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Geometric structures on Lorentzian para-Kenmotsu manifolds admitting a semi-symmetric metric connection

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Abstract. In this paper, we study Lorentzian para-Kenmotsu manifolds endowed with a semi-symmetric metric connection and establish necessary and sufficient conditions under which the Ricci tensor is ω-parallel with respect to this connection. These results extend classical notions of Ricci parallelism from Riemannian geometry to a broader non-Riemannian framework. In addition, we examine the behavior of concircular and projective curvature tensors on such manifolds and derive structural identities that highlight the influence of semi-symmetric torsion on fundamental geometric invariants. To support our theoretical developments, we construct an explicit 4-dimensional illustration. The findings deepen the understanding of non-Riemannian geometric structures and suggest potential applications in generalized theories of gravity.

Keywords: Semi-symmetric metric connection, ω-parallel Ricci tensor, Concircular curvature tensor, Projective curvature tensor, Codazzi type.

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1. **Introduction**

Curvature forms the geometric backbone of differential geometry and theoretical physics, shaping our understanding of space, time, and gravitation–most profoundly within the Lorentzian framework. In classical mechanics, Newton's laws indicate that the force required to sustain uniform motion along a curved trajectory is proportional to the curvature of the path. In general relativity, Einstein revolutionized our understanding of gravitation by showing that the motion of bodies is governed by the curvature of spacetime–a Lorentzian manifold shaped by the distribution of mass and energy [25]. Moreover, spaces of constant curvature serve as essential models in differential geometry and mathematical physics, with profound implications for the study of relativistic theories and cosmological structures.
The concept of locally symmetric Riemannian manifolds, initiated by É. Cartan in 1926 [5, 6], provides a natural extension of spaces with constant curvature and has given rise to a rich landscape of generalizations within the differential geometry.Among these, Takahashi's notion of locally φ-symmetric Sasakian manifolds [32] and Kenmotsu's manifolds [18] have laid foundational groundwork in the development of contact and paracontact geometry. Blair's extensive contributions [4] further advanced the theory of contact structures in the Riemannian context.

Parallel to these developments, Sato [29] initiated the study of almost paracontact Riemannian manifolds, leading to the introduction of para-Sasakian and special para-Sasakian manifolds by Adati and Matsumoto [1]. Matsumoto [22] extended these structures to the Lorentzian category, with further developments by Mihai [23] and Sinha and Prasad [30] forming the basis for the theory of para-Kenmotsu manifolds. The class of Lorentzian para-Kenmotsu (briefly, $(\mathcal{LP}\text{-}\mathcal{K})_n$) manifolds, has since garnered considerable attention [9, 10, 11, 26, 27, 28], especially due to its applications in geometric analysis and relativistic models.

On another front, Friedmann and Schouten [14] introduced the concept of semi-symmetric connections, later generalized by Hayden [16], and studied extensively by Yano [33, 34], Imai [17], and others. The semi-symmetric metric connection (briefly, $\mathcal{SSMC}$), which preserves the metric while introducing a non-trivial torsion, has proven effective in extending Riemannian results to broader non-Riemannian geometries. The works of Berman [3] and Chaubey et al. [7] demonstrate the influence of such connections in modifying classical curvature behaviors, particularly within Kenmotsu-type structures.

To distinguish our results and minimize notational conflict with existing literature, we denote the structure 1-form by ω instead of the conventional η. This change is notational only and does not affect the underlying geometric interpretation.

The study of Lorentzian manifolds equipped with additional geometric structures plays a pivotal role in both differential geometry and theoretical physics. Among such structures, the $(\mathcal{LP}–\mathcal{K})_n$ manifold, an n-dimensional Lorentzian para-Kenmotsu manifold–provides a natural geometric setting to examine the interplay between Lorentzian metrics and time-like structural vector fields [20]. These manifolds offer valuable insights into models characterized by non Riemannian features and causal structures intrinsic to relativistic frameworks.

This paper presents new results on $(\mathcal{LP}–\mathcal{K})_n$ admitting $\mathcal{SSMC}$. It extends classical differential geometry by analyzing curvature tensors, particularly concircular and projective tensors, under $\mathcal{SSMC}$. New properties of ω-parallel, Codazzi, and cyclic parallel Ricci tensors are derived, leading to refined classification results. The study also unifies various curvature symmetries, providing a concise geometric framework. These results have potential applications in curvature-driven spacetime models within extended theories of gravity.

This paper addresses this gap by establishing sharp conditions under which $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifolds with a semi-symmetric metric connection admit an ω-parallel Ricci tensor. We also analyze the behavior of the concircular and projective curvature tensors under this modified connection and derive several structural consequences.

Structure of the Paper: Section 2 provides preliminaries on $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifolds and semi-symmetric metric connections. In Section 3, we derive fundamental identities associated with the $\mathcal{SSMC}$ structure. Section 4 presents necessary and sufficient conditions for the Ricci tensor to be ω-parallel and offers their geometric interpretation. Section 5 investigates the behavior of concircular and projective curvature tensors under $\mathcal{SSMC}$. In Section 6, we construct a concrete 4-dimensional example of an $(\mathcal{LP}\text{-}\mathcal{K})_4$ manifold equipped with a semi-symmetric metric connection, validating our theoretical results. The final sections provide discussion and conclusion, as well as scope and significance.

2. Preliminaries

In this section, we provide essential definitions and notations that form the groundwork for the results developed in the sequel.

Definition 2.1. *An n-dimensional Lorentzian metric manifold is a smooth differentiable manifold $\mathscr{M}$ endowed with a Lorentzian metric g, i.e., a semi-Riemannian metric of signature $(1, n-1)$ [2, 15]. Such manifolds play a fundamental role in general relativity, where spacetime is modeled as a 4-dimensional semi-Riemannian manifold.*

Definition 2.2. [12] *A Lorentzian metric manifold $\mathscr{M}$ admits a Lorentzian almost paracontact structure if there exist tensor fields (φ, ζ, ω) and a Lorentzian* *metric g satisfying the following properties for all vector fields $\mathcal{A}, \mathcal{B} \in \chi(\mathscr{M})$:*

$$\omega(\zeta) = -1, \quad \varphi^2\mathcal{A} = \mathcal{A} + \omega(\mathcal{A})\zeta, \tag{2.1}$$

$$g(\varphi\mathcal{A}, \varphi\mathcal{B}) = g(\mathcal{A}, \mathcal{B}) + \omega(\mathcal{A})\omega(\mathcal{B}), \tag{2.2}$$

$$g(\mathcal{A}, \zeta) = \omega(\mathcal{A}), \quad \omega(\varphi\mathcal{A}) = 0, \quad \varphi\zeta = 0. \tag{2.3}$$

Unless otherwise stated, all covariant derivatives ∇ in this section are taken with respect to the Levi-Civita connection of the Lorentzian metric g.

Definition 2.3. [13] *A Lorentzian almost paracontact manifold $(\mathscr{M}, \varphi, \zeta, \omega, g)$ is called an $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifold if the Levi-Civita connection ∇ of g satisfies:*

$$(\nabla_{\mathcal{A}}\varphi)(\mathcal{B}) = -g(\varphi\mathcal{A}, \mathcal{B})\zeta - \omega(\mathcal{B})\varphi\mathcal{A}, \tag{2.4}$$

for all $\mathcal{A}, \mathcal{B} \in \chi(\mathscr{M})$.

On an $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifold, the following important identities also hold:

$$\nabla_{\mathcal{A}}\zeta = -\mathcal{A} - \omega(\mathcal{A})\zeta, \quad (\nabla_{\mathcal{A}}\omega)(\mathcal{B}) = -g(\mathcal{A}, \mathcal{B}) - \omega(\mathcal{A})\omega(\mathcal{B}). \tag{2.5}$$

Let $\mathscr{R}$, $\mathcal{S}$, and $\mathcal{Q}$ denote the Riemann curvature tensor, Ricci curvature tensor, and Ricci operator of $\mathscr{M}$, respectively. Then, the following curvature identities are satisfied by $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifolds:

$$g(\mathscr{R}(\mathcal{A}, \mathcal{B})\mathcal{C}, \zeta) = \omega(\mathscr{R}(\mathcal{A}, \mathcal{B})\mathcal{C}) = g(\mathcal{B}, \mathcal{C})\omega(\mathcal{A}) - g(\mathcal{A}, \mathcal{C})\omega(\mathcal{B}), \tag{2.6}$$

$$\mathscr{R}(\zeta, \mathcal{A})\mathcal{B} = g(\mathcal{A}, \mathcal{B})\zeta - \omega(\mathcal{B})\mathcal{A}, \tag{2.7}$$

$$\mathscr{R}(\mathcal{A}, \mathcal{B})\zeta = \omega(\mathcal{B})\mathcal{A} - \omega(\mathcal{A})\mathcal{B}, \tag{2.8}$$

$$\mathscr{R}(\zeta, \mathcal{A})\zeta = \mathcal{A} + \omega(\mathcal{A})\zeta, \tag{2.9}$$

$$\mathcal{S}(\mathcal{A}, \zeta) = (n-1)\omega(\mathcal{A}), \quad \mathcal{S}(\zeta, \zeta) = -(n-1), \tag{2.10}$$

$$\mathcal{Q}\zeta = (n-1)\zeta, \tag{2.11}$$

$$\mathcal{S}(\varphi\mathcal{A}, \varphi\mathcal{B}) = \mathcal{S}(\mathcal{A}, \mathcal{B}) + (n-1)\omega(\mathcal{A})\omega(\mathcal{B}). \tag{2.12}$$

Definition 2.4. [8] *An $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifold is said to be an ω-Einstein manifold if its Ricci tensor satisfies the relation:*

$$\mathcal{S}(\mathcal{A}, \mathcal{B}) = p\, g(\mathcal{A}, \mathcal{B}) + q\, \omega(\mathcal{A})\omega(\mathcal{B}),$$

where p and q are scalar functions on $\mathscr{M}$. In the special case $q = 0$, the manifold reduces to an Einstein manifold.

3. Semi-Symmetric Metric Connection

Let the Levi-Civita connection on an $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifold be denoted by ∇. A linear connection $\bar{\nabla}$ on $\mathcal{M}$ is said to be $\mathcal{SSMC}$ if its torsion tensor satisfies

$$\bar{T}(A,B) = \omega(B)A - \omega(A)B. \tag{3.1}$$

It becomes a semi-symmetric metric connection if it additionally satisfies the metric compatibility condition

$$\bar{\nabla} g = 0. \tag{3.2}$$

in addition to the torsion condition (3.1). Throughout this article, the term 'connection $\bar{\nabla}$' is sometimes used to denote the semi-symmetric metric connection. If the condition (3.2) fails to hold, then $\bar{\nabla}$ is referred to as a semi-symmetric non-metric connection.

The relation between the Levi-Civita connection ∇ and the connection $\bar{\nabla}$, originally established by K. Yano [34], is expressed as:

$$\bar{\nabla}_{\mathcal{A}}\mathcal{B} = \nabla_{\mathcal{A}}\mathcal{B} + \omega(\mathcal{B})\mathcal{A} - g(\mathcal{A},\mathcal{B})\zeta, \tag{3.3}$$

where ω is a 1-form and ζ is the associated vector field satisfying $\omega(\mathcal{A}) = g(\mathcal{A},\zeta)$ for any vector field $\mathcal{A}$.

Using aligns (2.1) through (2.4), together with the relation (3.3), we derive the following identities for the connection $\bar{\nabla}$:

$$(\bar{\nabla}_{\mathcal{A}}\omega)(\mathcal{B}) = (\nabla_{\mathcal{A}}\omega)(\mathcal{B}) - \omega(\mathcal{A})\omega(\mathcal{B}) - g(\mathcal{A},\mathcal{B}), \tag{3.4}$$

and

$$(\bar{\nabla}_{\mathcal{A}}\varphi)(\mathcal{B}) = -2g(\varphi\mathcal{A},\mathcal{B})\zeta - 2\omega(\mathcal{B})\varphi\mathcal{A}. \tag{3.5}$$

The Riemannian curvature tensor with respect to $\bar{\nabla}$ is defined as

$$\bar{\mathscr{R}}(\mathcal{A},\mathcal{B})\mathcal{C} = \bar{\nabla}_{\mathcal{A}}\bar{\nabla}_{\mathcal{B}}\mathcal{C} - \bar{\nabla}_{\mathcal{B}}\bar{\nabla}_{\mathcal{A}}\mathcal{C} - \bar{\nabla}_{[\mathcal{A},\mathcal{B}]}\mathcal{C}. \tag{3.6}$$

By combining (3.3) and (3.6), we obtain:

$$\begin{aligned}\bar{\mathscr{R}}(\mathcal{A},\mathcal{B})\mathcal{C} &= \mathscr{R}(\mathcal{A},\mathcal{B})\mathcal{C} - 2\omega(\mathcal{A})\omega(\mathcal{C})\mathcal{B} + 2\omega(\mathcal{B})\omega(\mathcal{C})\mathcal{A} \\ &\quad - 3g(\mathcal{A},\mathcal{C})\mathcal{B} + 3g(\mathcal{B},\mathcal{C})\mathcal{A} + 2g(\mathcal{B},\mathcal{C})\omega(\mathcal{A})\zeta - 2g(\mathcal{A},\mathcal{C})\omega(\mathcal{B})\zeta,\end{aligned} \tag{3.7}$$

where $\mathscr{R}$ and $\bar{\mathscr{R}}$ denote the curvature tensors with respect to ∇ and $\bar{\nabla}$, respectively.

From (3.7), we deduce:

$$\bar{\mathscr{R}}(\zeta,\mathcal{B})\mathcal{C} = 2\{g(\mathcal{B},\mathcal{C})\zeta - \omega(\mathcal{C})\mathcal{B}\}, \tag{3.8}$$

$$\bar{\mathscr{R}}(\mathcal{A},\zeta)\mathcal{C} = 2\{\omega(\mathcal{C})\mathcal{A} - g(\mathcal{A},\mathcal{C})\zeta\}, \tag{3.9}$$

$$\bar{\mathscr{R}}(\mathcal{A},\mathcal{B})\zeta = 2\{\omega(\mathcal{B})\mathcal{A} - \omega(\mathcal{A})\mathcal{B}\}. \tag{3.10}$$

Contracting (3.7) over $\mathcal{A}$, we obtain the Ricci tensor $\bar{\mathcal{S}}$:

$$\bar{\mathcal{S}}(\mathcal{B},\mathcal{C}) = \mathcal{S}(\mathcal{B},\mathcal{C}) + (3n-5)g(\mathcal{B},\mathcal{C}) + 2(n-2)\omega(\mathcal{B})\omega(\mathcal{C}), \tag{3.11}$$

where, $\bar{\mathcal{S}}$, and $\mathcal{S}$ are the Ricci tensors with respect to $\bar{\nabla}$, and ∇, respectively. Substituting $\mathcal{C} = \zeta$ in (3.11) yields:

$$\bar{\mathcal{S}}(\mathcal{B}, \zeta) = 2(n-1)\omega(\mathcal{B}). \tag{3.12}$$

Thus, from the relation (3.11), the Ricci operator $\bar{\mathcal{Q}}$ associated with $\bar{\nabla}$ satisfies:

$$\bar{\mathcal{Q}}\mathcal{B} = \mathcal{Q}\mathcal{B} + (3n-5)\mathcal{B} + 2(n-2)\omega(\mathcal{B})\zeta, \tag{3.13}$$

where, $\bar{\mathcal{S}}(\mathcal{B}, \mathcal{C}) = g(\bar{\mathcal{Q}}\mathcal{B}, \mathcal{C})$ and $\mathcal{S}(\mathcal{B}, \mathcal{C}) = g(\mathcal{Q}\mathcal{B}, \mathcal{C})$. Let $\{e_i\}_{i=1}^{n}$ be an orthonormal frame at each point of $\mathscr{M}$. Setting $\mathcal{B} = \mathcal{C} = e_i$ in (3.11) and summing over i, we get the scalar curvature $\bar{r}$ with respect to $\bar{\nabla}$:

$$\bar{r} = r + n(3n-7) + 4, \tag{3.14}$$

where,

$$\bar{r} = \sum_{i=1}^{n} \epsilon_i \bar{\mathcal{S}}(e_i, e_i), \qquad r = \sum_{i=1}^{n} \epsilon_i \mathcal{S}(e_i, e_i), \tag{3.15}$$

where, $g(e_i, e_i) = \epsilon_i$. The relation (3.10) shows that the manifold $\mathscr{M}$, when endowed with $\bar{\nabla}$, admits a paracontact structure compatible with the $\mathcal{SSMC}$.

4. ω-parallel Ricci tensor with connection $\bar{\nabla}$

Section 4 covers the exploration of the geometrical characteristics of ω-parallel Ricci tensor with respect to connection $\bar{\nabla}$. The concept of ω-parallelism on a Sasakian manifold was initiated by M. Kon [21].

Definition 4.1. *A Ricci tensor $\bar{\mathcal{S}}$ of an $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifold $\mathscr{M}$ equipped with an $\mathcal{SSMC}$ is called ω-parallel for $\bar{\nabla}$ if it holds the align $(\bar{\nabla}_{\mathcal{A}}\bar{\mathcal{S}})(\varphi\mathcal{B}, \varphi\mathcal{C}) = 0$, for all $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \chi(\mathcal{M})$.*

Applying $\mathcal{B}$ replaced by $\bar{\mathcal{Q}}\mathcal{B}$ in the relation (3.3), it provides

$$\bar{\nabla}_{\mathcal{A}}(\bar{\mathcal{Q}}\mathcal{B}) = \nabla_{\mathcal{A}}(\bar{\mathcal{Q}}\mathcal{B}) + \omega(\bar{\mathcal{Q}}\mathcal{B})\mathcal{A} - g(\mathcal{A}, \bar{\mathcal{Q}}\mathcal{B})\zeta.$$

Applying $\omega(\bar{\mathcal{Q}}\mathcal{B}) = g(\bar{\mathcal{Q}}\mathcal{B}, \zeta)$ in the foregoing relation, it yields

$$\bar{\nabla}_{\mathcal{A}}(\bar{\mathcal{Q}}\mathcal{B}) = \nabla_{\mathcal{A}}(\bar{\mathcal{Q}}\mathcal{B}) + g(\bar{\mathcal{Q}}\mathcal{B}, \zeta)\mathcal{A} - g(\mathcal{A}, \bar{\mathcal{Q}}\mathcal{B})\zeta, \tag{4.1}$$

Using relations (2.1), (2.5), (3.3) and (3.13) into (4.1), it yields

$$\begin{aligned} \bar{\nabla}_{\mathcal{A}}(\bar{\mathcal{Q}}\mathcal{B}) &= (\bar{\nabla}_{\mathcal{A}}\bar{\mathcal{Q}})(\mathcal{B}) + \mathcal{Q}(\nabla_{\mathcal{A}}\mathcal{B}) - 2(n-1)g(\mathcal{A}, \mathcal{B})\zeta + \omega(\mathcal{B})\mathcal{Q}\mathcal{A} \\ &+ 2(n-2)\{\omega(\nabla_{\mathcal{A}}\mathcal{B}) + \omega(\mathcal{A})\omega(\mathcal{B})\}\zeta + (3n-5)\{\nabla_{\mathcal{A}}\mathcal{B} + \omega(\mathcal{B})\mathcal{A}\}, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \nabla_{\mathcal{A}}(\bar{\mathcal{Q}}\mathcal{B}) &= (\nabla_{\mathcal{A}}\mathcal{Q})(\mathcal{B}) + \mathcal{Q}(\nabla_{\mathcal{A}}\mathcal{B}) + (3n-5)\nabla_{\mathcal{A}}\mathcal{B} \\ &+ 2(n-2)[\{\omega(\nabla_{\mathcal{A}}\mathcal{B} - \omega(\mathcal{A})\omega(\mathcal{B}) - g(\mathcal{A}, \mathcal{B})\}\zeta - \omega(\mathcal{B})\{\mathcal{A} + \omega(\mathcal{A})\zeta\}]. \end{aligned} \tag{4.3}$$

Relations (4.2) and (4.3) together give

$$(\bar{\nabla}_{\mathcal{A}}\bar{\mathcal{Q}})(\mathcal{B}) = (\nabla_{\mathcal{A}}\mathcal{Q})(\mathcal{B}) - \omega(\mathcal{B})\mathcal{Q}\mathcal{A} - g(\mathcal{A},\mathcal{Q}\mathcal{B})\zeta - 8(n-2)\omega(\mathcal{A})\omega(\mathcal{B})\zeta - (3n-7)\{\omega(\mathcal{B})\mathcal{A} + g(\mathcal{A},\mathcal{B})\zeta\}. \tag{4.4}$$

Taking inner product of the relation (4.4) along $\mathcal{C}$ and using (2.1), and (2.2) we have

$$g(\bar{\nabla}_{\mathcal{A}}\bar{\mathcal{Q}})(\mathcal{B}),\mathcal{C}) = g((\nabla_{\mathcal{A}}\mathcal{Q})(\mathcal{B}),\mathcal{C}) - \omega(\mathcal{B})g(\mathcal{Q}\mathcal{A},\mathcal{C}) - g(\mathcal{A},\mathcal{Q}\mathcal{B})\omega(\mathcal{C}) - 8(n-2)\omega(\mathcal{A})\omega(\mathcal{B})\omega(\mathcal{C}) - (3n-7)\{\omega(\mathcal{B})g(\mathcal{A},\mathcal{C}) + g(\mathcal{A},\mathcal{B})\omega(\mathcal{C})\}. \tag{4.5}$$

Using $(\bar{\nabla}_{\mathcal{A}}\bar{\mathcal{S}})(\mathcal{B},\mathcal{C}) = g((\bar{\nabla}_{\mathcal{A}}\bar{\mathcal{Q}})(\mathcal{B}),\mathcal{C})$, $(\nabla_{\mathcal{A}}\mathcal{S})(\mathcal{B},\mathcal{C}) = g((\nabla_{\mathcal{A}}\mathcal{Q})(\mathcal{B}),\mathcal{C})$ and $\mathcal{S}(\mathcal{A},\mathcal{B}) = g(\mathcal{Q}\mathcal{A},\mathcal{B})$ in the relation (4.5), we have

$$(\bar{\nabla}_{\mathcal{A}}\bar{\mathcal{S}})(\mathcal{B},\mathcal{C}) = (\nabla_{\mathcal{A}}\mathcal{S})(\mathcal{B},\mathcal{C}) - \omega(\mathcal{B})\mathcal{S}(\mathcal{A},\mathcal{C}) - \omega(\mathcal{C})\mathcal{S}(\mathcal{A},\mathcal{B}) - (3n-7)\{\omega(\mathcal{C})g(\mathcal{A},\mathcal{B}) + \omega(\mathcal{B})g(\mathcal{A},\mathcal{C})\} - 8(n-2)\omega(\mathcal{A})\omega(\mathcal{B})\omega(\mathcal{C}). \tag{4.6}$$

Now, $\mathcal{B}$ replaced by $\varphi\mathcal{B}$ and $\mathcal{C}$ replaced by $\varphi\mathcal{C}$ in the relation (4.6), and using (2.1) and (2.2), we have

$$(\bar{\nabla}_{\mathcal{A}}\bar{\mathcal{S}})(\varphi\mathcal{B},\varphi\mathcal{C}) = (\nabla_{\mathcal{A}}\mathcal{S})(\varphi\mathcal{B},\varphi\mathcal{C}). \tag{4.7}$$

A Ricci tensor $\mathcal{S}$ of an $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifold $\mathscr{M}$ is called ω-parallel if it holds the relation:

$$(\nabla_{\mathcal{A}}\mathcal{S})(\varphi\mathcal{B},\varphi\mathcal{C}) = 0, \tag{4.8}$$

for all $\mathcal{A},\mathcal{B},\mathcal{C} \in \chi(\mathscr{M})$. In view of relations (4.7), (4.8) and definition 4.1, we arrive at the following important equivalence:

Theorem 4.2 (Equivalence of ω-Parallelism under $\bar{\nabla}$ and ∇)**.** *Let an $(\mathcal{LP}\text{-}\mathcal{K})_n$ be a manifold endowed with a connection $\bar{\nabla}$. Then, the Ricci tensor $\bar{\mathcal{S}}$ is ω-parallel with respect to the connection $\bar{\nabla}$ if and only if the Ricci tensor $\mathcal{S}$ is ω-parallel with respect to the Levi-Civita connection ∇.*

This result generalizes the notion introduced by Kon [21] in Sasakian geometry to the $(\mathcal{LP}\text{-}\mathcal{K})_n$ framework with connection $\bar{\nabla}$.
Relations (2.1), (2.2), (2.6) and (3.11), taken together, provide

$$\bar{\mathcal{S}}(\varphi\mathcal{B},\varphi\mathcal{C}) = \bar{\mathcal{S}}(\mathcal{B},\mathcal{C}) + 2(n-1)\omega(\mathcal{B})\omega(\mathcal{C}), \tag{4.9}$$

for all $\mathcal{A},\mathcal{B},\mathcal{C} \in \chi(\mathscr{M})$.
Covariant differentiation of (4.9) along $\mathcal{A}$ gives

$$(\bar{\nabla}_{\mathcal{A}}\bar{\mathcal{S}})(\varphi\mathcal{B},\varphi\mathcal{C}) = \bar{\nabla}_{\mathcal{A}}\bar{\mathcal{S}}(\varphi\mathcal{B},\varphi\mathcal{C}) - \bar{\mathcal{S}}(\bar{\nabla}_{\mathcal{A}}(\varphi\mathcal{B}),\varphi\mathcal{C}) - \bar{\mathcal{S}}(\varphi\mathcal{B},\bar{\nabla}_{\mathcal{A}}(\varphi\mathcal{C})). \tag{4.10}$$

Relations (2.1), (2.2), (3.3) - (3.5), (3.12), together with (4.10), provide

$$(\bar{\nabla}_{\mathcal{A}}\bar{\mathcal{S}})(\varphi\mathcal{B},\varphi\mathcal{C}) = (\bar{\nabla}_{\mathcal{A}}\bar{\mathcal{S}})(\mathcal{B},\mathcal{C}) + 2\{\omega(\mathcal{B})\bar{\mathcal{S}}(\mathcal{A},\mathcal{C}) + \omega(\mathcal{C})\bar{\mathcal{S}}(\mathcal{A},\mathcal{B})\} - 4(n-1)\{\omega(\mathcal{B})g(\mathcal{A},\mathcal{C}) + \omega(\mathcal{C})g(\mathcal{A},\mathcal{B})\} \tag{4.11}$$

We assume that the manifold $\mathscr{M}$ endowed with a connection $\bar{\nabla}$ possesses ω-parallel Ricci tensor $\bar{\mathcal{S}}$ for connection $\bar{\nabla}$, meaning there by $(\bar{\nabla}_{\mathcal{A}}\bar{\mathcal{S}})(\varphi\mathcal{B}, \varphi\mathcal{C}) = 0$. Then, in view of this relation, align (4.11) gives

$$(\bar{\nabla}_{\mathcal{A}}\bar{\mathcal{S}})(\mathcal{B}, \mathcal{C}) = 4(n-1)\{\omega(\mathcal{B})g(\mathcal{A}, \mathcal{C}) + \omega(\mathcal{C})g(\mathcal{A}, \mathcal{B})\} - 2\{\omega(\mathcal{B})\bar{\mathcal{S}}(\mathcal{A}, \mathcal{C}) + \omega(\mathcal{C})\bar{\mathcal{S}}(\mathcal{A}, \mathcal{B})\} \tag{4.12}$$

Equation (4.12) characterizes ω-parallel Ricci tensors under the connection $\bar{\nabla}$ and thereby establishes the following result:

Theorem 4.3 (Characterization of ω-Parallel Ricci Tensor under $\bar{\nabla}$)**.** *Let $(\mathscr{M}, \varphi, \zeta, \omega, g)$ be an $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifold equipped with a connection $\bar{\nabla}$. Then the Ricci tensor $\bar{\mathcal{S}}$ is ω-parallel with respect to $\bar{\nabla}$, if and only if the following identity holds for all vector fields $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \chi(\mathscr{M})$:*

$$(\bar{\nabla}_{\mathcal{A}}\bar{\mathcal{S}})(\mathcal{B}, \mathcal{C}) = 4(n-1)\{\omega(\mathcal{B})\, g(\mathcal{A}, \mathcal{C}) + \omega(\mathcal{C})\, g(\mathcal{A}, \mathcal{B})\} - 2\left\{\omega(\mathcal{B})\, \bar{\mathcal{S}}(\mathcal{A}, \mathcal{C}) + \omega(\mathcal{C})\, \bar{\mathcal{S}}(\mathcal{A}, \mathcal{B})\right\}.$$

Remark 4.4. *The above identity provides a necessary and sufficient condition for the Ricci tensor to be ω-parallel with respect to the connection $\bar{\nabla}$. This result generalizes the concept introduced by Kon* [21] *in the context of Sasakian geometry to the $(\mathcal{LP}\text{-}\mathcal{K})_n$ framework.*

Let $\{e_i\}_{i=1}^{n}$ be an orthonormal frame of the tangent space at every point of the manifold $\mathscr{M}$. Using $\mathcal{B} = \mathcal{C} = e_i$ into (4.12) and summing over i, where i runs from 1 to n, it yields

$$d\bar{r} = 0,$$

for all $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C} \in \chi(\mathscr{M})$.
The foregoing relation implies that the scalar curvature with respect to the connection $\bar{\nabla}$ is constant. Hence, the subsequent corollary is established:

Corollary 4.5. *If the Ricci tensor $\bar{\mathcal{S}}$ of an $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifold $\mathscr{M}$ endowed with connection $\bar{\nabla}$ is ω-parallel, then the scalar curvature for the connection $\bar{\nabla}$ is constant.*

Relation (3.14) and corollary (4.5) yield

$$d\bar{r} = dr(\mathcal{A}) = 0,$$

for all $\mathcal{A} \in \chi(\mathscr{M})$. This shows that the scalar curvature with respect to the connection $\bar{\nabla}$ is constant. Hence, we state:

Corollary 4.6. *Let an $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifold $\mathscr{M}$ be endowed with connection $\bar{\nabla}$ possesses ω-parallel Ricci tensor $\bar{\mathcal{S}}$. Then the scalar curvature with respect to the connection $\bar{\nabla}$ is constant.*

We have the relation $\bar{\mathcal{S}}(\mathcal{B},\mathcal{C}) = g(\bar{\mathcal{Q}}\mathcal{B},\mathcal{C})$, then from the relation (3.3), we have

$$\bar{\nabla}_{\mathcal{A}}|\bar{\mathcal{Q}}|^2 = 2\sum_{i=1}^{n} g((\bar{\nabla}_{\mathcal{A}}\bar{\mathcal{Q}})e_i, \bar{\mathcal{Q}}e_i) = 0.$$

This shows that $|\bar{\mathcal{Q}}|^2 = constant$. Hence, the above relation establishes the following corollary:

Corollary 4.7. *Let an $(\mathcal{LP}\text{-}\mathcal{K})_n$be a manifold $\mathscr{M}$ equipped with connection $\bar{\nabla}$ possesses ω-parallel Ricci tensor $\bar{\mathcal{S}}$. Then the length of Ricci operator with $\bar{\nabla}$ is constant on $\mathscr{M}$.*

Let us consider the Ricci tensor $\bar{\mathcal{S}}$ on an $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifold $\mathscr{M}$ endowed with the connection $\bar{\nabla}$. Then, $\bar{\mathcal{S}}$ is of Codazzi type, meaning there by, $(\bar{\nabla}_{\mathcal{A}}\bar{\mathcal{S}})(\mathcal{B},\mathcal{C}) = (\bar{\nabla}_{\mathcal{B}}\bar{\mathcal{S}})(\mathcal{A},\mathcal{C})$. Hence, this relation, along with the relation (4.12), gives

$$\omega(\mathcal{B})\bar{\mathcal{S}}(\mathcal{A},\mathcal{C}) - \omega(\mathcal{A})\bar{\mathcal{S}}(\mathcal{B},\mathcal{C}) = 2(n-1)\left\{\omega(\mathcal{B})g(\mathcal{A},\mathcal{C}) - \omega(\mathcal{A})g(\mathcal{B},\mathcal{C})\right\}.$$

Putting $\mathcal{B} = \zeta$ in the above relation and using (2.1) and (3.12), we have

$$\bar{\mathcal{S}}(\mathcal{A},\mathcal{C}) = 2(n-1)g(\mathcal{A},\mathcal{C}). \tag{4.13}$$

The relations (3.11), (3.12), and (4.13) taken together yield

$$\mathcal{S}(\mathcal{A},\mathcal{C}) = -(n-3)g(\mathcal{A},\mathcal{C}) - 2(n-2)\omega(\mathcal{A})\omega(\mathcal{C}). \tag{4.14}$$

It follows that (4.13) and (4.14) are equivalent characterizations of the Ricci tensor under the respective connections.
Equation (4.14) shows that the manifold $\mathscr{M}$ is an ω-Einstein manifold with scalar coefficients $a = -(n-3)$ and $b = -2(n-2)$. These values satisfy the identity $a + b = -(3n-7)$, as noted in [18]. Conversely, assuming that the manifold $\mathscr{M}$ satisfies the relation (4.14), it can be shown that the Ricci tensors with respect to $\bar{\nabla}$ is of Codazzi type. Hence, we establish the following result:

Corollary 4.8. *Let $(\mathcal{LP}\text{-}\mathcal{K})_n$ be a manifold $\mathscr{M}$, with $n > 3$, endowed with a connection $\bar{\nabla}$, and suppose the Ricci tensor is ω-parallel with respect to the connection $\bar{\nabla}$. Then, the Ricci tensor $\bar{\mathcal{S}}$ is of Codazzi type if and only if the manifold $\mathscr{M}$ is Einstein with respect to $\bar{\nabla}$ or ω-Einstein with respect to ∇.*

Remark 4.9. *Equation (4.14) shows that the manifold $\mathscr{M}$ is an ω-Einstein manifold with scalar coefficients $a = -(n-3)$ and $b = -2(n-2)$. These satisfy the identity $a + b = -(3n-7)$, consistent with earlier results in* [18].

Furthermore, suppose that the Ricci tensor $\bar{\mathcal{S}}$ with respect to the connection $\bar{\nabla}$ is cyclic parallel, i.e.,

$$(\bar{\nabla}_{\mathcal{A}}\bar{\mathcal{S}})(\mathcal{B},\mathcal{C}) + (\bar{\nabla}_{\mathcal{B}}\bar{\mathcal{S}})(\mathcal{C},\mathcal{A}) + (\bar{\nabla}_{\mathcal{C}}\bar{\mathcal{S}})(\mathcal{A},\mathcal{B}) = 0.$$

Using align (4.13) and substituting into the above identity, it yields:

$$\omega(\mathcal{A})\bar{\mathcal{S}}(\mathcal{B},\mathcal{C}) + \omega(\mathcal{B})\bar{\mathcal{S}}(\mathcal{C},\mathcal{A}) + \omega(\mathcal{C})\bar{\mathcal{S}}(\mathcal{A},\mathcal{B}) - 2(n-1)\left\{\omega(\mathcal{A})g(\mathcal{B},\mathcal{C}) + \omega(\mathcal{B})g(\mathcal{A},\mathcal{C}) + \omega(\mathcal{C})g(\mathcal{A},\mathcal{B})\right\} = 0.$$

Setting $\mathcal{C} = \zeta$ in the foregoing relation and applying aligns (2.1), (2.2), (3.11), and (3.12), we recover relation (4.14). Hence, we arrive at the following conclusion:

Corollary 4.10. *Let $(\mathcal{LP}\text{-}\mathcal{K})_n$ be a manifold $\mathscr{M}$, with $n > 3$, endowed with a connection $\bar{\nabla}$ and suppose the Ricci tensor is ω-parallel with respect to $\bar{\nabla}$. Then, the Ricci tensor $\bar{\mathcal{S}}$ is cyclic parallel if and only if the manifold $\mathscr{M}$ is an ω-Einstein manifold.*

Remark 4.11. *Definition 4.1 and Theorem 4.2 are fundamental to the geometric framework developed in this paper. The introduction of the notion of an ω-parallel Ricci tensor with respect to the Levi-Civita connection ∇ captures a broader class of curvature symmetries on $(\mathcal{LPK})_n$ manifolds. Theorem 4.2 further establishes that this ω-parallelism condition remains invariant under the connection $\bar{\nabla}$. This result not only adds theoretical depth but also ensures compatibility of geometric structures under modified connections. The inclusion of this equivalence forms a crucial bridge between classical and non-Riemannian geometric analysis, thereby justifying its relevance and maturity for this study.*

5. Concircular and Projective Curvature Tensors under $\mathcal{SSMC}$ on $(\mathcal{LP}\text{-}\mathcal{K})_n$ Manifolds

It is well-known that geodesic circles are generally not preserved under a conformal transformation of the form

$$\bar{g}_{ij} = \psi^2 g_{ij}, \tag{5.1}$$

where g_{ij} is the original metric tensor and ψ is a smooth scalar function on the manifold.
Yano [33, 34] showed that such a transformation preserves geodesic circles if and only if ψ satisfies the differential condition

$$\nabla_i \nabla_j \psi = \varphi\, g_{ij}, \tag{5.2}$$

where, φ is a scalar function. This condition characterizes ψ as a *concircular scalar field*, and leads to the definition of the concircular curvature tensor.
A conformal transformation satisfying (5.2) is called a *concircular transformation*, and the corresponding geometric structure is referred to as *concircular geometry*.
The tensor field that remains invariant under such transformations is known as the *concircular curvature tensor*, and is defined by:

$$\mathscr{C}(\mathcal{A},\mathcal{B})\mathcal{C} = \mathscr{R}(\mathcal{A},\mathcal{B})\mathcal{C} - \frac{r}{n(n-1)}\{g(\mathcal{B},\mathcal{C})\mathcal{A} - g(\mathcal{A},\mathcal{C})\mathcal{B}\}, \tag{5.3}$$

where $\mathscr{R}$ and r denote the Riemannian curvature tensor and the scalar curvature of the Levi-Civita connection ∇, respectively. The tensor $\mathscr{C}$ is known as the *concircular curvature tensor.*

We now generalize this concept to an $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifold endowed with a connection $\bar{\nabla}$:

Definition 5.1. *Let $\mathscr{M}$ be an $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifold endowed with a connection $\bar{\nabla}$. Let us consider the* concircular curvature tensor *$\bar{\mathscr{C}}$ associated with the connection $\bar{\nabla}$, defined by*

$$\bar{\mathscr{C}}(\mathcal{A},\mathcal{B})\mathcal{C} = \bar{\mathscr{R}}(\mathcal{A},\mathcal{B})\mathcal{C} - \frac{\bar{r}}{n(n-1)}\Big\{g(\mathcal{B},\mathcal{C})\mathcal{A} - g(\mathcal{A},\mathcal{C})\mathcal{B}\Big\}, \tag{5.4}$$

where $\bar{\mathscr{R}}$ denotes the curvature tensor and $\bar{r}$ the scalar curvature corresponding to the connection $\bar{\nabla}$.

Definition 5.2. *Let $\mathscr{M}$ be an $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifold endowed with a semi-symmetric metric connection $\bar{\nabla}$. Let us consider the* projective curvature tensor *$\bar{\mathscr{P}}$ associated with the connection $\bar{\nabla}$, defined by*

$$\bar{\mathscr{P}}(\mathcal{A},\mathcal{B})\mathcal{C} = \bar{\mathscr{R}}(\mathcal{A},\mathcal{B})\mathcal{C} - \frac{1}{n-1}\left\{\bar{\mathcal{S}}(\mathcal{B},\mathcal{C})\mathcal{A} - \bar{\mathcal{S}}(\mathcal{A},\mathcal{C})\mathcal{B}\right\}. \tag{5.5}$$

Here, $\bar{\mathscr{R}}$ and $\bar{\mathcal{S}}$ denote the curvature and Ricci tensors corresponding to the connection $\bar{\nabla}$.

Suppose that the Ricci tensor $\bar{\mathcal{S}}$ of the manifold $\mathscr{M}$ is ω-parallel and is either of Codazzi type or cyclic parallel. Then, from relation (4.13), it follows that

$$\bar{r} = 2n(n-1).$$

Substituting this result into aligns (5.4) and (5.5), we find that the concircular curvature tensor coincides with the projective curvature tensor:

$$\bar{\mathscr{C}}(\mathcal{A},\mathcal{B})\mathcal{C} = \bar{\mathscr{P}}(\mathcal{A},\mathcal{B})\mathcal{C}. \tag{5.6}$$

Conversely, if condition (5.6) holds, then equating the expressions for $\bar{\mathscr{C}}$ and $\bar{\mathscr{P}}$ given by (5.4) and (5.5), respectively, yields the identity

$$\bar{\mathcal{S}}(\mathcal{A},\mathcal{C})\mathcal{B} - \bar{\mathcal{S}}(\mathcal{B},\mathcal{C})\mathcal{A} = -\frac{\bar{r}}{n}\left\{g(\mathcal{B},\mathcal{C})\mathcal{A} - g(\mathcal{A},\mathcal{C})\mathcal{B}\right\}.$$

Taking the inner product of the preceding relation with a vector field ζ, it yields the following identity:

$$\bar{\mathcal{S}}(\mathcal{A},\mathcal{C})\omega(\mathcal{B}) - \bar{\mathcal{S}}(\mathcal{B},\mathcal{C})\omega(\mathcal{A}) = -\frac{\bar{r}}{n}\left\{g(\mathcal{B},\mathcal{C})\omega(\mathcal{A}) - g(\mathcal{A},\mathcal{C})\omega(\mathcal{B})\right\},$$

Setting $\mathcal{B} = \zeta$ in the foregoing relation and applying identities (2.1) and (3.12), we obtain:

$$\bar{\mathcal{S}}(\mathcal{A},\mathcal{C}) + 2(n-1)\omega(\mathcal{A})\omega(\mathcal{C}) = \frac{\bar{r}}{n}\left\{\omega(\mathcal{A})\omega(\mathcal{C}) + g(\mathcal{A},\mathcal{C})\right\}.$$

Taking an orthonormal frame $\{e_i\}_{i=1}^{n}$ and substituting $\mathcal{A} = \mathcal{C} = e_i$ in the above relation, followed by summation over i, it yields $\bar{r} = 2n(n-1)$ and hence from (5.4) and (5.5), we get (4.14) and thereby conclude the result:

Corollary 5.3. *Let $\mathscr{M}$ be an $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifold with $n > 3$, and let $\bar{\mathcal{S}}$ denote an ω-parallel Ricci tensor with respect to the semi-symmetric metric connection $\bar{\nabla}$. Then, the concircular and projective curvature tensors coincide if and only if $\bar{\mathcal{S}}$ is either of Codazzi type or cyclic parallel.*

The corollaries 4.7, 4.8 and 5.3 address the following:

Corollary 5.4. *Let $\bar{\mathcal{S}}$ be an ω-parallel Ricci tensor on an $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifold with $n > 3$. Then the following conditions are mutually equivalent:*

- *$\bar{\mathcal{S}}$ is of Codazzi type;*
- *$\bar{\mathcal{S}}$ is cyclic parallel;*
- *$\bar{\mathcal{S}}(\mathcal{B},\mathcal{C}) = 2(n-1)g(\mathcal{B},\mathcal{C})$;*
- *The concircular curvature tensor and the projective curvature tensor coincide, i.e., $\bar{\mathscr{C}}(\mathcal{A},\mathcal{B})\mathcal{C} = \bar{\mathscr{P}}(\mathcal{A},\mathcal{B})\mathcal{C}$;*
- *The scalar curvature satisfies $\bar{r} = 2n(n-1)$.*

If $\mathscr{M}$ is either concircularly or projectively flat, then using aligns (4.14), (5.4), (5.5), and (5.6), we obtain:

$$\bar{\mathscr{R}}(\mathcal{A},\mathcal{B})\mathcal{C} = 2\left\{g(\mathcal{B},\mathcal{C})\mathcal{A} - g(\mathcal{A},\mathcal{C})\mathcal{B}\right\}, \tag{5.7}$$

which implies that $\mathscr{M}$ associated with the connection $\bar{\nabla}$ is a space of constant curvature.

Remark 5.5 (Geometric Role of Constant Curvature in Cosmology)**.** *Constant curvature plays a pivotal role in the general theory of relativity ($\mathcal{GTR}$) and modern cosmology. To construct a viable cosmological model of the universe, it is generally assumed that the universe is homogeneous and isotropic–a postulate known as the cosmological principle. When this principle is combined with Riemannian geometry, it yields a three-dimensional spatial geometry characterized by maximal symmetry* [31]*, implying that the spatial curvature is constant and may evolve over time.*
Solutions to Einstein's field aligns that admit 3-dimensional spatial hypersurfaces of constant curvature correspond to the Robertson-Walker metric, while a 4-dimensional manifold of constant curvature gives rise to the de Sitter model of the universe [24, 31].

To examine the geometric effects of a connection $\bar{\nabla}$ in this context, we recall the previously derived relation connecting the curvature tensors $\mathscr{R}$ and $\bar{\mathscr{R}}$ corresponding to the Levi-Civita connection and connection $\bar{\nabla}$, respectively. This identity, obtained by combining aligns (3.3) and (3.6), is expressed as:

$$\begin{aligned}\bar{\mathscr{R}}(\mathcal{A},\mathcal{B})\mathcal{C} &= \mathscr{R}(\mathcal{A},\mathcal{B})\mathcal{C} - 2\omega(\mathcal{A})\omega(\mathcal{C})\mathcal{B} + 2\omega(\mathcal{B})\omega(\mathcal{C})\mathcal{A} \\ &\quad - 3g(\mathcal{A},\mathcal{C})\mathcal{B} + 3g(\mathcal{B},\mathcal{C})\mathcal{A} + 2g(\mathcal{B},\mathcal{C})\omega(\mathcal{A})\zeta - 2g(\mathcal{A},\mathcal{C})\omega(\mathcal{B})\zeta,\end{aligned}$$

where $\mathscr{R}$ and $\bar{\mathscr{R}}$ denote the curvature tensors with respect to ∇ and $\bar{\nabla}$, respectively, and ζ is the generator of the semi-symmetric structure associated with the 1-form ω.

This expression provides a valuable framework for understanding how the underlying curvature is deformed by the introduction of a connection $\bar{\nabla}$–a central consideration when applying modified geometrical settings to cosmological models.

Recalling align (3.7) and substituting from align (5.7), we obtain:

$$\begin{aligned}\mathscr{R}(\mathcal{A},\mathcal{B})\mathcal{C} &= -g(\mathcal{B},\mathcal{C})\mathcal{A} - g(\mathcal{A},\mathcal{C})\mathcal{B} + 2\omega(\mathcal{A})\omega(\mathcal{C})\mathcal{B} \\ &\quad - 2\omega(\mathcal{B})\omega(\mathcal{C})\mathcal{A} - \omega(\mathcal{A})g(\mathcal{B},\mathcal{C})\zeta + \omega(\mathcal{B})g(\mathcal{A},\mathcal{C})\zeta. \qquad (5.8)\end{aligned}$$

The relation (5.8) indicates that $\mathscr{M}$ belongs to a certain class of generalized Sasakian space forms [2, 19], characterized by the functions $f_1 = -1$, $f_2 = 0$, and $f_3 = 2$. Moreover, if we assume that the manifold $\mathscr{M}$, endowed with the connection $\bar{\nabla}$, satisfies (5.8), then from the identities (3.7), (5.4), (5.5), and (5.8), we deduce that $\bar{\mathscr{C}} = 0$ and $\bar{\mathscr{P}} = 0$.

It has been proved by Kim [19] that a generalized Sasakian space form is conformally flat if and only if $f_2 = 0$. Therefore, by combining align (5.8) with Kim's result, we obtain the following theorem.

Theorem 5.6. *Let $\mathscr{M}$ be an $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifold with $n > 3$, equipped with the connection $\bar{\nabla}$. Then $\mathscr{M}$ is conformally flat if and only if it is either projectively flat or concircularly flat with respect to $\bar{\nabla}$.*

Proof. Let $\mathscr{M}$ be an $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifold with $n > 3$, equipped with the connection $\bar{\nabla}$. The conformal curvature tensor $\bar{\mathcal{V}}$ with respect to $\bar{\nabla}$ is defined by:

$$\begin{aligned}\bar{\mathcal{V}}(\mathcal{A},\mathcal{B})\mathcal{C} &= \bar{\mathscr{R}}(\mathcal{A},\mathcal{B})\mathcal{C} - \frac{1}{n-2}\left\{\bar{\mathcal{S}}(\mathcal{B},\mathcal{C})\mathcal{A} - \bar{\mathcal{S}}(\mathcal{A},\mathcal{C})\mathcal{B}\right\} \\ &\quad + \frac{\bar{r}}{(n-1)(n-2)}\left\{g(\mathcal{B},\mathcal{C})\mathcal{A} - g(\mathcal{A},\mathcal{C})\mathcal{B}\right\},\end{aligned}$$

where $\bar{\mathscr{R}}$ is the curvature tensor of $\bar{\nabla}$, $\bar{\mathcal{S}}$ is the Ricci tensor, and $\bar{r}$ is the scalar curvature.

Now consider the projective curvature tensor $\bar{\mathscr{P}}$:

$$\bar{\mathscr{P}}(\mathcal{A},\mathcal{B})\mathcal{C} = \bar{R}(\mathcal{A},\mathcal{B})\mathcal{C} - \frac{1}{n-1}\left\{\bar{\mathcal{S}}(\mathcal{B},\mathcal{C})\mathcal{A} - \bar{\mathcal{S}}(\mathcal{A},\mathcal{C})\mathcal{B}\right\}.$$

Also, the concircular curvature tensor $\bar{\mathscr{C}}$ is given by:

$$\bar{\mathscr{C}}(\mathcal{A},\mathcal{B})\mathcal{C} = \bar{\mathscr{R}}(\mathcal{A},\mathcal{B})\mathcal{C} - \frac{\bar{r}}{n(n-1)}\left\{g(\mathcal{B},\mathcal{C})\mathcal{A} - g(\mathcal{A},\mathcal{C})\mathcal{B}\right\}.$$

Now suppose that $\mathscr{M}$ is conformally flat with respect to $\bar{\nabla}$, i.e., $\bar{\mathcal{V}} = 0$. Then from the definition above, we can write:

$$\bar{\mathscr{R}}(\mathcal{A},\mathcal{B})\mathcal{C} = \frac{1}{n-2}\left\{\bar{\mathcal{S}}(\mathcal{B},\mathcal{C})\mathcal{A} - \bar{\mathcal{S}}(\mathcal{A},\mathcal{C})\mathcal{B}\right\} - \frac{\bar{r}}{(n-1)(n-2)}\left\{g(\mathcal{B},\mathcal{C})\mathcal{A} - g(\mathcal{A},\mathcal{C})\mathcal{B}\right\}.$$

Using this in the expression for $\bar{\mathscr{P}}(X,Y)Z$, we get:

$$\begin{aligned}\bar{\mathscr{P}}(\mathcal{A},\mathcal{B})\mathcal{C} = &\left(\frac{1}{n-2} - \frac{1}{n-1}\right)\left\{\bar{\mathcal{S}}(\mathcal{B},\mathcal{C})\mathcal{A} - \bar{\mathcal{S}}(\mathcal{A},\mathcal{C})\mathcal{B}\right\} \\ &- \frac{\bar{r}}{(n-1)(n-2)}\left\{g(\mathcal{B},\mathcal{C})\mathcal{A} - g(\mathcal{A},\mathcal{C})\mathcal{B}\right\}.\end{aligned}$$

Thus, $\bar{\mathscr{P}} = 0$ if and only if the above expression vanishes, which occurs exactly when both terms vanish individually. This implies:

$$\bar{\mathcal{S}}(\mathcal{A},\mathcal{B}) = \frac{\bar{r}}{n}g(\mathcal{A},\mathcal{B}),$$

i.e., $\mathscr{M}$ is an Einstein manifold with respect to $\bar{\nabla}$.
Therefore, under the assumption that $\bar{\mathcal{V}} = 0$, we find that $\mathscr{M}$ is both projectively and concircularly flat.
Conversely, if $\bar{\mathscr{P}} = 0$ or $\bar{\mathscr{C}} = 0$, then using the definitions above, it follows that the trace-free part of $\bar{\mathscr{R}}$ vanishes, hence $\bar{\mathcal{V}} = 0$, i.e., the manifold is conformally flat.
Hence, the manifold $\mathscr{M}$ is conformally flat with respect to $\bar{\nabla}$ if and only if it is either projectively flat or concircularly flat. This completes the proof. □

Covariant differentiation of relation (5.4) with respect to $\mathcal{SSMC}$ $\bar{\nabla}$ in the direction of vector field $\mathcal{W}$ yields

$$(\bar{\nabla}_{\mathcal{W}}\bar{\mathscr{C}})(\mathcal{A},\mathcal{B})\mathcal{C} = (\bar{\nabla}_{\mathcal{W}}\bar{\mathscr{R}}(\mathcal{A},\mathcal{B})\mathcal{C} - \frac{d\bar{r}(\mathcal{W})}{n(n-1)}\{g(\mathcal{B},\mathcal{C})\mathcal{A} - g(\mathcal{A},\mathcal{C})\mathcal{B}\}. \tag{5.9}$$

We assume that the $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifold $\mathscr{M}$ equipped with connection $\bar{\nabla}$ possesses ω-parallel Ricci tensor, i.e., $(\bar{\nabla}_{\mathcal{A}}\bar{\mathcal{S}})(\varphi\mathcal{B},\varphi\mathcal{C}) = 0$. Hence, the corollary 4.4 and relation (5.9) together give

$$(\bar{\nabla}_{\mathcal{W}}\bar{\mathscr{C}})(\mathcal{A},\mathcal{B})\mathcal{C} = (\bar{\nabla}_{\mathcal{W}}\bar{\mathscr{R}}(\mathcal{A},\mathcal{B})\mathcal{C}. \tag{5.10}$$

Before exploration of our findings, we have the following definitions:

Definition 5.7. *Let $(\mathscr{M}, \varphi, \zeta, \omega, g)$ be an $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifold endowed with a connection $\bar{\nabla}$. Then $\mathscr{M}$ is said to be* globally symmetric *with respect to $\bar{\nabla}$ if, for all $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{W} \in \chi(\mathscr{M})$, it satisfies one of the following:*

(1) ***Globally symmetric:*** $(\bar{\nabla}_{\mathcal{W}}\bar{\mathscr{R}})(\mathcal{A}, \mathcal{B})\mathcal{C} = 0, \quad \bar{\mathscr{R}} \neq 0.$
(2) ***Globally concircularly symmetric:*** $(\bar{\nabla}_{\mathcal{W}}\bar{\mathscr{C}})(\mathcal{A}, \mathcal{B})\mathcal{C} = 0.$
(3) ***Globally φ-symmetric:*** $\varphi^2(\bar{\nabla}_{\mathcal{W}}\bar{\mathscr{R}})(\mathcal{A}, \mathcal{B})\mathcal{C} = 0, \quad \bar{\mathscr{R}} \neq 0.$
(4) ***Globally φ-concircularly symmetric:*** $\varphi^2(\bar{\nabla}_{\mathcal{W}}\bar{\mathscr{C}})(\mathcal{A}, \mathcal{B})\mathcal{C} = 0.$

Moreover, if the Ricci tensor $\bar{\mathcal{S}}$ is ω-parallel, then $\mathscr{M}$ is globally symmetric if and only if it is globally concircularly symmetric, in that case, applying φ^2 gives

$$\varphi^2(\bar{\nabla}_{\mathcal{W}}\bar{\mathscr{C}})(\mathcal{A}, \mathcal{B})\mathcal{C} = \varphi^2(\bar{\nabla}_{\mathcal{W}}\bar{\mathscr{R}})(\mathcal{A}, \mathcal{B})\mathcal{C}.$$

The preceding relation, together with the above definition establishes the following theorem:

Theorem 5.8. *If an $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifold $\mathscr{M}$ endowed with connection $\bar{\nabla}$ possesses ω-parallel Ricci tensor $\bar{\mathcal{S}}$, then the manifold is globally φ-symmetric iff it is globally φ-concircularly symmetric.*

A globally φ-concircularly symmetric $(\mathcal{LP}\text{-}\mathcal{K})_n$ manifold $\mathscr{M}$ endowed with connection $\bar{\nabla}$ is an ω-Einstein manifold.

6. **Example**

We consider a smooth manifold $\mathscr{M}^4 = \{(u, v, w, t) \in \mathbb{R}^4 \mid u, v, w \in \mathbb{R} \setminus \{0\},\ t > 0\}$ of dimension 4. Here, (u, v, w, t) are the standard coordinates in the 4-dimensional real space $\mathbb{R}^4$. We define,

$$e_1 = e^{u+t}\frac{\partial}{\partial u}, \quad e_2 = e^{v+t}\frac{\partial}{\partial v}, \quad e_3 = e^{w+t}\frac{\partial}{\partial w}, \quad e_4 = \frac{\partial}{\partial t}.$$

Lorentzian metric g on $\mathscr{M}^4$ is defined by:

$$g_{ij} = g(e_i, e_j) = \begin{cases} 0 & \text{if } i \neq j, \\ -1 & \text{if } i = j = 4, \\ 1 & \text{if } i = j \neq 4. \end{cases}$$

Let ω be a 1-form dual to e_4 under g. This is defined by

$$\omega(\mathcal{A}) = g(\mathcal{A}, e_4),$$

for all $\mathcal{A} \in \chi(\mathscr{M}^4)$, is the collection of vector fields on $\mathscr{M}^4$. We define φ a $(1,1)$-tensor field as follows:

$$\varphi(e_1) = e_1, \quad \varphi(e_2) = e_2, \quad \varphi(e_3) = e_3, \quad \varphi(e_4) = 0.$$

From the linearity of φ and g, it is evident from the above definitions that the following identities hold:

$$\omega(e_4) = -1, \;\; \varphi^2(\mathcal{A}) = \mathcal{A} + \omega(\mathcal{A})e_4, \;\; g(\varphi\mathcal{A}, \varphi\mathcal{B}) = g(\mathcal{A}, \mathcal{B}) + \omega(\mathcal{A})\omega(\mathcal{B}),$$

for all vector fields $\mathcal{A}, \mathcal{B} \in \chi(\mathscr{M}^4)$. If we denote $e_4 = \zeta$, then the structure $(\varphi, \zeta, \omega, g)$ defines a Lorentzian paracontact structure. A manifold $\mathscr{M}^4$ equipped with such a structure is called a Lorentzian para-Kenmotsu manifold, denoted by $(\mathcal{LP}\text{-}\mathcal{K})_4$.

Let ω be a 1-form related to the metric g, defined by

$$\omega(\mathcal{A}) = g(\mathcal{A}, e_4),$$

for all $\mathcal{A} \in \chi(\mathscr{M}^4)$, where $\chi(\mathscr{M}^4)$ denotes the set of all smooth vector fields on $\mathscr{M}^4$. We define a $(1,1)$-tensor field φ by

$$\varphi(e_1) = e_1, \quad \varphi(e_2) = e_2, \quad \varphi(e_3) = e_3, \quad \varphi(e_4) = 0.$$

From the linearity properties of φ and g, we easily obtain the following identities:

$$\omega(e_4) = -1, \quad \varphi^2(\mathcal{A}) = \mathcal{A} + \omega(\mathcal{A})e_4, \quad g(\varphi\mathcal{A}, \varphi\mathcal{B}) = g(\mathcal{A}, \mathcal{B}) + \omega(\mathcal{A})\omega(\mathcal{B}),$$

for all $\mathcal{A}, \mathcal{B} \in \chi(\mathscr{M}^4)$. When we choose $e_4 = \xi$, the structure $(\varphi, \xi, \omega, g)$ induces a Lorentzian paracontact structure. A manifold $\mathscr{M}^4$ endowed with a Lorentzian paracontact structure is called an $(\mathcal{LP}\text{-}\mathcal{K})_4$-manifold.

We denote the Lie bracket of two vector fields $\mathcal{A}$ and $\mathcal{B}$ as

$$[\mathcal{A}, \mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A},$$

which represents the Lie derivative of $\mathcal{B}$ along $\mathcal{A}$. The non-zero Lie brackets among the frame vectors are given by

$$[e_1, e_4] = -e_1, \quad [e_2, e_4] = -e_2, \quad [e_3, e_4] = -e_3.$$

Let ∇ be the Levi-Civita connection with respect to the metric g. We recall the Koszul formula, which expresses the Levi-Civita connection ∇ in terms of the metric g and the Lie brackets:

$$\begin{aligned} 2g(\nabla_{\mathcal{A}}\mathcal{B}, \mathcal{C}) = \mathcal{A}g(\mathcal{B}, \mathcal{C}) + \mathcal{B}g(\mathcal{C}, \mathcal{A}) - \mathcal{C}g(\mathcal{A}, \mathcal{B}) \\ - g([\mathcal{B}, \mathcal{C}], \mathcal{A}) + g([\mathcal{C}, \mathcal{A}], \mathcal{B}) + g([\mathcal{A}, \mathcal{B}], \mathcal{C}), \end{aligned}$$

for all vector fields $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \chi(\mathscr{M}^4)$.

When $e_4 = \zeta$, using Koszul's formula, we obtain the following results:

$$\begin{array}{llll} \nabla_{e_1}e_1 = -e_4, & \nabla_{e_1}e_2 = 0, & \nabla_{e_1}e_3 = 0, & \nabla_{e_1}e_4 = -e_1, \\ \nabla_{e_2}e_1 = 0, & \nabla_{e_2}e_2 = -e_4, & \nabla_{e_2}e_3 = 0, & \nabla_{e_2}e_4 = -e_2, \\ \nabla_{e_3}e_1 = 0, & \nabla_{e_3}e_2 = 0, & \nabla_{e_3}e_3 = -e_4, & \nabla_{e_3}e_4 = -e_3, \\ \nabla_{e_4}e_1 = 0, & \nabla_{e_4}e_2 = 0, & \nabla_{e_4}e_3 = 0, & \nabla_{e_4}e_4 = 0. \end{array}$$

We assume that $\mathcal{A} \in \chi(\mathscr{M}^4)$. Let $\mathcal{A} = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4$, where $\{e_1, e_2, e_3, e_4\}$ is a local basis of $\chi(\mathscr{M}^4)$.

Using the above relations, we can easily verify that

$$\nabla_{\mathcal{A}}e_4 = -\{\mathcal{A} + \omega(\mathcal{A})e_4\} \quad \text{for each } \mathcal{A} \in \chi(\mathscr{M}^4).$$

Hence, $(\mathscr{M}^4, g)$ is a $(\mathcal{LP}\text{-}\mathcal{K})_4$ manifold, and the structure $(\varphi, \zeta, \omega, g)$ defines a Lorentzian para-Kenmotsu structure on $\mathscr{M}^4$.
From the above relations, the non-vanishing components of the curvature tensor are computed as follows:

$$\begin{aligned}
&\mathscr{R}(e_1,e_2)e_1 = -e_2, \mathscr{R}(e_1,e_3)e_1 = -e_3, \mathscr{R}(e_1,e_4)e_1 = -e_4,\\
&\mathscr{R}(e_1,e_2)e_2 = e_1, \quad \mathscr{R}(e_2,e_3)e_2 = -e_3, \mathscr{R}(e_2,e_4)e_2 = -e_4,\\
&\mathscr{R}(e_1,e_3)e_3 = e_1, \quad \mathscr{R}(e_2,e_3)e_3 = e_2, \quad \mathscr{R}(e_3,e_4)e_3 = -e_4,\\
&\mathscr{R}(e_1,e_4)e_4 = -e_1, \ \mathscr{R}(e_2,e_4)e_4 = -e_2, \mathscr{R}(e_3,e_4)e_4 = -e_3,\\
&\mathcal{S}(e_1,e_1) = 3, \qquad \mathcal{S}(e_2,e_2) = 3, \qquad \mathcal{S}(e_3,e_3) = 3,\\
&\mathcal{S}(e_4,e_4) = -3, \qquad \mathcal{S}(\varphi e_1,\varphi e_1) = 3, \quad \mathcal{S}(\varphi e_2,\varphi e_2) = 3,\\
&\mathcal{S}(\varphi e_3,\varphi e_3) = 3, \quad \mathcal{S}(\varphi e_4,\varphi e_4) = 0, \quad \mathcal{S}(\varphi e_i,\varphi e_j) = 0,
\end{aligned}$$

for all $i, j = 1, 2, 3, 4$ $(i \neq j)$. The foregoing relations help in computing

$$(\nabla_{\mathcal{A}}\mathcal{S})(\varphi e_i, \varphi e_j) = 0, \quad \forall\, \mathcal{A} \in \chi(\mathscr{M}^4), \quad i, j = 1, 2, 3, 4.$$

Hence, this result implies that the manifold $\mathscr{M}$ is ω-parallel.
In view of align (3.3) and the above relations, the following results can be easily computed:

$$\begin{aligned}
&\bar{\nabla}_{e_1}e_1 = -2e_4, \ \bar{\nabla}_{e_1}e_2 = 0, \qquad \bar{\nabla}_{e_1}e_3 = 0, \qquad \bar{\nabla}_{e_1}e_4 = -2e_1,\\
&\bar{\nabla}_{e_2}e_1 = 0, \qquad \bar{\nabla}_{e_2}e_2 = -2e_4, \ \bar{\nabla}_{e_2}e_3 = 0, \qquad \bar{\nabla}_{e_2}e_4 = -2e_2,\\
&\bar{\nabla}_{e_3}e_1 = 0, \qquad \bar{\nabla}_{e_3}e_2 = 0, \qquad \bar{\nabla}_{e_3}e_3 = -2e_4, \ \bar{\nabla}_{e_3}e_4 = -2e_3,\\
&\bar{\nabla}_{e_4}e_1 = 0, \qquad \bar{\nabla}_{e_4}e_2 = 0, \qquad \bar{\nabla}_{e_4}e_3 = 0, \qquad \bar{\nabla}_{e_4}e_4 = 0.
\end{aligned}$$

Moreover, the torsion tensor $\bar{\mathcal{T}}$ is given by the components listed below.

$$\begin{aligned}
&\bar{\mathcal{T}}(e_1,e_1) = 0, \quad \bar{\mathcal{T}}(e_1,e_2) = 0, \quad \bar{\mathcal{T}}(e_1,e_3) = 0, \quad \bar{\mathcal{T}}(e_1,e_4) = -e_1,\\
&\bar{\mathcal{T}}(e_2,e_1) = 0, \quad \bar{\mathcal{T}}(e_2,e_2) = 0, \quad \bar{\mathcal{T}}(e_2,e_3) = 0, \quad \bar{\mathcal{T}}(e_2,e_4) = -e_2,\\
&\bar{\mathcal{T}}(e_3,e_1) = 0, \quad \bar{\mathcal{T}}(e_3,e_2) = 0, \quad \bar{\mathcal{T}}(e_3,e_3) = 0, \quad \bar{\mathcal{T}}(e_3,e_4) = -e_3,\\
&\bar{\mathcal{T}}(e_4,e_1) = -e_1, \bar{\mathcal{T}}(e_4,e_2) = -e_1, \bar{\mathcal{T}}(e_4,e_3) = -e_3, \bar{\mathcal{T}}(e_4,e_4) = 0.
\end{aligned}$$

Thus, from the above discussion, the relation (3.1) indicates that the linear connection introduced in (3.3) is a connection $\bar{\nabla}$ on $(\mathscr{M}^4, g)$. Furthermore, the following relations hold:

$$(\bar{\nabla} e_1 g)(e_i, e_j) = 0, \ (\bar{\nabla} e_2 g)(e_i, e_j) = 0, \ \ (\bar{\nabla} e_3 g)(e_i, e_j) = 0, \ \ (\bar{\nabla} e_4 g)(e_i, e_j) = 0,$$

for all $i, j = 1, 2, 3, 4$. This confirms that the condition (3.2) is satisfied on $\mathscr{M}^4$. Consequently, the linear connection defined in (3.3) is indeed an $\mathcal{SSMC}$ on $\mathscr{M}^4$. Hence, it follows that the manifold $(\mathscr{M}^4, g)$ is an $(\mathcal{LP}\text{-}\mathcal{K})_4$ manifold endowed with the connection $\bar{\nabla}$ given by (3.3).

The foregoing relations facilitate the computation of the curvature and Ricci tensors with respect to $\mathcal{SSMC}$ as follows:

$$\begin{array}{lll}
\bar{\mathscr{R}}(e_1, e_2)e_3 = 0, & \bar{\mathscr{R}}(e_1, e_3)e_3 = 4e_1, & \bar{\mathscr{R}}(e_3, e_2)e_2 = 4e_3, \\
\bar{\mathscr{R}}(e_3, e_1)e_1 = 4e_3, & \bar{\mathscr{R}}(e_2, e_1)e_1 = 4e_2, & \bar{\mathscr{R}}(e_2, e_3)e_3 = 4e_2, \\
\bar{\mathscr{R}}(e_1, e_2)e_2 = 4e_1, & \bar{\mathscr{R}}(e_3, e_1)e_2 = 0, & \bar{\mathscr{R}}(e_3, e_4)e_3 = -2e_4, \\
\bar{\mathcal{S}}(e_1, e_1) = 6, & \bar{\mathcal{S}}(e_2, e_2) = 6, & \bar{\mathcal{S}}(e_3, e_3) = 6, \\
\bar{\mathcal{S}}(e_4, e_4) = -6, & \bar{r} = 36. &
\end{array}$$

We can compute other components with the aid of symmetric and skew-symmetric properties. The relations [(4.7) - (4.14)] can easily be confirmed. Furthermore,

$$\bar{\mathcal{S}}(\varphi e_1, \varphi e_1) = 10, \ \bar{\mathcal{S}}(\varphi e_2, \ \varphi e_2) = 10, \ \bar{\mathcal{S}}(\varphi e_3, \varphi e_3) = 10,$$
$$\bar{\mathcal{S}}(\varphi e_4, \varphi e_4) = 0, \quad \bar{\mathcal{S}}(\varphi e_i, \varphi e_j) = 0$$

for all $i, j = 1, 2, 3, 4$ $(i \neq j)$.
From the above explorations, we say that $(\bar{\nabla}_{\mathcal{A}}\bar{\mathcal{S}})(\varphi e_i, \varphi e_j) = 0$, for all $\mathcal{A} \epsilon \chi(\mathscr{M}^4)$, where, i and j run from 1 to 4. From here, the manifold equipped with connection $\bar{\nabla}$ possesses ω-parallel Ricci tensor for connection $\bar{\nabla}$. Thus, from the foregoing explanations, we arrive at the subsequent fact:
"If the manifold $\mathscr{M}^4$ possesses ω-parallel Ricci tensor with respect to Levi-Civita connection, then it also contains ω-parallel Ricci tensor with respect to $\mathcal{SSMC}$ ".
Thus, the theorem 5.2 verified. The relation (4.14) gives $\bar{r} = 36$, which is a constant. Corollary 5.4 also holds on $\mathscr{M}^4$.

7. Discussion & Conclusion

A central outcome of this study is the observation that equipping a $(\mathcal{LP}-\mathcal{K})_n$ manifold with a semi-symmetric metric connection $\bar{\nabla}$ leads to enriched geometric structures and modified curvature conditions. The presence of such a connection alters the intrinsic and extrinsic properties of the manifold in ways that highlight the flexibility and depth of the Lorentzian para-Kenmotsu framework.
This investigation not only clarifies the influence of concircular and projective symmetries under the semi-symmetric connection but also opens avenues for further geometric analysis. Future work may explore higher-dimensional analogues, classifications under additional curvature constraints, or the role of such structures in broader geometric contexts.
Overall, this work advances the understanding of Lorentzian para-Kenmotsu geometry with alternative connections, contributing to the ongoing development of generalized geometric structures in differential geometry.

8. Scope and Significance

This paper presents new results on Lorentzian para-Kenmotsu manifolds with semi-symmetric metric connections, focusing on curvature structures such as concircular and projective tensors. We derive properties of ω-parallel, Codazzi, and cyclic parallel Ricci tensors, leading to refined classifications and deeper understanding of their geometric behavior.

The findings are relevant to mathematical physics, as Lorentzian manifolds model spacetime in relativity, and semi-symmetric connections provide flexible frameworks for curvature-based models in extended gravity theories.

Overall, the work bridges advanced differential geometry and gravitational theory, offering a precise framework for studying curvature symmetries and non-Riemannian structures.

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