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Finite topological type of complete gradient shrinking GRF system solitons

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Abstract. This paper investigates the properties and topological implications of gradient shrinking general Ricci flow (GRF) system solitons. A GRF system soliton is a solution that evolves through a one-parameter family of diffeomorphisms or scaling transformations. Under specific geometric constraints, such as bounded Ricci curvature or positive injectivity radius, we establish a lower bound for the potential function associated with these solitons. Furthermore, we demonstrate that any complete gradient shrinking GRF system soliton exhibits finite topological type. These results extend the understanding of geometric flows, linking them to broader applications in differential geometry and topology.

Keywords: general Ricci flow, soliton, shrinking, finite topological type.

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1. Introduction

Geometric flows transform an initial metric into a more insightful one, facilitating the analysis of a manifold's topology and supporting the resolution of various geometric conjectures. These flows find applications in diverse fields, including physics, mechanics, and practical challenges like 3D face recognition in computer science. In 1982, Hamilton introduced the Ricci flow for Riemannian manifolds, governed by the evolution equation:

$$\frac{\partial}{\partial t} g_{ij} = -2Ric_{ij}, \quad g(t=0) := g_0.$$

The Ricci flow, which evolves a Riemannian metric based on its Ricci curvature, serves as a natural counterpart to the heat equation for metrics. In 2008, B. List extended the Ricci flow system as follows

$$\begin{cases} \partial_t g = -2Rc + 2\alpha_n du \otimes du, \\ \partial_t u = \Delta u, \end{cases}$$

where $u(t)$ is a scalar function and α_n a constant depending only on the dimension $n \geq 3$. Via the extended Ricci flow, B. List has shown that Hamilton's Ricci flow and the static Einstein vacuum equations are closely connected. The extended Ricci flow provides a strong link between problems in low-dimensional geometry and topology and questions in physics, especially general relativity (see [9]). In 2018, J.Y. Wu introduce a general Ricci flow system which is closely linked with the Ricci flow and the renormalization group flow. Let M be a smooth manifold and $g(t)$ a family of Riemannian metrics on M and $H(t)$ a 1-parameter family of closed 3-forms on M. The triple $(M, g(t), H(t))$ is called a general Ricci flow system, if we have the following equations:

$$\begin{cases} \partial_t g = -2Ric + h + 2\alpha_n du \otimes du, \\ \partial_t H = \Delta_{LB} H, \\ \partial_t u = \Delta u, \end{cases}$$

where, Ric is the Ricci curvature, h a 2-form, written in a local coordinates as $h_{ij} = g^{kl} g^{mn} H_{ikm} H_{jln}$ (see [10]). J.Y. Wu established the short-time existence, entropy functionals, higher-derivative estimates, and compactness theorems for this system on closed Riemannian manifolds. M. Ishida proved that the shrinking entropy functional is monotonic along the generalized Ricci flow with a perturbed term and remains constant on a shrinking generalized Ricci soliton-type solution (see [7]). Moreover, it has been shown that any complete shrinking Ricci soliton on Finslerian manifold and generalized Ricci flow system soliton in Riemannian geometry possesses a finite fundamental group (see [1, 2, 12]). Also, it is proved that a complete non-compact shrinking Rieman or Finsler Yamabe soliton has finite fundamental group (see [3, 4]). In this paper, we further demonstrate that such solitons have finite topological type. General Ricci flow system solitons, which extend the notion of classical Ricci solitons, have also been the subject of extensive study (see [6, 8, 10]).

Fang, Man, and Zhang showed that a complete gradient shrinking Ricci soliton has finite topological type if its scalar curvature is bounded, (see [5]). Also the first peresent author obtain a finite topological type property on complete gradient Finsler Yamabe solitons (see [13]). Similarly, on Finslerian spaces, forward complete gradient shrinking Ricci solitons exhibit finite topological type under conditions such as bounded Ricci scalar or positive injectivity radius (see [14]).

This paper establishes a lower bound for the potential function of a gradient shrinking general Ricci flow system soliton and proves that any such soliton has finite topological type under appropriate conditions.

2. A lower bound for potential function of gradient shrinking GRF system solitons

The general Ricci flow system soliton (GRF system soliton) is defined by the following equations:

$$\begin{cases} \partial_t g = \mathcal{L}_X g + cg, \\ \partial_t H = \mathcal{L}_X H, \\ \partial_t u = \mathcal{L}_X u, \end{cases}$$

where X is a vector field on $M \times [0, T)$ and c is a real-valued function on $[0, T)$. If $X = \nabla f$ is the gradient of a smooth function f, the soliton is called a gradient soliton. We say that the soliton is shrinking, steady or expanding, if $c < 0$, $c = 0$ or $c > 0$, respectively (see [10]).

Here, we consider extended gradient shrinking GRF system solitons as follows:

$$\begin{cases} Ric + \nabla^2 f - \frac{1}{2}h - \alpha_n du \otimes du \geq \lambda g, \\ \mathcal{L}_{\nabla f} H = \Delta_{LB} H, \\ \mathcal{L}_{\nabla f} u = \Delta u, \end{cases} \tag{2.1}$$

and considering some conditions on this and obtain a lower bound for potential function f.

Theorem 2.1. *Let (M, g) be a complete Riemannian manifold satisfying (2.1), where $\lambda > 0$, $h \geq 0$ and ∇u is bounded. Also, if either the Ricci tensor Ric is bounded above or $Ric \geq \delta^{-1} g$ and the injectivity radius $inj(M, g) \geq \delta > 0$ for some $\delta > 0$. Then, for a fixed point $p \in M$, the following inequality is satisfied.*

$$f(x) \geq \frac{\lambda}{4} d(p, x)^2 - (\|\nabla f\|_p + \frac{F}{2}) d(p, x) + f(p) \tag{2.2}$$

where F is a real constant.

Proof. Let $p \in M$ be a fix point and let θ be a minimal geodesic parametrized by arc length s joining p to any point $x \in M$. Note that $s := d(p, x)$. Then along θ we have

$$Ric(\theta', \theta') + \nabla^2 f(\theta', \theta') - \frac{1}{2} h(\theta', \theta') - \alpha_n g(\theta', \nabla u)^2 \geq \lambda. \tag{2.3}$$

On the other hand

$$\nabla^2 f(\theta', \theta') = \mathcal{L}_{\nabla f} g(\theta', \theta') = 2g(\nabla_{\theta'} \nabla f, \theta') = 2\frac{d}{ds} g(\nabla f, \theta').$$

By integration of both sides inequality (2.3) and using from last relation, we have

$$\int_0^t Ric(\theta', \theta')ds + 2g(\nabla f, \theta'(s))\Big|_0^t - \frac{1}{2}\int_0^t h(\theta', \theta')ds - \alpha_n \int_0^t g(\theta'(s), \nabla u)^2 ds \geq \lambda t.$$

Therefore

$$\begin{aligned} &\int_0^t Ric(\theta', \theta')ds + 2g(\nabla f, \theta'(s))\Big|_0^t \geq \\ &\lambda t + \frac{1}{2}\int_0^t h(\theta', \theta')ds + \alpha_n \int_0^t g(\theta'(s), \nabla u)^2 ds, \end{aligned}$$

Since $h \geq 0$, we have $\int_0^t h(\theta', \theta')ds \geq 0$ and consequently

$$\begin{aligned} &\int_0^t Ric(\theta', \theta')ds + 2g(\nabla f, \theta'(s))\Big|_0^t \geq \\ &\lambda t + \frac{1}{2}\int_0^t h(\theta', \theta')ds + \alpha_n \int_0^t g(\theta'(s), \nabla u)^2 ds \geq \\ &\lambda t + \alpha_n \int_0^t g(\theta'(s), \nabla u)^2 ds, \end{aligned}$$

So

$$\int_0^t Ric(\theta', \theta')ds + 2g(\nabla f, \theta'(t)) - 2g(\nabla f, \theta'(0)) \geq \lambda t + \alpha_n \int_0^t g(\theta'(s), \nabla u)^2 ds,$$

and consequently

$$2g(\nabla f, \theta'(t)) \geq \lambda t + 2g(\nabla f, \theta'(0)) - \int_0^t Ric(\theta', \theta')ds + \alpha_n \int_0^t g(\theta'(s), \nabla u)^2 ds.$$

Now, by using $\frac{d}{dt} f(\theta(t)) = g(\nabla f, \theta'(t))$, yields

$$2\frac{d}{dt} f(\theta(t)) \geq \lambda t + 2g(\nabla f, \theta'(0)) - \int_0^t Ric(\theta', \theta')ds + \alpha_n \int_0^t g(\theta'(s), \nabla u)^2 ds. \tag{2.4}$$

By applying the Cauchy-Schwarz inequality $|g(\theta'(s), \nabla f)| \leq \|\nabla f\|_{\theta(s)}$ and the inequality (2.4), we get

$$2\frac{d}{dt}f(\theta(t)) \geq \lambda t + 2g(\nabla f, \theta'(0)) - \int_0^t Ric(\theta', \theta')ds + \alpha_n \int_0^t g(\theta'(s), \nabla u)^2 ds \geq$$
$$\lambda t - 2\|\nabla f\|_p - \int_0^t Ric(\theta', \theta')ds + \alpha_n \int_0^t g(\theta'(s), \nabla u)^2 ds. \quad (2.5)$$

By Lemma 4 of [5], the integral $\int_0^t Ric(\theta', \theta')ds$ is bounded above by a real constant C, provided that $Ric \geq \delta^{-1}g$ and the injectivity radius $inj(M, g) \geq \delta > 0$ for some $\delta > 0$. Also, applying Lemma 2.2 of [11] the integral $\int_0^t Ric(\theta', \theta')ds$ has an upper bound $2(n-1) + 2\Lambda$, whenever the Ricci tensor Ric is bounded above by a constant Λ. In general, the integral $\int_0^t Ric(\theta', \theta')ds$ has an upper bound under the assumptions of the theorem, that is, $\int_0^t Ric(\theta', \theta')ds < D$ for a positive real constat D. Now, according to inequality (2.5), we get

$$2\frac{d}{dt}f(\theta(t)) \geq \lambda t - 2\|\nabla f\|_p - D + \alpha_n \int_0^t g(\theta'(s), \nabla u)^2 ds. \quad (2.6)$$

On the other hand, if $\alpha_n \geq 0$, then $2\alpha_n \int_0^t g(\theta'(s), \nabla u)^2 ds \geq 0$ and from (2.6) we conclude that

$$2\frac{d}{dt}f(\theta(t)) \geq \lambda t - 2\|\nabla f\|_p - D.$$

By applying the Cauchy-Schwarz inequality we get $g(\theta'(s), \nabla u)^2 \leq \|\nabla u\|^2$. Also, since $\|\nabla u\| \leq E$ for some constant E, via inequality (2.6) we have

$$2\frac{d}{dt}f(\theta(t)) \geq \lambda t - 2\|\nabla f\|_p - D + \alpha_n E^2,$$

if $\alpha_n \leq 0$. Therefore, for any α_n, there exist a constant F such that

$$2\frac{d}{dt}f(\theta(t)) \geq \lambda t - 2\|\nabla f\|_p - F, \quad (2.7)$$

By integration of both sides respect to t from 0 to s, we conclude that

$$2(f(\theta(s)) - f(\theta(0))) \geq \frac{\lambda}{2}s^2 - (2\|\nabla f\|_p + F)s,$$

and therefore

$$f(\theta(s)) \geq \frac{\lambda}{4}s^2 - (\|\nabla f\|_p + \frac{F}{2})s + f(p).$$

Since $s = d(p, x)$, we get

$$f(x) \geq \frac{\lambda}{4}d(p,x)^2 - (\|\nabla f\|_p + \frac{F}{2})d(p,x) + f(p),$$

as we have claimed. □

Here, via obtained lower bound (2.2), we show that M has finite topological type. That is, M is homeomorphic to interior of a compact manifold with boundary.

Theorem 2.2. *Let (M, g) be a complete Riemannian manifold satisfying (2.1). Then M is of finite topological type if either $Ric \geq \delta^{-1} g$ and the injectivity radius $inj(M, g) \geq \delta > 0$ for some $\delta > 0$ or the Ricci tensor Ric is bounded above.*

Proof. In order to prove which M is of finite topological type it suffices to show that the function f is a proper function and has no critical points outside of a large compact domain. The deformation lemma (Isotopy Lemma) of Morse theory leads to M has finite topological type. Since f is a smooth function so it's continuous and consequently $f^{-1}\big([a, b]\big)$ is closed for any $a, b < \infty$. On the other hand, for any $x \in f^{-1}\big([a, b]\big)$ we have $f(x) \in [a, b]$ and therefore $|f(x)| \leq \max\{|a|, |b|\}$. Applying Theorem 2.1, we have

$$|f(x)| \geq f(x) \geq \frac{\lambda}{8} d(p, x)^2 - (\|\nabla f\|_p + \frac{F}{4}) d(p, x) + f(p).$$

By $|f(x)| \leq \max\{|a|, |b|\}$, we conclude that

$$\max\{|a|, |b|\} \geq \frac{\lambda}{8} d(p, x)^2 - (\|\nabla f\|_p + \frac{F}{4}) d(p, x) + f(p).$$

Thus for any $x \in f^{-1}\big([a, b]\big)$, the distance function $d(p, x)$ is bounded and consequently $f^{-1}\big([a, b]\big)$ is a bounded set. So far, we have shown that $f^{-1}\big([a, b]\big)$ is closed and bounded in the complete manifold M. Therefore, by Hoph-Rinow Theorem, we conclude that $f^{-1}\big([a, b]\big)$ is compact and so f is a proper function. In the following, one can easily peruse that f has no critical points outside of a compact set. For this purpose, suppose that $x \in M$ be a critical point of function f. By using relation (2.6) in the proof of Theorem 2.1, we consider a compact set $\bar{B}(p, \frac{2(4\|\nabla f\|_p + \frac{F}{4})}{\lambda})$. □

As a consequence of Theorem 2.2, it can be seen that any complete shrinking gradient general Ricci flow system soliton has finite topological type as follows.

Corollary 2.3. *Any complete shrinking gradient general Ricci flow system soliton under the assumptions of Theorem 2.2, has finite topological type.*

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