f(x, y, z) = f(y, z)$), the connection simplifies to:

$$\nabla_{\partial_y}\partial_z = \frac{1}{2}f_y\partial_x, \quad \nabla_{\partial_z}\partial_z = \frac{1}{2}f_z\partial_x - \frac{\epsilon}{2}f_y\partial_y. \tag{2.3}$$

The non-zero components of the curvature tensor R are:

$$\begin{aligned}
R(\partial_x, \partial_z)\partial_x &= \frac{1}{2}f_{xx}\partial_x, \quad R(\partial_x, \partial_z)\partial_y = \frac{1}{2}f_{xy}\partial_x,\\
R(\partial_y, \partial_z)\partial_y &= -\frac{1}{2}f_{yy}\partial_x, \quad R(\partial_y, \partial_z)\partial_x = \frac{1}{2}f_{xy}\partial_x,\\
R(\partial_x, \partial_z)\partial_z &= \frac{1}{2}ff_{xx}\partial_x - \frac{\epsilon}{2}ff_{xy}\partial_y - \frac{1}{2}ff_{xx}\partial_z,\\
R(\partial_y, \partial_z)\partial_z &= \frac{1}{2}ff_{xy}\partial_x - \frac{\epsilon}{2}f_{yy}\partial_y - \frac{1}{2}f_{xy}\partial_z.
\end{aligned}$$

When a null parallel vector field exists (i.e., $f = f(y, z)$), the Christoffel symbols and curvature components further reduce:

$$\Gamma^1_{23} = \Gamma^1_{32} = \frac{1}{2}f_y, \quad \Gamma^1_{33} = \frac{1}{2}f_z, \quad \Gamma^2_{33} = -\frac{\epsilon}{2}f_y \tag{2.4}$$

and

$$R(\partial_y, \partial_z)\partial_y = -\frac{1}{2}f_{yy}\partial_x, \quad R(\partial_y, \partial_z)\partial_z = -\frac{\epsilon}{2}f_{yy}\partial_y. \tag{2.5}$$

2.2. **Vector Product.** Let (e_1, e_2, e_3) be the canonical basis of $\mathbb{R}^3$. For vectors u, v in M, the vector product $u \times v$ relative to g^ϵ_f is defined by:

$$g^\epsilon_f(u \times v, w) = \det(u, v, w) \tag{2.6}$$

for any vector $w \in M$. For $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, the explicit form is:

$$u \times v = \left(\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} - f\begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix}\right) e_1 - \epsilon\begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} e_2 + \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} e_3.$$

3. Fundamental Equations for Semi-Riemannian Submanifolds

Consider a pseudo-Riemannian manifold (M, g) equipped with its unique Levi-Civita connection ∇. For convenience, the metric g is also represented by the inner product $\langle\cdot,\cdot\rangle$. Let Σ denote a semi-Riemannian surface embedded in M. We define ξ as a unit normal vector field on Σ with a causal character given by $\varepsilon_1 = \langle\xi, \xi\rangle = \pm 1$.

Let D represent the Levi-Civita connection associated with the induced metric on Σ via the inclusion map $i : \Sigma \hookrightarrow M$. For any tangent vector fields X, Y, Z on Σ, the Gauss and Weingarten formulas are expressed as:

$$\nabla_X Y = D_X Y + h(X, Y)\xi \tag{3.1}$$

$$-\nabla_X \xi = SX. \tag{3.2}$$

In these expressions, h denotes the second fundamental form and S is the shape operator. These two entities are linked by the following identity:

$$g_f^\epsilon(SX,Y) = h(X,Y)\varepsilon_1 = g_f^\epsilon(\nabla_X Y,\xi)\cdot \tag{3.3}$$

Furthermore, if R^M and R represent the curvature tensors of (M,g) and (Σ, i^*g) respectively, they satisfy the Gauss and Codazzi equations:

$$\begin{aligned}\langle R^M(X,Y)Z,W\rangle = \langle R(X,Y)Z,W\rangle + \varepsilon_1\Big(&h(Y,Z)h(X,W)\\ &- h(X,Z)h(Y,W)\Big)\end{aligned} \tag{3.4}$$

$$\langle R^M(X,Y)Z,\xi\rangle = \varepsilon_1\Big((\nabla h)(Y,X,Z) - (\nabla h)(X,Y,Z)\Big) \tag{3.5}$$

where the covariant derivative of the second fundamental form is defined as:

$$(\nabla h)(X,Y,Z) = X(h(Y,Z)) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z). \tag{3.6}$$

3.1. **Parametrized Surfaces and Isometric Immersions.** We now consider the classical formulation of equation (3.4) for surfaces in (M, g_f^ϵ) viewed as isometric immersions. Let $\mathcal{D} \subset \mathbb{R}^2$ be an open region where any horizontal or vertical line intersects $\mathcal{D}$ in at most a single interval. A two-parameter map $\varphi : \mathcal{D} \to M$ generates two families of parameter curves: the u-curves $u \mapsto \varphi(u, v_0)$, and the v-curves $v \mapsto \varphi(u_0, v)$.

The partial velocities are given by $\varphi_u = d\varphi(\partial_u)$ and $\varphi_v = d\varphi(\partial_v)$. For a smooth vector field Z defined along φ, its partial covariant derivatives $Z_u = \frac{DZ}{\partial u}$ and $Z_v = \frac{DZ}{\partial v}$ represent the covariant rates of change. In a local coordinate system, Z_u is expanded as:

$$Z_u = \sum_k \left\{ \frac{\partial Z^k}{\partial u} + \sum_{i,j} \Gamma_{ij}^k Z^i \frac{\partial x^j}{\partial u} \right\} \partial_k. \tag{3.7}$$

The second-order partial derivatives φ_{uu} and φ_{vv} represent the accelerations, while the mixed derivative is:

$$\varphi_{uv} = \sum_k \left\{ \frac{\partial^2 x^k}{\partial v \partial u} + \sum_{i,j} \Gamma_{ij}^k \frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial v} \right\} \partial_k\cdot \tag{3.8}$$

3.2. **Geometric Curvatures.** For an isometric immersion φ, the coefficients of the first fundamental form are:

$$E = g_f(\varphi_u,\varphi_u), \quad F = g_f(\varphi_u,\varphi_v), \quad G = g_f(\varphi_v,\varphi_v) \tag{3.9}$$

The second fundamental form coefficients are defined by:

$$L = \varepsilon_1 g_f(\varphi_{uu},\xi), \quad M = \varepsilon_1 g_f(\varphi_{uv},\xi), \quad N = \varepsilon_1 g_f(\varphi_{vv},\xi) \tag{3.10}$$

The mean curvature H is computed as:

$$H = \frac{\varepsilon_1}{2}\left(\frac{LG - 2MF + NE}{EG - F^2}\right). \tag{3.11}$$

The sectional curvature K of the surface φ relates to the sectional curvature K^M of the ambient manifold via:

$$K(\varphi_u, \varphi_v) = K^M(\varphi_u, \varphi_v) + \varepsilon_1 \frac{LN - M^2}{EG - F^2} \tag{3.12}$$

The Gauss-Codazzi relations (see [9]) can also be expressed as:

$$\begin{aligned} \varphi_{uuv} - \varphi_{uvu} &= R^M(\varphi_u, \varphi_v)\varphi_u \\ \varphi_{vuv} - \varphi_{vvu} &= R^M(\varphi_v, \varphi_u)\varphi_v. \end{aligned} \tag{3.13}$$

Finally, for the Walker metric, a curve $\gamma(t)$ is a geodesic if it satisfies the following system derived from (2.4):

$$\begin{cases} \frac{d^2\gamma_1}{dt^2} = f_y \frac{d\gamma_2}{dt}\frac{d\gamma_3}{dt} + \frac{1}{2} f_z \left(\frac{d\gamma_3}{dt}\right)^2, \\ \frac{d^2\gamma_2}{dt^2} = -\frac{\epsilon}{2} f_y \left(\frac{d\gamma_3}{dt}\right)^2, \\ \frac{d^2\gamma_3}{dt^2} = 0. \end{cases} \tag{3.14}$$

4. **Ruled surfaces in Walker 3-Manifold**

In this section, we give the geometry of ruled surfaces in a strict Walker 3-manifold.

Definition 4.1. *Let α and β be two solutions of the geodesic equations (3.14). A ruled surface in the Walker manifold (M, g_f^{ϵ}) is defined by*

$$R(s,t) = \alpha(s) + t\beta(s). \tag{4.1}$$

Let S be a surface parametrized by the equation (4.1). To calculate the coefficients of the second fundamental form, let us first determine the following derivation of R:

$$R_s = \alpha^{'}(s) + t\beta^{'}(s) = ((\alpha_1^{'}(s) + t\beta_1^{'}), (\alpha_2^{'}(s) + t\beta_2^{'}), (\alpha_3^{'}(s) + t\beta_3^{'}))$$

and

$$R_t = \beta(s) = (\beta_1(s), \beta_2(s), \beta_3(s)) \tag{4.2}$$

Then, one has:

$$\begin{aligned} E &= g_f^{\epsilon}(R_s, R_s) \\ &= \alpha_1'\alpha_3' + t\alpha_1'\beta_3' + t\alpha_3'\beta_1' + t^2\beta_1'\beta_3' \\ &\quad + \varepsilon\left((\alpha_2')^2 + 2t\alpha_2'\beta_2' + t^2(\beta_2')^2\right) \\ &\quad + \alpha_3'\alpha_1' + t\alpha_3'\beta_1' + t\alpha_1'\beta_3' + t^2\beta_1'\beta_3' \\ &\quad + f\left((\alpha_3')^2 + 2t\alpha_3'\beta_3' + t^2(\beta_3')^2\right). \end{aligned}$$

Since $\beta_3^{'}$ and $\alpha_3^{'}$ constant, one can put :

$$a = \alpha_3^{'},\ b = \beta_3^{'}$$

and we have

$$E = E_2t^2 + E_1t + E_0$$

where

$$\begin{aligned}
E_2 &= 2\beta_1'b + \varepsilon(\beta_2')^2 + fb^2,\\
E_1 &= 2(\alpha_1'b + \alpha_3'\beta_1') + 2\varepsilon\alpha_2'\beta_2' + 2fab,\\
E_0 &= 2\alpha_1'\alpha_3' + \varepsilon(\alpha_2')^2 + fa^2.
\end{aligned}$$

The same computation gives the second and third coefficients:

$$\begin{aligned}
F &= \underbrace{\beta_1bt + \epsilon\beta_2'\beta_2t + \beta_1'\beta_3t + fb\beta_3t}_{F_1t} + \underbrace{fa\beta_3 + \beta_3\alpha_1' + \epsilon\alpha_2'\beta_2 + \beta_1a}_{F_0}\\
&= F_1t + F_0,
\end{aligned}$$

where

$$\begin{aligned}
F_1 &= \beta_1b + \epsilon\beta_2'\beta_2 + \beta_1'\beta_3 + fb\beta_3\\
F_0 &= fa\beta_3 + \beta_3\alpha_1' + \epsilon\alpha_2'\beta_2 + \beta_1a,
\end{aligned}$$

and

$$G = \epsilon\beta_2^2 + f\beta_3^2 + 2\beta_1\beta_2 = G_0.$$

To calculate the coefficients of the second fundamental form, we first calculate the following second partial derivative:

By putting $R_{st} = \begin{pmatrix} R^1_{st}\\ R^2_{st}\\ R^3_{st}\end{pmatrix}$ we have:

$$\begin{aligned}
R^1_{st} &= \beta_1^{'} + \frac{1}{2}(a+tb)(f_y\beta_2 + f_z\beta_3),\\
R^2_{st} &= \beta_2^{'} - \frac{\epsilon}{2}f_y(a\beta_3 + tb\beta_3),\\
R^3_{st} &= \beta_3^{'} = b.
\end{aligned}$$

Then we have

$$R_{st} = \begin{pmatrix} \beta_1^{'} + \frac{1}{2}(a+tb)(f_y\beta_2 + f_z\beta_3)\\ \beta_2^{'} - \frac{\epsilon}{2}f_y(a\beta_3 + tb\beta_3)\\ b \end{pmatrix}. \tag{4.3}$$

Similary computation gives

$$R_{tt} = \begin{pmatrix} \frac{1}{2}f_y\beta_3\beta_2 + \frac{1}{2}f_y(\beta_3)^2\\ -\frac{\epsilon}{2}f_y(\beta_3)^2\\ 0 \end{pmatrix} \tag{4.4}$$

and

$$R_{ss} = \begin{pmatrix} \alpha_1'' + t\beta_1'' + (a+tb)(f_y + \frac{1}{2}\ f_z a + \frac{1}{2}\ f_z tb) \\ \alpha_2'' + t\beta_2'' + (a+tb)^2 \\ a+tb \end{pmatrix}. \tag{4.5}$$

To determine the normal vector of the surfaces, we need to compute the following product:

$$R_s \times R_t = \left(\begin{vmatrix} \alpha_1' + t\beta_1' & \beta_1' \\ \alpha_2' + t\beta_2' & \beta_2' \end{vmatrix} - f \begin{vmatrix} \alpha_2' + t\beta_2 & \beta_2' \\ \alpha_3' + t\beta_3' & \beta_3' \end{vmatrix} \right) e_1 - \epsilon \begin{vmatrix} \alpha_1' + t\beta_1' & \beta_1' \\ \alpha_3' + t\beta_3' & \beta_3' \end{vmatrix} e_2 + \begin{vmatrix} \alpha_2' + t\beta_2' & \beta_2' \\ \alpha_3' + t\beta_3' & \beta_3' \end{vmatrix} e_3$$

$$\begin{aligned} R_s \times R_t &= \Big((\alpha_1' + t\beta_1')\beta_2' - \beta_1'(\alpha_2' + t\beta_2') - f\beta_3'(\alpha_2' + t\beta_2') + f\beta_2'(\alpha_3' + t\beta_3'))\Big) e_1 \\ &- \epsilon \Big((\alpha_1' + t\beta_1')\beta_3' - (\alpha_3' + t\beta_3')\beta_1'\Big) e_2 \\ &+ \Big((\alpha_2' + t\beta_2')\beta_3' - (\alpha_3' + t\beta_3')\beta_2'\Big) e_3. \end{aligned}$$

After some simplification, we have

$$\begin{aligned} R_s \times R_t &= \Big(\alpha_1'\beta_2' - \beta_1'\alpha_2' - fb\alpha_2' + f\beta_2' a)\Big) e_1 \\ &+ \Big(-b\epsilon\alpha_1' + a\beta_1'\Big) e_2 \\ &+ \Big(\alpha_2' b - a\beta_2'\Big) e_3 \end{aligned}$$

where

$$e_1 = \partial_y, \quad e_2 = \frac{2-f}{2\sqrt{2}}\partial_x + \frac{1}{\sqrt{2}}\partial_z, \quad e_3 = \frac{2+f}{2\sqrt{2}}\partial_x - \frac{1}{\sqrt{2}}\partial_z,$$

constitute a local pseudo-orthonormal frame field on M, satisfying:

$$g_f^\epsilon(e_1, e_1) = \epsilon, \quad g_f^\epsilon(e_2, e_2) = 1, \quad \text{and} \quad g_f^\epsilon(e_3, e_3) = -1. \tag{4.6}$$

Thus, the signature of the metric g_f^ϵ is $(\epsilon, 1, -1)$. By putting: $A_1 = \alpha_1'\beta_2' - \beta_1'\alpha_2' - fb\alpha_2' + f\beta_2' a$, $A_2 = -b\epsilon\alpha_1' + a\beta_1'$ and $A_3 = \alpha_2' b - a\beta_2'$ we have:

$$R_s \times R_t = A_1 e_1 + A_2 e_2 + A_3 e_3.$$

Using the equations (4.6), we get

$$||R_s \times R_t|| = \sqrt{|\epsilon A_1^2 + A_2^2 - A_3^2|}$$

and the unit normal vector is given by:

$$\xi = \frac{A_1 e_1 + A_2 e_2 + A_3 e_3}{\sqrt{|\epsilon A_1^2 + A_2^2 - A_3^2|}}. \tag{4.7}$$

With $\Delta = \sqrt{|\epsilon A_1^2 + A_2^2 - A_3^2|}$, and an easy computation, the coefficients L, M and N of the second fundamental form are:

$$L = L_2 t^2 + L_1 t + L_0$$

where

$$\begin{cases} L_2 = \frac{\epsilon_1}{\Delta}\left(\epsilon b^2 A_2 + \frac{1}{2}b^2 A_3\right), \\ L_1 = \frac{\epsilon_1}{\Delta}\left(bA_1 + \epsilon\beta_2^{"} A_2 + 2a\epsilon b A_2 + \beta_1^{"} A_3 + \frac{1}{2}f_z abA_3 + bf_y A_3 + \frac{1}{2}f_z abA_3 + bfA_3\right), \\ L_0 = \frac{\epsilon_1}{\Delta}\left(aA_1 + \epsilon\alpha_2^{"} A_2 + \epsilon a^2 A_2 + \alpha_1^{"} A_3 + af_y A_3\right), \end{cases}$$

$$M = M_1 t + M_0,$$

where

$$\begin{aligned} M_1 &= \frac{\epsilon_1}{2\Delta}\Big(-\epsilon^2 f_y b\beta_3 A_2 + b(f_y\beta_2 + f_z\beta_3)A_3\Big), \\ M_0 &= \frac{\epsilon_1}{2\Delta}\Big(2\beta_2' A_2 - \epsilon^2 f_y a\beta_3 A_2 + a(f_y\beta_2 + f_z\beta_3)A_3 + 2bA_1 + 2(\beta_1' + bf)A_3\Big), \end{aligned}$$

and

$$N = -\frac{\epsilon^2}{2} f_y \beta_3^2 A_2 + \frac{1}{2} f_y A_3 \left(\beta_3\beta_2 + \beta_3^2\right) = N_0.$$

Using the above equations, we get

$$EG - F^2 = (E_2G_0 - F_1^2)t^2 + (E_1G_0 - 2F_1F_0)t + (E_0G_0 - F_0^2). \tag{4.8}$$

Theorem 4.2. *The ruled surface in the Walker manifold (M, g_f^{ϵ}) defined by (4.1) is non-degenerate if*

$$\begin{cases} E_2G_0 \neq F_1^2 \\ (E_1G_0 - 2F_0F_1)^2 < 4\left(E_2G_0 - F_1^2\right)\left(E_0G_0 - 2F_0^2\right). \end{cases} \tag{4.9}$$

Proof. The expression $EG - F^2$ is quadratic in t. For it to be non-zero, its discriminant must be negative, and the coefficient of t^2 must be non-zero. □

Now, let us calculate the mean curvature. We have

$$\begin{aligned} LG - 2MF + NE &= (L_2G_0t^2 + L_1G_0t + L_0G_0) - 2(M_1F_1t^2 + (M_1F_0 + M_0F_1)t + M_0F_0) \\ &\quad + (E_2N_0t^2 + E_1N_0t + E_0N_0) \\ &= (L_2G_0 - 2M_1F_1 + E_2N_0)t^2 \\ &\quad + (L_1G_0 - 2(M_1F_0 + M_0F_1) + E_1N_0)t \\ &\quad + (L_0G_0 - 2M_0F_0 + E_0N_0). \end{aligned}$$

Now using the formula (3.11), one can get Finally, the mean curvature:

$$H = \frac{\varepsilon_1}{2}\frac{(L_2G_0 - 2M_1F_1 + E_2N_0)t^2 + (L_1G_0 - 2(M_1F_0 + M_0F_1) + E_1N_0)t + (L_0G_0 - 2M_0F_0 + E_0N_0)}{(E_2G_0 - F_1^2)t^2 + (E_1G_0 - 2F_1F_0)t + (E_0G_0 - F_0^2)}.$$

Theorem 4.3. *The ruled surface in a Walker manifold* (M, g_f^ϵ) *defined by* (4.1) *is minimal if we have the following*

$$\begin{cases} L_2G_0 - 2M_1F_1 + E_2N_0 = 0, \\ L_1G_0 - 2(M_1F_0 + M_0F_1) + E_1N_0 = 0, \\ L_0G_0 - 2M_0F_0 + E_0N_0 = 0. \end{cases}$$

For the Gauss curvature, we need to calculate LN and M^2. And we have

$$\begin{aligned} LN &= \left(L_2t^2 + L_1t + L_0\right) N_0 \\ &= L_2N_0t^2 + L_1N_0t + L_0N_0 \end{aligned}$$

and

$$\begin{aligned} M^2 &= \left(M_1t + M_0\right)^2 \\ &= M_1^2t^2 + M_0^2 + 2M_1M_0t. \end{aligned}$$

Then using the formula (3.12), we have the Gauss curvature as

$$K(X_s, X_t) = K^M(X,Y) + \varepsilon_1 \frac{(L_2N_0 - M_1^2)\, t^2 + (L_1N_0 - 2M_1M_0)\, t + (L_0N_0 - M_0^2)}{(E_2G_0 - F_1^2)\, t^2 + (E_1G_0 - 2F_1F_0)\, t + (E_0G_0 - F_0^2)}.$$

Let us compute $K^M(X,Y)$. We put

$$X = X^1\partial_x + X^2\partial_y + X^3\partial_z, \qquad Y = Y^1\partial_x + Y^2\partial_y + Y^3\partial_z.$$

By the equations (2.5) and the bilinearity, we have

$$R(X,Y) = (X^2Y^3 - X^3Y^2)\, R(\partial_y, \partial_z)$$

and

$$\begin{aligned} R^M(X,Y)X &= (X^2Y^3 - X^3Y^2)\, R(\partial_y, \partial_z)X \\ &= (X^2Y^3 - X^3Y^2)\Big(X^2R(\partial_y, \partial_z)\partial_y + X^3R(\partial_y, \partial_z)\partial_z\Big) \\ &= -\tfrac{1}{2} f_{yy}\, (X^2Y^3 - X^3Y^2)\big(X^2\partial_x + \varepsilon X^3\partial_y\big). \end{aligned}$$

Then we have by using the metric that
$g(\partial_x, Y) = Y^3$ et $g(\partial_y, Y) = \varepsilon Y^2$ and we obtain

$$\begin{aligned} g\big(R^M(X,Y)X,\, Y\big) &= -\tfrac{1}{2} f_{yy}\, (X^2Y^3 - X^3Y^2)\Big(X^2g(\partial_x, Y) + \varepsilon X^3g(\partial_y, Y)\Big) \\ &= -\tfrac{1}{2} f_{yy}\, (X^2Y^3 - X^3Y^2)\Big(X^2Y^3 + X^3Y^2\Big) \\ &= -\tfrac{1}{2} f_{yy}\, \Big((X^2Y^3)^2 - (X^3Y^2)^2\Big). \end{aligned}$$

The sectional curvature of the ambient space is

$$K^M(X,Y) = -\frac{f_{yy}}{2} \times \frac{(X^2Y^3)^2 - (X^3Y^2)^2}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

Recall the following notations.

$$R_s = (\alpha_1' + t\beta_1',\ \alpha_2' + t\beta_2',\ a + tb), \qquad R_t = (\beta_1',\ \beta_2',\ b),$$

where $a = \alpha_3',\ b = \beta_3'$. If we put $X = R_s$ and $Y = R_t$ then we have

$$X^2 = \alpha_2' + t\beta_2', \qquad X^3 = a + tb,$$
$$Y^2 = \beta_2', \qquad Y^3 = b.$$

And then we have

$$\begin{aligned} X^2Y^3 - X^3Y^2 &= (\alpha_2' + t\beta_2')b - (a + tb)\beta_2' \\ &= b\alpha_2' - a\beta_2'. \end{aligned}$$

Similarly we have

$$X^2Y^3 + X^3Y^2 = (\alpha_2' + t\beta_2')b + (a + tb)\beta_2' = b\alpha_2' + a\beta_2' + 2tb\beta_2'.$$

We found,

$$K^M(R_s, R_t) = -\frac{f_{yy}}{2}\,\frac{(b\alpha_2' - a\beta_2')\big(b\alpha_2' + a\beta_2' + 2tb\beta_2'\big)}{(E_2G_0 - F_1^2)\,t^2 + (E_1G_0 - 2F_1F_0)\,t + (E_0G_0 - F_0^2)}.$$

If we put

$$\eta = \frac{f_{yy}}{2}\Big((b\alpha_2')^2 - (a\beta_2')^2 + 2tb\beta_2'(b\alpha_2' - a\beta_2')\Big)$$

we obtain

$$K(X_s, X_t) = \frac{-\eta + \varepsilon_1\big((L_2N_0 - M_1^2)t^2 + (L_1N_0 - 2M_1M_0)t + (L_0N_0 - M_0^2)\big)}{(E_2G_0 - F_1^2)t^2 + (E_1G_0 - 2F_1F_0)t + (E_0G_0 - F_0^2)}.$$

Then we have the following theorem.

Theorem 4.4. *The ruled surface in the Walker manifold (M, g_f^ϵ) defined by (4.1) is flat if and only if*

$$\begin{cases} \eta = 0 \\ L_2N_0 - M_1^2 = 0, \\ L_1N_0 - 2M_1M_0 = 0, \\ L_0N_0 - M_0^2 = 0. \end{cases} \tag{4.10}$$

Example 4.5. *Let M a Walker 3-manifold where $f \neq 2$. Consider the following geodesic system*

$$\begin{cases} \dfrac{d^2\alpha_1(t)}{dt^2} &= f_y\dfrac{d\alpha_2}{dt}\dfrac{d\alpha_3}{dt} + \dfrac{1}{2}f_z\left(\dfrac{d\alpha_3}{dt}\right)^2, \\ \dfrac{d^2\alpha_2(t)}{dt^2} &= -\dfrac{\varepsilon}{2}f_y\left(\dfrac{d\alpha_3}{dt}\right)^2, \\ \dfrac{d^2\alpha_3(t)}{dt^2} &= 0. \end{cases} \tag{4.11}$$

We have

$$\frac{d^2\alpha_3}{dt^2} = 0 \quad \Longrightarrow \quad \frac{d\alpha_3}{dt} = C_3 \quad (constant).$$

By integration, we have,

$$\alpha_3(t) = C_3 t + C_3',$$

where C_3' is constant.
If we replace $\dfrac{d\alpha_3}{dt} = C_3$ the second equation we have:

$$\frac{d^2\alpha_2}{dt^2} = -\frac{\varepsilon}{2} f_y \big(C_3\big)^2 = -\frac{\varepsilon}{2} f_y C_3^2 \quad \textit{(constant)}.$$

By integration, we have

$$\alpha_2(t) = -\frac{\varepsilon}{4} f_y C_3^2\, t^2 + C_2 t + C_2',$$

with C_2 and C_2' are constants. We have $\dfrac{d\alpha_3}{dt} = C_3$ and $\dfrac{d\alpha_2}{dt} = -\frac{\epsilon}{2} f_y C_3^2 t + C_2$. Therefore

$$\frac{d^2\alpha_1(t)}{dt^2} \quad = \quad f_y\left(-\frac{\epsilon}{2} f_y C_3^2 t + C_2\right) C_3 + \frac{1}{2} f_z C_3^2$$

Let's integrate once D_1 and we have:

$$\frac{d\alpha_1}{dt} = -\frac{\varepsilon}{4} f_y^2 C_3^3\, t^2 + \big(f_y C_1 C_3 + \tfrac{1}{2} f_z C_3^2\big) t + D_1.$$

$$\begin{aligned}
\frac{d^2\alpha_1(t)}{dt^2} &= f_y\left(-\frac{\epsilon}{2} f_y C_3^2 t + C_2\right) C_3 + \frac{1}{2} f_z C_3^2 \\
&= -\frac{\epsilon}{2} f_y^2 C_3^3 t + C_2 f_y C_3 + \frac{1}{2} f_z C_3^2, \\
\frac{d\alpha_1(t)}{dt} &= -\frac{\epsilon}{4} f_y^2 C_3^3 t^2 + C_2 f_y C_3 t + \frac{1}{2} f_z C_3^2 t + C_4, \\
\alpha_1(t) &= -\frac{\epsilon}{12} f_y^2 C_3^3 t^3 + \frac{1}{2} C_2 f_y C_3 t^2 + \frac{1}{4} f_z C_3^2 t^2 + C_4 t + C_5
\end{aligned}$$

$$\boxed{\begin{aligned}
&\alpha_3(t) = C_3 t + C_2, \\
&\alpha_2(t) = -\frac{\varepsilon}{4} f_y C_3^2\, t^2 + C_1 t + C_0, \\
&\alpha_1(t) = -\frac{\epsilon}{12} f_y^2 C_3^3 t^3 + \frac{1}{2} C_2 f_y C_3 t^2 + \frac{1}{4} f_z C_3^2 t^2 + C_4 t + C_5
\end{aligned}}$$

where C_3, C_2, C_4, C_5 are constants.

By analogy, we find the solutions of the system

$$\begin{cases} \dfrac{d^2\beta_1(t)}{dt^2} &= f_y \dfrac{d\beta_2}{dt}\dfrac{d\beta_3}{dt} + \dfrac{1}{2} f_z \left(\dfrac{d\beta_3}{dt}\right)^2, \\ \dfrac{d^2\beta_2(t)}{dt^2} &= -\dfrac{\varepsilon}{2} f_y \left(\dfrac{d\beta_3}{dt}\right)^2, \\ \dfrac{d^2\beta_3(t)}{dt^2} &= 0. \end{cases} \tag{4.12}$$

$$\boxed{\begin{aligned} &\beta_3(t) = D_3 t + D_2, \\ &\beta_2(t) = -\frac{\varepsilon}{4} f_y D_3^2\, t^2 + D_1 t + D_0, \\ &\beta_1(t) = -\frac{\epsilon}{12} f_y^2 D_3^3 t^3 + \frac{1}{2} D_2 f_y D_3 t^2 + \frac{1}{4} f_z D_3^2 t^2 + D_4 t + D_5 \end{aligned}}$$

where D_3, D_2, D_4, D_5 constants of integration.

If we take $C_1 = D_2 = D_5 = 1$ and $C_0 = C_2 = C_3 = C_4 = C_5 = D_1 = D_3 = D_4 = D_0 = 0$, then we get two geodesics given by

$$\alpha(s) = (0, s, 0), \qquad \beta(s) = (1, 0, 1),$$

and the corresponding ruled surface is

$$R(s,t) = \alpha(s) + t\,\beta(s) = (t,\ s,\ t).$$

The tangent vectors to this surface are

$$R_s = (0, 1, 0), \qquad R_t = (1, 0, 1).$$

The coefficients of the first fundamental form are then.

$$E = g(R_s, R_s) = \epsilon,$$

$$F = g(R_s, R_t) = 0,$$

$$G = g(R_t, R_t) = 2 + f.$$

Therefore

$$EG - F^2 = (2 + f)\epsilon \neq 0, \quad \textit{since } f \textit{ is not constant}$$

which shows that the surface is non-degenerate. We have

$R_{ss} = R_{st} = 0$ and $R_{tt} = \left(\frac{1}{2} f_y, -\frac{\epsilon}{2} f_y, 0\right)$. It follows that the coefficients of the second fundamental form are

$$L = M = 0.$$

and

$$N = N_0 \neq 0.$$

By the Gauss equation, the Gauss curvature of the surface is given by

$$K(X_s, X_t) = K^M(X_s, X_t) + \varepsilon_1 \frac{LN - M^2}{EG - F^2}.$$

Since $LN - M^2 = (2 + f)\epsilon \neq 0$,

$$K^M(X_s, X_t) = 0$$

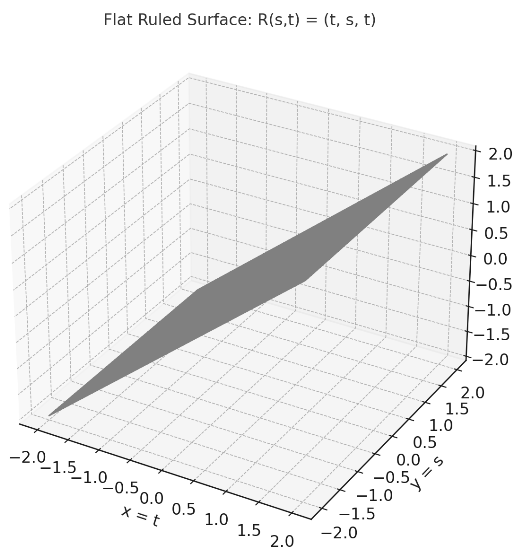
we conclude that

$$K(X_s, X_t) = 0.$$

Hence, the ruled surface

$$\boxed{R(s,t) = (t,\ s,\ t)}$$

is a flat surface in the strict Walker 3*-manifold* M.



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