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On a class of projectively flat (α,β) metrics

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Abstract. Given any Finsler metric F on a smooth manifold, its symmetrized metric $\hat{F}(x,y) := \frac{1}{2}\big(F(x,y)+F(x,-y)\big)$ may inherit some geometric properties of F. We examine this fact for the Matsumoto metric $F=\alpha^2/(\alpha-\beta)$ and prove that if the Matsumoto metric is locally projectively flat then, so is its symmetrized metric $\hat{F}=\alpha^3/(\alpha^2-\beta^2)$. In particular, the converse result also holds.

Keywords: (α,β) metrics, Matsumoto metric, projectively flat, symmetrized metrics.

1. **Introduction**

The solutions to Hilbert's fourth problem in the regular distance functions with straight geodesics are projectively flat Finsler metrics on a convex open set in $\mathbb{R}^n$[9]. A natural question arises, if a given Finsler metric F possesses a certain property, does its symmetrized metric also share that same property? This fundamental question can be posed for a wide array of geometric properties. In the present work, we are going to investigate this question specifically for the property of projective flatness. It is known that if a Randers metric

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$\alpha+\beta$, is locally projectively flat then, so is its symmetrized metric, namely, the Riemannian metric α. A similar result holds for the Berwald metric

$$F=\frac{(\alpha+\beta)^2}{\alpha}.$$

The locally projectively flat Berwald metric was studied in following theorem.

Theorem 1.1. *(cf. [3]) Let $k\neq 0$. Then $F=\alpha+\epsilon\beta+\frac{k\beta^2}{\alpha}$ is projectively flat if and only if*
(i) $b_{i|j}=\tau((k^{-1}+2b^2)a_{ij}-3b_ib_j)$,
(ii) $G^i_\alpha=\theta y^i-\tau\alpha^2 b^i$,
where $\tau=\tau(x)$ and $\theta=a_i(x)y^i$. In this case,

$$G^i=\{\theta+\tau\chi\alpha\}y^i,$$

where

$$\chi:=\frac{(\epsilon+2ks)(1-2ks^2)}{2k(1+\epsilon s+ks^2)}-s,\ \ s=\frac{\beta}{\alpha}$$

It not a hard stuff to see that if $F=\frac{(\alpha+\beta)^2}{\alpha}=\alpha+2\beta+\frac{\beta^2}{\alpha}$, $(\epsilon=2,k=1)$ is locally projectively flat then, so is its symmetrized metric $\hat{F}=\frac{\alpha^2+\beta^2}{\alpha}=\alpha+\frac{\beta^2}{\alpha}$, $(\epsilon=0,k=1)$.

The locally projectively flat Matsumoto metrics have been studied through a series of works [2, 11, 12, 13] where, the main proven fact is that the locally projectively flat Matsumoto metrics are locally Minkowskian. Our objective is to analyze this same characterization for the symmetrized Matsumoto metric to determine the resulting conclusions. We prove the following theorem.

Theorem 1.2. *The symmetrized Matsumoto metric $F=\frac{\alpha^3}{\alpha^2-\beta^2}$ is locally projectively flat if and only if*
(i) β is parallel with respect to α,
(ii) α is locally projectively flat, i.e., of constant curvature.

We apply Theorem 1.2 to prove that the local projective flatness is logically equivalent for both Matsumoto and symmetrized Matsumoto metrics.

2. Preliminaries

Let M be an n-dimensional C^∞ connected manifold. The tangent space of M at $x\in M$ is denoted by T_xM and the tangent manifold of M is the disjoint union of tangent spaces $TM:=\bigcup_{x\in M}T_xM$. Every element of TM is a pair (x,y) where $x\in M$ and $y\in T_xM$. Denote the slit tangent manifold by $TM_0=TM\backslash\{o\}$, where o denotes the zero section of the tangent bundle. The natural projection $\pi:TM\to M$ given by $\pi(x,y):=x$ makes TM a vector bundle of rank n over M and TM_0 a fiber bundle over M with fiber type $\mathbb{R}^n\backslash\{o\}$.

A Finsler metric on M is a function $F : TM \to [0, \infty)$ satisfying following conditions: (i) F is C^∞ on TM_0, (ii) $F(x, y)$ is positively 1-homogeneous y and (iii) for each $y \in T_X M$, the following quadratic form g_y on $T_x M$ is positive definite,

$$g_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y+su+tv)\Big]|_{s,t=0},\ \ u,v \in T_x M.$$

For a given Finsler metric $F = F(x, y)$, the geodesic equations are given by:

$$\ddot{x}^i + 2G^i(x,\dot{x}) = 0,$$

where G^i are the spray coefficients, computed from F.

Finsler geometry generalizes Riemannian geometry by replacing the quadratic form defining the metric with a more general norm on each tangent space. Among the diverse classes of Finsler metrics, (α, β)-metrics constitute a fundamental family, defined in the form

$$F = \alpha\,\phi\,(s)\,,\ \ s = \frac{\beta}{\alpha}$$

where α is a Riemannian metric, β is a 1-form, and $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth function. The spray coefficients of F and α, respectively, G^i and G^i_α are given by:

$$G^i = \frac{g^{il}}{4}\left\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\right\},\quad G^i_\alpha = \frac{a^{il}}{4}\left\{[\alpha^2]_{x^k y^l} y^k - [\alpha^2]_{x^l}\right\},$$

where: g^{ij} and a^{ij} are the inverse matrices of g_{ij} and a_{ij}, respectively , $(g_{ij}) := \left(\frac{1}{2}[F^2]_{y^i y^j}\right)$ and $(a^{ij}) := (a_{ij})^{-1}$.

In Finsler geometry, the study of projectively flat provides deep insights into the geodesic structure of spaces that generalize Riemannian manifolds. A Finsler metric F on a smooth manifold M is called projectively flat if its geodesics coincide with straight lines in some local coordinate system, meaning the geodesic spray coefficients G^i satisfy

$$G^i(x,y) = P(x,y)y^i,$$

where $P(x, y)$ is a 1-homogeneous function in y. This condition ensures that the space is locally projectively flat equivalent to a Minkowski space. We have the following.

Lemma 2.1. *The spray coefficients G^i of a Finsler metric F are related to Riemannian (G^i_α) by:*

$$G^i = G^i_\alpha + \alpha Q s^i_0 + J\{-2Q\alpha s_0 + r_{00}\}\frac{y^i}{\alpha} + H\{-2Q\alpha s_0 + r_{00}\}\{b^i - s\frac{y^i}{\alpha}\}, \tag{2.1}$$

where:

$$\begin{aligned} Q &:= \frac{\phi'}{\phi - s\phi'}, \\ J &:= \frac{\phi'(\phi - s\phi')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')}, \\ H &:= \frac{\phi''}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')}, \end{aligned}$$

$s = \beta/\alpha$, $b = \|\beta_x\|_\alpha$, $s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i})$, $s_{l0} = s_{li}y^i$, $s_0 = s_{l0}b^l$, $r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i})$ *and* $r_{00} = r_{ij}y^iy^j$.

The formula (2.1) was given in [7, 4] and a different version of its was given in [5, 8].

Also Hamel in [6] due that a Finsler metric F on $\mathcal{U} \subset \mathbb{R}^n$ is projectively flat if and only if

$$F_{x^k y^l} y^k - F_{x^l} = 0. \tag{2.2}$$

From the above equation, We have the following lemma.

Lemma 2.2. *(Shen-Yildrim,[3]) An (α, β)-metric $F = \alpha\phi(s)$, where $s = \beta/\alpha$ is projectively flat on an open subset $\mathcal{U} \subset \mathbb{R}^n$ if and only if the following condition holds:*

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \alpha^3 Q s_{l0} + H\alpha(-2\alpha Q s_0 + r_{00})(b_l\alpha - s y_l) = 0, \tag{2.3}$$

where:

$$s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i}), \quad s_{l0} = s_{li}y^i, \quad s_0 = s_{l0}b^l, \quad r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i}), \quad r_{00} = r_{ij}y^iy^j.$$

3. Proof of Main Theorems

In this section, we consider the symmetrized Matsumoto metric

$$F = \frac{\alpha^3}{\alpha^2 - \beta^2} = \alpha\phi(s) = \frac{1}{1 - s^2}, \; s = \frac{\beta}{\alpha}, \tag{3.1}$$

where $s < 1$, so that ϕ must be a positive function.

By Lemma 2.1, the spray coefficients G^i of F are given by (2.1) with

$$\begin{aligned} Q &= \frac{2s}{1 - 3s^2} = \frac{2\alpha\beta}{\alpha^2 - 3\beta^2}, \\ J &= \frac{s - 3s^3}{1 - 6s^2 - 3s^4 + 6b^2s^2 + 2b^2} = \frac{\alpha^3\beta - 3\beta^3\alpha}{(1 + 2b^2)\alpha^4 - 3\beta^4 6(b^2 - 1)\alpha^2\beta^2}, \\ H &= \frac{1 + 3s^2}{1 - 6s^2 - 3s^4 + 6b^2s^2 + 2b^2} = \frac{\alpha^4 + 3\beta^2\alpha^2}{(1 + 2b^2)\alpha^4 - 3\beta^4 + 6(b^2 - 1)\alpha^2\beta^2}. \end{aligned}$$

Equation (2.3) is reduced to the following equation:

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \alpha^3 \frac{2\alpha\beta}{\alpha^2 - 3\beta^2} S_{l0} \\ + \frac{\alpha^4 + 3\alpha^2\beta^2}{(1+2b^2)\alpha^4 - 3\beta^4 + 6(b^2-1)\alpha^2\beta^2}\alpha(-2\alpha\frac{2\alpha\beta}{\alpha^2 - 3\beta^2}S_0 + r_{00})(b_l\alpha - \frac{\beta}{\alpha}y_l) = 0 \tag{3.2}$$

Lemma 3.1. [2] *If* $(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha = 0$, *then* α *is projectively flat.*

By (3.2), Li in [2] proved the following.

Theorem 3.2. *The Matsumoto metric* $F = \frac{\alpha^2}{\alpha-\beta}$ *is locally projectively flat if and only if*
(i) β *is parallel with respect to* α,
(ii) α *is locally projectively flat, i.e., of constant curvature.*

Now we can establish this condition for the symmetrized Matsumoto metric.

3.1. **Proof of Theorem 1.2.**

Proof. Suppose that the symmetrized Matsumoto metric $F = \frac{\alpha^3}{\alpha^2-\beta^2}$ is locally projectively flat. Now, we rewrite (2.3) as a polynomial in y^i and α. We obtain

$$(\alpha^2 - 3\beta^2)((1+2b^2)\alpha^4 - 3\beta^4 + 6(b^2-1)\alpha^2\beta^2)(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha \\ + 2\beta\alpha^4((1+2b^2)\alpha^4 - 3\beta^4 + 6(b^2-1)\alpha^2\beta^2)s_{l0} \\ + (\alpha^4 + 3\alpha^2\beta^2)(-4\alpha^2\beta s_0 + (\alpha^2 - 3\beta^2)r_{00})(b_l\alpha^2 - \beta y_l) = 0. \tag{3.3}$$

Simplifying the above equation yields

$$\Big\{(4s_{l0}b^2 - 4s_0b_l + 2s_{l0})\beta + a_{ml}(2b^2+1)G^m_\alpha + b_l r_{00}\Big\}\alpha^8 \\ + \Big\{(12s_{l0}b^2 - 12s_0b_l - 12s_{l0})\beta^3 + (-9G^m_\alpha a_{ml} + 4s_0y_l)\beta^2 - \beta y_l r_{00} \\ - y_l G^m_\alpha(2b^2+1)y_m\Big\}\alpha^6 - \Big\{6\beta^3 s_{l0} + (a_{ml}(18b^2-15)G^m_\alpha - 12s_0y_l + 9b_l r_{00})\beta^2 \\ - 9y_l y_m G^m_\alpha \beta^2\Big\}\alpha^4 + \beta^4\Big\{9\beta^2 G^m_\alpha a_{ml} + 9\beta y_l r_{00} + y_l y_m G^m_\alpha(18b^2-15)\Big\}\alpha^2 \\ - 9G^m_\alpha\beta^6 y_l y_m = 0. \tag{3.4}$$

From (3.4), we can see that $9G^m_\alpha\beta^6 y_l y_m$ has the factor α^2 and since α^2 does not divide $\beta^6 y_l y_m$ then we have

$$G^m_\alpha = \mu^m\alpha^2, \tag{3.5}$$

where μ^m $(m = 1, ..., n)$ is a function of x.
Substituting (3.5) into (3.4), we can get the following equation

$$a_{10}\alpha^8 + a_8\alpha^6 + a_6\alpha^4 + a_4\alpha^2 + a_2 = 0, \tag{3.6}$$

where

$$\begin{aligned}
&a_{10} = (2b^2+1)a_{ml}\mu^m,\\
&a_8 = (-9\beta^2 a_{ml} - y_m(2b^2+1)y_l)\mu^m + (4s_{l0}b^2 - 4s_0 b_l + 2s_{l0})\beta + b_l r_{00},\\
&a_6 = -\beta(18\mu^m a_{ml}b^2\beta^3 - 15\mu^m a_{ml}\beta^3 - 12s_{l0}b^2\beta^2 - 9\mu^m\beta y_l y_m + 12s_0\beta^2 b_l\\
&\ - 4s_0\beta y_l + 12s_{l0}\beta^2 + r_{00}y_l),\\
&a_4 = ((9\beta^2 a_{ml} + y_l y_m(18b^2-15))\mu^m - 9b_l r_{00} - 6\beta s_{l0} + 12s_0 y_l)\beta^4,\\
&a_2 = 9\beta^5 y_l(r_{00} - y_m\mu^m\beta).
\end{aligned}$$

From (3.6), we see that a_2 has the factor α^2 and since α^2 does not divide $\beta^5 y_l$ then we get

$$r_{00} = c(x)\alpha^2 + y_m\mu^m\beta, \tag{3.7}$$

where $c(x)$ is a function of x.

Substituting (3.5) and (3.7) into (3.3) and contracting with b^l yields

$$\begin{aligned}
&(\alpha^2-3\beta^2)((1+2b^2)\alpha^4 - 3\beta^4 + 6(b^2-1)\alpha^2\beta^2)(b_m\alpha^2 - y_m\beta)\mu^m\alpha^2\\
&+ 2\beta\alpha^4((1+2b^2)\alpha^4 - 3\beta^4 + 6(b^2-1)\alpha^2\beta^2)s_0\\
&+ (\alpha^4+3\alpha^2\beta^2)(-4\alpha^2\beta s_0 + (\alpha^2-3\beta^2)(c(x)\alpha^2 + y_m\mu^m\beta))(b^2\alpha^2-\beta^2) = 0.
\end{aligned} \tag{3.8}$$

Simplifying the above equation, we can get

$$m_{10}\alpha^6 + m_8\alpha^4 + m_6\alpha^2 + m_4 = 0, \tag{3.9}$$

where

$$\begin{aligned}
&m_{10} = c(x)b^2 + (2b^2+1)b_m\mu^m,\\
&m_8 = -\beta(b^2 y_m\mu^m + 9\beta b_m\mu^m + y_m\mu^m + c(x)\beta - 2s_0),\\
&m_6 = -\beta^3((b_m(18b^2-15)\beta - 8y_m)\mu^m + 9c(x)b^2\beta + 8s_0),\\
&m_4 = 3\beta^5(3\mu^m b^2 y_m + 3\mu^m b_m\beta + 3c(x)\beta - 5\mu^m y_m + 2s_0).
\end{aligned}$$

From the above equation, we can see that m_4 has the factor α^2. The fact that α^2 does not divide β^5, implies that α^2 must divide $3b^2\mu^m y_m - 5\mu^m y_m + 3\mu^m b_m\beta + 3c(x)\beta + 2s_0$ which is of degree one polynomial and conclude that

$$3b^2\mu^m y_m - 5\mu^m y_m + 3\mu^m b_m\beta + 3c(x)\beta + 2s_0 = 0, \tag{3.10}$$

then

$$m_4 = 0. \tag{3.11}$$

Substituting (3.11) in (3.9) and simplifying, we get

$$m_{10}\alpha^4 + m_8\alpha^2 + m_6 = 0. \tag{3.12}$$

Note that m_6 has the factor α^2 and we know that α^2 does not divide β^3, implies that α^2 must divide $(b_m(18b^2-15)\beta-8y_m)\mu^m+9c(x)b^2\beta+8s_0$ which is of degree one polynomial and yields

$$(b_m(18b^2-15)\beta-8y_m)\mu^m+9c(x)b^2\beta+8s_0=0, \tag{3.13}$$

then

$$m_6=0. \tag{3.14}$$

Substituting (3.14) in (3.12) and simplifying, we have

$$m_{10}\alpha^2+m_8=0. \tag{3.15}$$

Thus m_8 has the factor α^2. Since α^2 does not divide β, it must divide $b^2\mu^m y_m+\mu^m y_m+9\mu^m b_m\beta+c(x)\beta-2s_0$, which is of degree one polynomial. Consequently, we obtain

$$b^2\mu^m y_m+\mu^m y_m+9\mu^m b_m\beta+c(x)\beta-2s_0=0, \tag{3.16}$$

thus

$$m_8=0. \tag{3.17}$$

Substituting (3.17) in (3.15) yields

$$m_{10}\alpha^2=0, \tag{3.18}$$

then we get

$$c(x)b^2+(2b^2+1)b_m\mu^m=0. \tag{3.19}$$

From (3.16), we have

$$2s_0=b^2\mu^m y_m+\mu^m y_m+9\mu^m b_m\beta+c(x)\beta. \tag{3.20}$$

Substituting (3.20) into (3.10), we obtain

$$b^2\mu^m y_m-\mu^m y_m+3b_m\mu^m\beta+c(x)\beta=0 \tag{3.21}$$

From the above equation we get

$$\mu^m y_m=\chi\beta, \tag{3.22}$$

where

$$\chi=\frac{3b_m\mu^m+c(x)}{1-b^2}.$$

Substituting (3.22) into (3.20) yields

$$s_0=\tau\beta, \tag{3.23}$$

where

$$\tau=\frac{(b^2+1)\chi+9\mu^m b_m+c(x)}{2}.$$

By differentiating (3.23) with respect to y^i, we have

$$s_i = \tau b_i \tag{3.24}$$

By contracting the above equation with b^i yields

$$s_i b^i = s_{ji} b^i b^j = \tau \left\|\beta\right\|^2 . \tag{3.25}$$

Note that $s_{ji}b^i b^j = 0$, thus $\tau = 0$ and

$$s_0 = 0. \tag{3.26}$$

By multiplying equation (3.16) by 3 and then subtracting it from equation (3.10), we obtain

$$\mu^m (y_m + 3b_m \beta) = 0. \tag{3.27}$$

Let us for a while that $(\mu^1, ..., \mu^n) \neq (0, ..., 0)$; replacing μ^m by y^m in (3.27) yields

$$\mu^m (\mu_m + 3b_m b_k \mu^k) = 0. \tag{3.28}$$

After simplifying the above equation, we get

$$\left\|\mu\right\|^2 + 3(b_k \mu^k)^2 = 0, \tag{3.29}$$

where $\left\|\mu\right\|^2 = \mu_m \mu^m$, thus $(\mu^1, ..., \mu^n) = (0, ..., 0)$ that contradicts with the assumption $(\mu^1, ..., \mu^n) \neq (0, ..., 0)$. Thus, $(\mu^1, ..., \mu^n) = (0, ..., 0)$ and it follows from the equation (3.19) that

$$c(x) = \frac{2b^2 + 1}{b^2} b_m \mu^m = 0, \tag{3.30}$$

and following (3.7) it results:

$$r_{00} = 0. \tag{3.31}$$

Substituting $r_{00} = 0$ and $\mu^m = 0, (m = 1, ..., n)$ in a_2 in (3.6) yields

$$a_2 = 0.$$

Then by substituting $r_{00} = 0$, $\mu^m = 0, (m = 1, ..., n)$ and $s_0 = 0$ in a_4 in (3.6) and taking into account that a_4 has the factor α^2 and the fact that α^2 does not divide $\beta^5 s_{l0}$, it results that

$$\beta^5 s_{l0} = 0, \tag{3.32}$$

thus

$$s_{l0} = 0. \tag{3.33}$$

Then, by (3.26) and (3.33) we get $b_{i;j} = 0$. Thus β is parallel with respect to α.

Also by Lemma 2.1 and according to equations (3.26), (3.33) and (3.31), we have

$$G^i = G^i_\alpha = P y^i, \tag{3.34}$$

and by Lemma 3.1 satisfying, α is projectively flat.

Conversely, suppose that β is parallel with respect to α and α is locally projectively flat, then by Lemma 2.1, we can easily conclude that F is locally projectively flat. ☐

Remark 3.3. *Comparing conditions (i) and (ii) in Theorem 1.2 and Theorem 3.2, we can conclude that the Matsumoto metric is projectively flat if and only if its symmetrized metric is projectively flat. This stands in contrast to the behavior of Randers-type metrics, for which this equivalence does not hold in general.*

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