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Remarks on four-dimensional locally symmetric Walker manifolds

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Abstract. In this paper, we examine certain geometric properties of the curvature tensor for a special case of the Walker metric, assuming $g_{33} = g_{44} = k \neq 0$, where k is a constant, on a 4-dimensional manifold. Finally, we investigate the necessary and sufficient conditions for the 4-dimensional manifold with this special case of the Walker metric to be locally symmetric.

Keywords: Curvature tensor, Einstein, Locally symmetric, Walker metric.

1. Introduction

To gain a deeper understanding of pseudo-Riemannian manifolds, it is essential to study the curvature properties of specific classes of these spaces. Consequently, comparing the results from Riemannian geometry with their pseudo-Riemannian counterparts enhances our comprehension of which features are more closely related to the signature of the metric tensor and which are more general.

A Walker n-manifold is a pseudo-Riemannian manifold that admits a parallel null r-plane field, with $r \leq \frac{n}{2}$. The canonical forms of the metrics were first introduced and studied by Walker A. G. (1950) in [11]. Following this, even-dimensional Walker manifolds (with $n = 2m$) that possess half-dimensional parallel null fields ($r = m$) received particular attention. Specifically, a 4-dimensional pseudo-Riemannian manifold M is called a Walker manifold if its

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metric signature is $(2,2)$ and it admits a parallel, totally isotropic 2-plane field [11]. The canonical form of this metric involves three functions $a(x,y,z,t)$, $b(x,y,z,t)$ and $c(x,y,z,t)$, such that, in suitable coordinates, the metric g can be expressed as:

$$g = 2dx \otimes dz + 2dy \otimes dt + adz \otimes dz + 2cdz \otimes dt + bdt \otimes dt \tag{1.1}$$

for some functions $a = a(x,y,z,t)$, $b = b(x,y,z,t)$ and $c = c(x,y,z,t)$.

Recently, many authors have examined various properties of Walker manifolds [1, 5, 7, 8, 9, 10]. To our knowledge, much of this research has concentrated on the curvature properties of the Walker metric. For example, in [6], the curvature properties of four-dimensional Walker metrics with a specific case of (1.1) have been investigated. In [3], other geometric properties of Walker metrics assuming that $c = c(x,y,z,t)$ is non-zero but constant have been analyzed. In [4], properties of four-dimensional manifolds with Walker metrics were studied under the condition that $a = a(x,y,z,t)$ and $b = b(x,y,z,t)$ are both zero. Furthermore, in [2], some geometric properties of the Walker metric (1.1), where $a = a(x,y,z,t)$ and $b = b(x,y,z,t)$ are non-zero and equal to k, with k being constant, were explored.

In this paper, we investigate certain characteristics of four-dimensional manifolds equipped with a special type of Walker metric that has not been previously studied in [2]. Specifically, we consider Walker metrics under the condition that $a = a(x,y,z,t)$ and $b = b(x,y,z,t)$ are both non-zero but equal to a constant k, so that in suitable coordinates, the metric is expressed as:

$$g_c(x,y,z,t) = 2(dx \circ dz + dy \circ dt + cdz \circ dt) + kdz \circ dz + kdt \circ dt \tag{1.2}$$

and its matrix form as:

$$g_c(x,y,z,t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & k & c \\ 0 & 1 & c & k \end{pmatrix}$$

where c is a smooth function on M. Additionally, the inverse matrix of the metric form (1.2) satisfies

$$g_c^{-1}(x,y,z,t) = \begin{pmatrix} -k & -c & 1 & 0 \\ -c & -k & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{1.3}$$

Our main objective is to generalize the findings of the research in [4], taking into account the metric (1.2).

This article is structured into four parts. In part 2, we will explore certain properties of the curvature of the Walker metric (1.2). The third and fourth parts will examine Einstein-like and locally symmetric Walker metrics (1.2), respectively.

2. **Curvature tensor of Walker metrics** (1.2)

The curvature tensor R of the pseudo-Riemannian metric g, which is defined in terms of the Levi-Civita connection ∇, as follows:

$$R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]. \tag{2.1}$$

With standard calculations and utilizing (1.2) and (1.3), it is possible to ascertain the non-zero components of Christoffel's symbols Γ^k_{ij}, which are:

$$\begin{aligned}
&\Gamma^1_{14} = \frac{1}{2}\{c_x + c_t\}, \quad \Gamma^1_{24} = \frac{1}{2}\{c_y + c_t\}, \quad \Gamma^1_{34} = \frac{1}{2}\{kc_x + cc_y + c_t\}\\
&\Gamma^1_{43} = \frac{1}{2}\{kc_n + cc_y - c_z\}, \qquad \Gamma^1_{44} = c_t\\
&\Gamma^2_{13} = \frac{1}{2}\{c_x + c_z\}, \qquad \Gamma^2_{23} = \frac{1}{2}\{c_y + c_z\},\\
&\Gamma^2_{33} = c_z, \qquad \Gamma^2_{34} = \frac{1}{2}\{cc_x + kc_y - c_t\},\\
&\Gamma^2_{43} = \frac{1}{2}\{cc_x + kc_y + c_z\},\\
&\Gamma^3_{34} = \Gamma^3_{43} = -\frac{1}{2}c_x, \qquad \Gamma^4_{34} = \Gamma^4_{43} = -\frac{1}{2}c_y.
\end{aligned} \tag{2.2}$$

Similarly, the non-vanishing covariant derivatives of coordinates vector fields are given as follows:

$$\begin{aligned}
&\nabla_{X_1}X_3 = \frac{1}{2}c_xX_2, \quad \nabla_{X_2}X_3 = \frac{1}{2}c_yX_2, \quad \nabla_{X_1}X_4 = \frac{1}{2}c_xX_1,\\
&\nabla_{X_2}X_4 = \frac{1}{2}c_yX_1, \quad \nabla_{X_3}X_3 = c_zX_2\\
&\nabla_{X_3}X_4 = \frac{1}{2}(cc_y + kc_x)X_1 + \frac{1}{2}(kc_y + cc_x)X_2 - \frac{1}{2}c_xX_3 - \frac{1}{2}c_yX_4, \quad \nabla_{X_4}X_4 = c_tX_1.
\end{aligned} \tag{2.3}$$

Using (2.3) into (2.1) and from (1.2) we can completely obtain the possibly non-vanishing components of the curvature tensor, then:

$$\begin{aligned}
&R(X_3, X_1)X_1 = \frac{1}{2}c_{xx}X_2\\
&R(X_3, X_1)X_2 = \frac{1}{2}c_{xy}X_2\\
&R(X_3, X_1)X_3 = \frac{1}{4}(2c_{xz} - c_xc_y)X_2\\
&R(X_3, X_1)X_4 = \frac{1}{4}(2cc_{xy} + 2kc_{xx} - 2c_{xz} + c_xc_y)X_1 + \frac{1}{2}(kc_{xy} + cc_{xx})X_2\\
&-\frac{1}{2}c_{xx}X_3 - \frac{1}{2}c_{xy}X_4
\end{aligned}$$

$$
\begin{aligned}
R(X_4,X_1)X_1 &= \frac{1}{2}c_{xx}X_1\\
R(X_4,X_1)X_2 &= \frac{1}{2}c_{xy}X_1\\
R(X_4,X_1)X_3 &= \frac{1}{2}(cc_{xy}+kc_{xx})X_1+\frac{1}{4}(2kc_{xy}+c_x^2+2cc_{xx}-2c_{xt})X_2\\
&\quad -\frac{1}{2}c_{xx}X_3-\frac{1}{2}c_{xy}X_4,\\
R(X_4,X_1)X_4 &= \frac{1}{4}(2c_{xt}-c_x^2)X_1\\
R(X_3,X_2)X_1 &= \frac{1}{2}c_{xy}X_1\\
R(X_3,X_2)X_2 &= \frac{1}{2}c_{yy}X_2\\
R(X_3,X_2)X_3 &= \frac{1}{4}(2c_{yz}-c_y^2)X_2\\
R(X_3,X_2)X_4 &= \frac{1}{4}(2kc_{xy}+2cc_{yy}+c_y^2-2c_{yz})X_1+\frac{1}{2}(kc_{yy}+cc_{xy})X_2\\
&\quad -\frac{1}{2}c_{xy}X_3-\frac{1}{2}c_{yy}X_4\\
R(X_4,X_2)X_1 &= \frac{1}{2}c_{xy}X_1\\
R(X_4,X_2)X_2 &= \frac{1}{2}c_{yy}X_2\\
R(X_4,X_2)X_3 &= \frac{1}{2}(cc_{yy}+kc_{xy})X_1+\frac{1}{4}(-2c_{yt}+2kc_{yy}+c_xc_y+2cc_{xy})X_2\\
&\quad -\frac{1}{2}c_{xy}X_3-\frac{1}{2}c_{yy}X_4\\
R(X_4,X_2)X_4 &= \frac{1}{4}(2c_{yt}-c_xc_y)X_1\\
R(X_4,X_3)X_1 &= \frac{1}{4}(2c_{xz}-c_xc_y)X_1+\frac{1}{4}(c_x^2-2c_{xt})X_2\\
R(X_4,X_3)X_2 &= \frac{1}{4}(2c_{yz}-c_y^2)X_1+\frac{1}{4}(c_xc_y-2c_{yt})X_2\\
R(X_4,X_3)X_3 &= \frac{1}{4}(2cc_{yz}+2kc_{xz}-cc_y^2-kc_xc_y)X_1+\frac{1}{4}(kc_x^2+cc_xc_y\\
&\quad +2kc_{yz}+2cc_{xz}-4c_{zt})X_2+\frac{1}{4}(c_xc_y-2c_{xz})X_3+\frac{1}{4}(c_y^2-2c_{yz})X_4\\
R(X_4,X_3)X_4 &= \frac{1}{4}(4c_{zt}-2cc_{yt}-2kc_{xt}-cc_xc_y-kc_y^2)X_1+\frac{1}{4}(cc_x^2+kc_xc_y\\
&\quad -2kc_{yt}-2cc_{xt})X_2+\frac{1}{4}(2c_{xt}-c_x^2)X_3+\frac{1}{4}(2c_{yt}-c_xc_y)X_4. \qquad (2.4)
\end{aligned}
$$

Now, consider the metric (1.2), we determine all the nonzero components of the $(0,4)$-curvature tensor R by straightforward computations. which are given as follows:

$$
\begin{aligned}
R(X_3,X_1,X_1,X_4) &= \frac{1}{2}c_{xx},\\
R(X_3,X_1,X_2,X_4) &= \frac{1}{2}c_{xy},\\
R(X_3,X_2,X_2,X_4) &= \frac{1}{2}c_{yy},\\
R(X_3,X_1,X_4,X_3) &= \frac{1}{4}(c_xc_y-2c_{xz})\\
R(X_4,X_1,X_3,X_4) &= \frac{1}{4}(c_x^2-2c_{xt})\\
R(X_3,X_2,X_3,X_4) &= \frac{1}{4}(2c_{yz}-c_y^2)\\
R(X_4,X_2,X_4,X_3) &= \frac{1}{4}(2c_{yt}-c_xc_y)\\
R(X_4,X_3,X_3,X_4) &= \frac{1}{4}(kc_x^2+kc_y^2+2cc_xc_y-4c_{zt})\\
R(X_4,X_2,X_2,X_4) &= \frac{1}{2}c_{yy}.
\end{aligned}
\tag{2.5}
$$

Now, we can calculate the components $Ric(X_i,X_j)$ with respect to $\{Xi\}$, $i=1,2,3,4$ of the Ricci tensor Ric of M. We obtain

$$
Ric = \begin{pmatrix}
0 & 0 & \frac{1}{2}c_{xy} & \frac{1}{2}c_{xx}\\
0 & 0 & \frac{1}{2}c_{yy} & \frac{1}{2}c_{xy}\\
\frac{1}{2}c_{xy} & \frac{1}{2}c_{yy} & \frac{1}{2}(c_{zy}-c_y^2) & \frac{1}{2}A(x,y)\\
\frac{1}{2}c_{xx} & \frac{1}{2}c_{xy} & \frac{1}{2}A(x,y) & \frac{1}{2}(2c_{xt}-c_x^2)
\end{pmatrix}, \tag{2.6}
$$

where $A(x,y) := kc_{xx}+kc_{yy}+2cc_{xy}-c_{xz}-c_{yt}+c_xc_y$. Now, from (1.2) and (2.6) also determine the component of the Ricci operator. Then we obtain

$$
\hat{Ric} = \begin{pmatrix}
\frac{1}{2}c_{xy} & \frac{1}{2}c_{yy} & \frac{1}{2}(2c_{yz}-c_y^2-kc_{xy}-cc_{yy}) & \frac{1}{2}B(x,y)\\
\frac{1}{2}c_{xx} & \frac{1}{2}c_{xy} & \frac{1}{2}B(x,y) & \frac{1}{2}(2c_{xt}-c_x^2-kc_{xy}-cc_{xx})\\
0 & 0 & \frac{1}{2}c_{xy} & \frac{1}{2}c_{xx}\\
0 & 0 & \frac{1}{2}c_{yy} & \frac{1}{2}c_{xy}
\end{pmatrix}, \tag{2.7}
$$

where $B(x,y) := kc_{yy}+cc_{xy}-c_{xz}-c_{yt}+c_xc_y$. At this stage, it is now possible to calculate of the covariant derivative ∇Ric of the metric (1.2). We can proceed to compute the covariant derivative ∇Ric of the metric (1.2). By utilizing (2.3) and (2.7) the following proposition can be proven:

Proposition 2.1. *The non-vanishing components*

$$\nabla_{X_i}Ric(X_j,X_k) = (\nabla_{X_i}Ric)(X_j,X_k)$$

of the covariant derivative ∇Ric *of Walker metric* (1.2), *are given by*

$$
\begin{aligned}
\nabla_{X_1} Ric(X_1,X_3) &= \nabla_{X_1} Ric(X_2,X_4) = \nabla_{X_1} Ric(X_3,X_1) = \nabla_{X_1} Ric(X_4,X_2) \\
&= \nabla_{X_2} Ric(X_1,X_4) = \nabla_{X_2} Ric(X_4,X_1) = \frac{1}{2}c_{xxy}, \\
\nabla_{X_1} Ric(X_1,X_4) &= \nabla_{X_1} Ric(X_1,X_4) = \frac{1}{2}c_{xxx}, \\
\nabla_{X_1} Ric(X_2,X_3) &= \nabla_{X_1} Ric(X_3,X_2) = \nabla_{X_2} Ric(X_1,X_3) = \nabla_{X_2} Ric(X_3,X_1) \\
&= \nabla_{X_2} Ric(X_2,X_4) = \nabla_{X_2} Ric(X_4,X_2) = \frac{1}{2}c_{xyy}, \\
\nabla_{X_2} Ric(X_2,X_3) &= \nabla_{X_2} Ric(X_3,X_2) = \frac{1}{2}c_{yyy}, \\
\nabla_{X_1} Ric(X_3,X_3) &= c_{xyz} - c_y c_{xy} - \frac{1}{2}c_x c_{yy}, \\
\nabla_{X_1} Ric(X_3,X_4) &= \nabla_{X_1} Ric(X_4,X_3) = \frac{1}{2}(kc_{xxx} + kc_{yyy} - c_{xxz} - c_{xyt} + c_y c_{xx}) \\
&\quad + c_x c_{xy} + cc_{xxy}, \\
\nabla_{X_1} Ric(X_4,X_4) &= c_{xxt} - \frac{3}{2}c_x c_{xx}, \\
\nabla_{X_2} Ric(X_2,X_3) &= \nabla_{X_2} Ric(X_3,X_2) = \frac{1}{2}c_{yyy}, \\
\nabla_{X_2} Ric(X_3,X_3) &= c_{yyz} - \frac{3}{2}c_y c_{yy}, \\
\nabla_{X_2} Ric(X_3,X_4) &= \nabla_{X_2} Ric(X_4,X_3) = \frac{1}{2}(kc_{xxy} + kc_{yyy} - c_{xxy} - c_{yyt} + c_x c_{yy}) \\
&\quad + c_y c_{xy} + cc_{xyy}, \\
\nabla_{X_2} Ric(X_4,X_4) &= c_{xyt} - c_x c_{xy} - \frac{1}{2}c_y c_{xx}, \\
\nabla_{X_3} Ric(X_1,X_3) &= \nabla_{X_3} Ric(X_3,X_1) = \frac{1}{2}c_{xyz} - \frac{1}{4}c_x c_{yy}, \\
\nabla_{X_3} Ric(X_1,X_4) &= \nabla_{X_3} Ric(X_4,X_1) = \frac{1}{2}c_{xxz} + \frac{1}{4}c_y c_{xx}, \\
\nabla_{X_3} Ric(X_2,X_3) &= \nabla_{X_3} Ric(X_3,X_2) = \frac{1}{2}c_{yyz} - \frac{1}{4}c_y c_{yy}, \\
\nabla_{X_3} Ric(X_2,X_4) &= \nabla_{X_3} Ric(X_4,X_2) = \frac{1}{2}c_{xyz} + \frac{1}{4}c_x c_{yy}, \\
\nabla_{X_3} Ric(X_3,X_3) &= c_{yzz} - c_y c_{yz} - c_z c_{yy}, \\
\nabla_{X_3} Ric(X_3,X_4) &= \nabla_{X_3} Ric(X_4,X_3) = \frac{1}{4}(c_y c_{xz} - cc_y c_{xy} - kc_x c_{xy} - cc_x c_{yy} \\
&\quad + kc_y c_{xx} - c_y c_{yt}) + \frac{1}{2}(kc_{xxz} + kc_{yyz} + c_z c_{xy} - c_{xzz} - c_{yzt} + cc_y c_{xy}) \\
&\quad + cc_{xyz} + c_x c_{yz}, \\
\nabla_{X_3} Ric(X_4,X_4) &= \frac{1}{2}(-3c_x c_{xz} - cc_y c_{xx} - kc_y c_{xy} + cc_x c_{xy} + kc_x c_{yy} - c_x c_{yt}) \\
&\quad + c_{xzt} + c_y c_{xt}, \\
\nabla_{X_4} Ric(X_1,X_3) &= \nabla_{X_4} Ric(X_3,X_1) = \frac{1}{2}c_{xyt} + \frac{1}{4}c_y c_{xx},
\end{aligned}
\tag{2.8}
$$

$$\nabla_{X_4} Ric(X_1, X_4) = \nabla_{X_4} Ric(X_4, X_1) = \frac{1}{2} c_{xxt} - \frac{1}{4} c_x c_{xx},$$
$$\nabla_{X_4} Ric(X_2, X_3) = \nabla_{X_4} Ric(X_3, X_2) = \frac{1}{2} c_{yyt} + \frac{1}{4} c_x c_{yy},$$
$$\nabla_{X_4} Ric(X_2, X_4) = \nabla_{X_4} Ric(X_4, X_2) = \frac{1}{2} c_{xyt} - \frac{1}{4} c_y c_{xx}$$
$$\nabla_{X_4} Ric(X_3, X_3) = \frac{1}{2}(-3c_y c_{yt} + c c_y c_{xy} - k c_x c_{xy} - c c_x c_{yy} + k c_y c_{xx} - c_{xz} c_y) + c_{yzt} + c_x c_{yz},$$
$$\nabla_{X_4} Ric(X_3, X_4) = \nabla_{X_4} Ric(X_4, X_3) = \frac{1}{4}(c_x c_{yt} - c c_y c_{xx} - k c_y c_{xy} + c c_x c_{xy} + k c_x c_{yy} - c_x c_{xz}) + \frac{1}{2}(k c_{xxt} + k c_{yyt} + c_t c_{xy} - c_{xzt} - c_{ytt}) + c c_{xyt} + c_y c_{xt},$$
$$\nabla_{X_4} Ric(X_4, X_4) = c_{xtt} - c_x c_{xt} - c_t c_{xx}.$$

3. Einstein-like Walker metrics g_c

Einstein-like metrics are generalization of Einstein metrics which introduced and studied by Gray[5]. Through three interesting classes $\mathcal{P}, \mathcal{A}$, and $\mathcal{B}$. Also, in [1], some of geometric properties of Walker metrics g_c, are investigated. Therefore, according to our purpose, in this part, we focus on $\mathcal{P}$-metric, $\mathcal{A}$-metric and $\mathcal{B}$-metric. Suppose (M, g) be a Pseudo-Riemannian manifold:

(i) The necessary and sufficient condition for M to belong to the class of manifolds with parallel Ricci tensor $(\mathcal{P})$ is that its Ricci tensor is parallel, i.e.
$$\nabla_X Ric(Y, Z) = 0 \tag{3.1}$$
for all vector fields X, Y, Z tangent to M.

(ii) A necessary and sufficient condition for M to belong to class $\mathcal{A}$ is that its Ricci tensor is cyclic-parallel, i.e.
$$\nabla_X Ric(Y, Z) + \nabla_Y Ric(Z, X) + \nabla_Z Ric(X, Y) = 0 \tag{3.2}$$
for all vector fields X, Y, Z tangent to M. (3.2) is equivalent to requiring that ∇ is killing tensor, that is,
$$\nabla_X Ric(X, X) = 0 \tag{3.3}$$

(iii) A necessary and sufficient condition for M to belong to class $\mathcal{B}$ is that its Ricci tensor is a Codazzi tensor, i.e.
$$(\nabla_X)(Y, Z) = \nabla_Y Ric(X, Z). \tag{3.4}$$

Because Einstein-like manifolds be between the classes of Pseudo-Rimannian manifolds with parallel Ricc tensor $(\mathcal{P})$ and the class of Pseudo-Rimannian manifolds with constant scalar curvature $(\mathcal{C})$, that is $\mathcal{P} = \mathcal{A} \cap \mathcal{B} \subset \mathcal{A} \cup \mathcal{B} \subset \mathcal{C}$.

In [1], [3], [4], [6], Einstein-like Pseudo-Riemannian metrics have been studied by several authors.

In this part, taking into account the metric (1.2) and force the defining function c to be of the following special form:

$$c(x,y,z,t) = xp(z,t) + yq(z,t) + s(z,t), \tag{3.5}$$

where p, q and s are C^∞ real-valued functions, we obtain the following result by applying (3.1), (3.3) and (3.4) respectively to the components of ∇Ric described in proposition (2.1):

Theorem 3.1. *The necessary and sufficient condition for the metric described in* (1.2)

i) *to be Ricci-parallel, is that c is in the form of* (3.5) *and where p and q are real-valued functions of C which in* (3.6) *satisfying.*

$$\begin{aligned} &p^2 = 2p_t + e(z), \qquad q^2 = 2q_z + h(t), \\ &qp_z - qq_t - 2(p_{zz} + q_{zt}) + 4pq_z = 0, \\ &pq_t - pp_z - 2(p_{zt} + q_{tt}) + 4qp_t = 0, \\ &3pp_z + pq_t - 2p_{zt} - 2qp_t = 0, \\ &3qq_t + qp_z - 2q_{zt} - 2pq_z = 0, \end{aligned} \tag{3.6}$$

for two arbitrary smooth functions h and e.

ii) *to belong to class $\mathcal{A}$ is that*

$$p^2 = 2p_t + e(z), \qquad q^2 = 2q_z + h(t), \tag{3.7}$$

for two arbitrary smooth functions h and e.

iii) *belongs to class $\mathcal{B}$ if c is in the form of* (3.5) *and p and q are satisfied in the following relations*

$$3qp_z + 5qq_t - 2p_{zz} - 6q_{zt} = 0, \quad 3pq_t + 5pp_z - 2q_{tt} - 6p_{zt} = 0. \tag{3.8}$$

4. **Locally symmetric Walker metrics** (1.2)

We know that a connected pseudo-Riemannian manifold M, for it to be a symmetric space, its geodesic symmetries must be isometric. Also, if a manifold is isometric with a symmetric space, that manifold is called locally symmetric. A Pseudo-Riemannian manifold (M,g) is locally symmetric if and only if $\nabla R = 0$. In particular, a locally symmetric space is Ricci-parallel. Now, consider a Walker metric (1.2) and by straightforward calculation, we obtain the possibly non-vanishing components $\nabla_R R_{ijem} = (\nabla_{X_k} R)(X_i, X_j, X_e, X_m)$ of the covariant derivative of R, $i,j,k,l \in \{1,2,3,4\}$, are given by:

$$
\begin{aligned}
&\nabla_1 R_{3114} = \frac{1}{2}c_{xxx}, \qquad \nabla_1 R_{3124} = \frac{1}{2}c_{xxy}, \qquad \nabla_1 R_{3223} = \frac{1}{2}c_{xyy},\\
&\nabla_2 R_{3114} = \frac{1}{2}c_{xxy}, \qquad \nabla_2 R_{3124} = \frac{1}{2}c_{xyy}, \qquad \nabla_2 R_{3223} = \frac{1}{2}c_{yyy},\\
&\nabla_3 R_{3114} = \frac{1}{2}c_{xxz}, \qquad \nabla_3 R_{3124} = \frac{1}{2}c_{xyz}, \qquad \nabla_3 R_{3223} = \frac{1}{2}c_{yyz},\\
&\nabla_4 R_{3114} = \frac{1}{2}c_{xxt}, \qquad \nabla_4 R_{3124} = \frac{1}{2}c_{xyt}, \qquad \nabla_4 R_{3223} = \frac{1}{2}c_{yyt},\\
&\nabla_1 R_{3143} = \frac{1}{4}(c_{xxy} + 2c_x c_{xy} - 2c_{xxz}), \qquad \nabla_2 R_{3143} = \frac{1}{4}(2c_{xy}c_y + c_x c_{yy} - 2c_{xyz}),\\
&\nabla_3 R_{3143} = \frac{1}{2}(c_x c_{yz} - c_{xzz} + c_z c_{xy}),\\
&\nabla_4 R_{3143} = \frac{1}{4}(-2c_{xzt} + c_x c_{yt} + c_x c_{xz} + cc_y c_{xx} + kc_x c_{xx} + kc_y c_{xy} + cc_x c_{xy}),\\
&\nabla_1 R_{4134} = \frac{1}{2}(c_x c_{xx} - c_{xxt}), \qquad \nabla_2 R_{4134} = \frac{1}{2}(c_x c_{xy} - c_{xyt}),\\
&\nabla_3 R_{4134} = \frac{1}{4}(2c_x c_{xz} - 2c_{xzt} + c_x c_{yt} - c_y c_{xt} + cc_y c_{xx} + cc_x c_{xy} + kc_x c_{xx} + kc_y c_{xy}),\\
&\nabla_4 R_{4134} = \frac{1}{2}(c_x c_{xt} - c_{xtt} - c_t c_{xx}),\\
&\nabla_1 R_{3234} = \frac{1}{2}(c_{xyz} - c_y c_{xy}), \qquad \nabla_2 R_{3234} = \frac{1}{2}(c_{yyz} - c_y c_{yy}),\\
&\nabla_3 R_{3234} = \frac{1}{2}(c_{yzz} - c_y c_{yz} - c_z c_{yy}),\\
&\nabla_4 R_{3234} = \frac{1}{4}(2c_{yzt} - 3c_y c_{yt} - c_y c_{xz} + 2c_x c_{yz} - kcc_{yy} - cc_x c_{yy}),\\
&\nabla_1 R_{4243} = \frac{1}{4}(2c_{xyt} - c_{xx}c_y - c_x c_{xy}),\\
&\nabla_2 R_{4243} = \frac{1}{4}(2c_{yyt} - 2c_y c_{xy} - c_x c_{yy} + c_y c_{yy}),\\
&\nabla_3 R_{4243} = \frac{1}{4}(-kc_x c_{xy} - kc_y c_{yy} + 2c_{yzt} - 2c_x c_{yz}\\
&\qquad + 2c_z c_{yy} - c_{xz}c_y + c_y c_{yt} - cc_x c_{yy} - cc_y c_{xy}),\\
&\nabla_4 R_{4243} = \frac{1}{4}(2c_{ytt} - 2c_{xt}c_y - kc_y c_{yy} + cc_x c_{yy} - 2c_t c_{xy}),\\
&\nabla_1 R_{4334} = -\frac{1}{2}(kc_x c_{xx} + kc_y c_{xy} + cc_{xx}c_y + cc_x c_{xy} - 2c_{xzt} + c_x c_{yt} + c_x c_{xz}),\\
&\nabla_2 R_{4334} = \frac{1}{2}(kc_x c_{xy} + kc_y c_{yy} + cc_{xy}c_y + cc_x c_{yy} + c_y c_{yt} + c_y c_{xz} - 2c_{yzt}),\\
&\nabla_3 R_{4334} = kc_x c_{xz} + kc_y c_{yz} + c_z c_{yt} + cc_{xz}c_y + cc_x c_{yt} - c_y c_{zt} - 2c_{zzt},\\
&\nabla_4 R_{4334} = kc_x c_{xt} + kc_y c_{yt} + c_t c_{xz} + cc_{xt}c_y + cc_x c_{yt} - c_x c_{zt} - c_{ztt}).
\end{aligned}
\tag{4.1}
$$

When Walker metric (1.2) is Ricci-parallel, referring to Theorem 3.1, we know that the function c defined in (3.5) satisfies (3.6). Therefore, the non-zero components of the covariant derivative R, from (4.1), will be as follows:

$$
\begin{aligned}
\nabla_4 R_{3143} &= \frac{1}{4}(2p_{zt} - pq_t - pp_z) = \nabla_1 R_{4334},\\
\nabla_3 R_{1334} &= \frac{1}{2}(pq_z - p_{zz}),\\
\nabla_3 R_{4134} &= \frac{1}{4}(2pp_z - 2p_{zt} + pq_t - qp_t),\\
\nabla_4 R_{4134} &= \frac{1}{2}(pp_t - p_{tt}),\\
\nabla_4 R_{3234} &= \frac{1}{4}(2q_{zt} - 3qq_t - qp_z + 2pq_z),\\
\nabla_3 R_{3234} &= \frac{1}{2}(q_{zz} - qq_z),\\
\nabla_3 R_{4243} &= \frac{1}{4}(2q_{zt} - p_z q - 2pq_z + 2qq_t),\\
\nabla_4 R_{4243} &= \frac{1}{2}(q_{tt} - p_t q),\\
\nabla_1 R_{4334} &= \frac{1}{2}(-2p_{zt} + pp_z + pq_t),\\
\nabla_2 R_{4334} &= \frac{1}{2}(kc_x c_{xy} + kc_y c_{yy} + cc_y c_{xy} + cc_x c_{yy} + c_y c_{yt} + c_y c_{xz} - 2c_{yzt}),\\
\nabla_3 R_{3434} &= x(p_{zzt} - p(p_z q + pq_z) + qp_{zt} - p_z q_t) + y(q_{zzt} - q(qp_z + pq_z)\\
&+ qq_{zt} - q_z q_t) + s_{zzt} + qs_{zt} - s_z q_t - spq_z - sqp_z - kqq_z - kpp_z,\\
\nabla_4 R_{3434} &= x(p_{ztt} - p(pq_t + qp_t) + pq_{zt} - pp_z) + y(q_{ztt} - q(pq_t + qp_t)\\
&+ pq_{zt} - q_t p_z) + s_{ztt} + ps_{zt} + sp_z - sqp_t - spq_t - kpp_t - kqq_t.
\end{aligned}
\tag{4.2}
$$

Again, taking into account the fact that g is Ricci-parallel, therefore, the necessary and sufficient condition for the metric (1.2) to be locally symmetric, is that c is of the special form (3.5) and in relations (3.6) also applies.

$$
\begin{gathered}
q_{zz} - qq_z = 0, \qquad p_{tt} - pp_t = 0, \qquad q_{tt} - qp_t = 0, \qquad p_{zz} - pq_z = 0, \quad (4.3)\\
2p_{zt} - pq_t - pp_z = 0, \qquad 2q_{zt} - qp_z - qq_t = 0, \qquad 2q_{zt} - p_z q - 2pq_z + qq_t = 0,\\
2q_{zt} - 3qq_t + 2pq_z - qp_z = 0, \qquad p_{zzt} - p(p_z q + pq_z) + qp_{zt} - p_z q_t = 0,\\
q_{zzt} - q(qp_z + pq_z) + qq_{zt} - q_z q_t = 0,\\
p_{ztt} - p(pq_t + qp_t) + pq_{zt} - pp_z = 0, \qquad q_{ztt} - q(pq_t + qp_t) + pq_{zt} - q_t p_z = 0,\\
s_{zzt} + qs_{zt} - s_z q_t - s(pq_z + qp_z) - kpp_z - kqq_z = 0,\\
s_{ztt} + ps_{zt} + sp_z - s(qp_t + pq_t) - kpp_t - kqq_t = 0.
\end{gathered}
$$

Using both (3.6) and (4.3) by standard calculations we obtain the following:

Theorem 4.1. *The Walker metric g, defined in (1.2), is locally symmetric if and only if c is of the special form given in (3.5), and the functions p, q and s which are smooth (C^∞) and real-valued satisfying one of the following sets of conditions:*

i) $q = ap$ *and*

$$p_z = \frac{a}{2}p^2 + k, \qquad p_t = \frac{1}{2}p^2 + \frac{k}{a},$$
$$s_{zzt} + aps_{zt} - s_t p_z - 2app_z s - ka^2 pp_z - kpp_z = 0$$

for two real constant $a \neq 0$, *and* k.

ii) $q = 0,\ p = p(t),\ s_{zt} = G(t)$ *and* $s_{ztt} + ps_{zt} - kpp_t = 0$

iii) $p = 0,\ q = q(z)$, *and* $s_{zt} = H(z)$ *and* $s_{zzt} - kqq_z + qs_{zt} = 0$

iv) $p = q = 0$, *and* $s_{zt} = \gamma$, γ *is a real constant.*

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