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On generalized silver Finsler metrics

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Abstract. In this paper, we present a coordinate-free investigation of the generalized silver Finsler metric. Specifically, for a Finsler manifold (M, L) and a 1-form $\mathfrak{B}$, we study various geometric structures associated with the Finsler metric

$$\widetilde{L}(x,y) = L\,\phi(s), \quad \text{where} \quad s := \frac{\mathfrak{B}}{L}, \quad \phi(s) := s^2 - 2s - 1.$$

The function $\phi(s)$ has roots $s_1 = 1-\sqrt{2}$ and $s_2 = 1+\sqrt{2}$, where the positive root represents the so-called the *silver ratio*. Assuming that L is a Finsler metric, we refer to $\widetilde{L}$ as the generalized silver Finsler metric. We derive the associated metric and Cartan tensors, along with other fundamental geometric objects. The non-degeneracy condition of the metric tensor of $\widetilde{L}$ is characterized. We compute the geodesic spray, Barthel connection, and Berwald connection of $\widetilde{L}$, when the 1-form $\mathfrak{B}$ arises from a concurrent π-vector field. Furthermore, we determine the curvature of the Barthel connection associated with $\widetilde{L}$. An illustrative example is also provided.

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connection.

1. Introduction

Irrational numbers possess intriguing properties that rival those of rational numbers. Among the most captivating is the golden ratio, denoted by $\xi = \frac{1+\sqrt{5}}{2}$, which is the positive root of the quadratic equation $\xi^2 - \xi - 1 = 0$. Motivated by this equation, Hrețcanu [5] introduced a golden structure on a manifold M through a tensor field φ of type $(1,1)$, which satisfies the identity $\varphi^2 = \varphi + I$.

Similarly, another irrational number of comparable significance is the silver ratio $\theta = 1 + \sqrt{2}$, or known as silver mean, which is the positive root of the equation $\xi^2 - 2\xi - 1 = 0$. This constant has found applications in architecture, physics, and design. M. Ozkan and B. Peltek [8], for instance, introduced the Silver Riemannian Manifolds.

Finsler geometry, as a natural generalization of Riemannian geometry, provides a rich and flexible framework for studying differential geometry, particularly through its ability to model direction-dependent structures. Among the many classes of Finsler metrics introduced in the literature, significant attention has been given to those defined through modifications of a base metric L via auxiliary scalar functions or 1-forms. This allows for the construction of new metrics with interesting geometric and physical properties.

A notable line of research in this direction involves the use of ratios or transformations involving 1-forms. Inspired by classical number theory and continued fraction expansions, the silver ratio $s = 1 + \sqrt{2}$, serves as a novel motivator for defining a new class of Finsler metrics. In this paper, we introduce and study a Finsler metric of the form:

$$\widetilde{L}(x,y) = L\,\phi(s), \quad \text{where} \quad s = \frac{\mathfrak{B}}{L}, \quad \phi(s) = s^2 - 2s - 1.$$

The function $\phi(s)$ has roots $s_1 = 1 - \sqrt{2}$ and $s_2 = 1 + \sqrt{2}$, linking the structure of the metric to the silver ratio. We refer to $\widetilde{L}$ as the *generalized silver Finsler metric*. This study opens the door to further exploration of Finsler metrics constructed via special number-theoretic functions and their potential applications in both pure and applied settings.

Following the pullback formalism to Finsler geometry, we investigate a coordinate-free study of generalized silver Finsler metric with special one π-form. First, by the concept of generalized silver Finsler metric we mean the deformation of a Finsler metric L (not necessarily Riemannian) by a one form $\mathfrak{B}$, that is,

$$\widetilde{L}(x,y) = L\,\phi(s) = \frac{\mathfrak{B}^2(x,y)}{L(x,y)} - 2\mathfrak{B}(x,y) - L(x,y), \tag{1.1}$$

with $s := \frac{\mathfrak{B}}{L}$, $\phi(s) := s^2 - 2s - 1$, whose roots $s_1 = \left(1-\sqrt{2}\right)$, $s_2 = \left(1+\sqrt{2}\right)$ (silver ratio), and L is Finslerian. We study the geometric objects of $\widetilde{L}$, for example, the metric and Cartan tensors. Moreover, we characterize the non-degenerate property of the metric tensor $\widetilde{g}$, that is, $\widetilde{g}$ is non-degenerate if and only if

$$L^2(1-2p^2) + 3\mathfrak{B}^2 \neq 0,$$

where $p^2 := g(\overline{p}, \overline{p})$, $\mathfrak{B} := g(\overline{p}, \overline{\eta})$, and $\overline{p}$ is a concurrent π-vector field over (M, L). It should be noted that $\widetilde{L}$ represents a regular Finsler structure if the above non-degenerate property is satisfied together with the inequality $1-\sqrt{2} > s > 1+\sqrt{2}$. Moreover, if the inequality $1-\sqrt{2} > s > 1+\sqrt{2}$ is not satisfied, then $\widetilde{L}$ represents a pseudo-Finsler structure ($\widetilde{L}$ is negative).

When (M, L) that admits a concurrent π-vector field $\overline{p}$, then we have the corresponding π-form $\mathbf{B} := i_{\overline{p}}\, g$, where g is the metric tensor of L. Hence, we get the associated one form $\mathfrak{B}(x,y) := \mathbf{B}(\overline{\eta})$. Within the generalized silver Finsler metric (1.1), we calculate intrinsically the relationship between the related canonical sprays as well as the two Barthel connections Γ and $\widetilde{\Gamma}$. The transformation of the curvature tensor field of the Barthel connection is obtained. For example, the associated canonical sprays G and $\widetilde{G}$ are related by

$$\widetilde{G} = G - \frac{2L^2\left(-2\mathfrak{B}^3 + 3\mathfrak{B}^2 L + L^3\right)}{\left(\mathfrak{B}^2 - 2\mathfrak{B}L - L^2\right)\left(3\mathfrak{B}^2 + L^2\left(1-2p^2\right)\right)}\,\mathcal{C} - \frac{2L^4}{3\mathfrak{B}^2 + L^2\left(1-2p^2\right)}\,\gamma\overline{p},$$

where, $\mathcal{C}$ is the Liouville vector field and γ is the canonical inclusion map.

As an example of a Finsler metric (M, L) that admits a concurrent vector field. Let $M = \{(x^1, x^2, x^3, x^4) \in U \subset \mathbb{R}^4 \mid x^1,\, x^2 \neq 0\}$; U is an open subset in $\mathbb{R}^4$ and L be a conic Finsler metric given by

$$L = \sqrt{(x^1)^2\left(\frac{(x^2)^2(y^4)^2 + 2\,y^2\,y^4}{y^2}\right)^2 + (y^1)^2 + (y^3)^2},$$

where $(x^1, x^2, x^3, x^4;\, y^1, y^2, y^3, y^4) \in \mathcal{T}U \subset \mathbb{R}^4 \times \mathbb{R}^4$; $\mathcal{T}U := TU \setminus \{0\}$.

Moreover, the components of the corresponding π-form $\mathbf{B}$ are given by $\mathbf{B}_1 = x^1$, $\mathbf{B}_2 = 0$, $\mathbf{B}_3 = x^3$, $\mathbf{B}_4 = 0$, and hence the associated one form $\mathfrak{B}(x,y)$ becomes $\mathfrak{B}(x,y) = x^1y^1 + x^3y^3$. Therefore, we have

$$\begin{aligned}\widetilde{L}(x,y) \;&=\; \frac{(x^1y^1 + x^3y^3)^2}{\sqrt{(x^1)^2\left(\frac{(x^2)^2(y^4)^2+2\,y^2\,y^4}{y^2}\right)^2 + (y^1)^2 + (y^3)^2}} - 2(x^1y^1 + x^3y^3)\\ &\quad - \sqrt{(x^1)^2\left(\frac{(x^2)^2(y^4)^2 + 2\,y^2\,y^4}{y^2}\right)^2 + (y^1)^2 + (y^3)^2}.\end{aligned}$$

Hence, the given Finsler structure $\widetilde{L}$ defines a conic Finsler structure over M, under the condition of non-degeneracy given by (4.1).

2. **Notations and Preliminaries**

Let M be a smooth manifold of dimension n, and consider its tangent bundle (TM, π, M) with the corresponding tangent bundle of TM denoted by $(TTM, d\pi, TM)$. The vertical subbundle $V(TM)$ of TM is defined by $V(TM) := \ker(d\pi)$, while the pullback bundle is denoted by $\pi^{-1}(TM)$, and $\mathcal{T}M := TM \setminus \{0\}$ denotes the slit tangent bundle. There exists a short exact sequence of vector bundle morphisms [3]:

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} TTM \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

where γ is the canonical inclusion map, and $\rho := (\pi_{TM}, d\pi)$.

The *almost tangent structure* (also called the *vertical endomorphism*) J on TM is given by $J = \gamma \circ \rho$. We denote by $C^\infty(TM)$ the space of smooth functions on TM, and let $\mathfrak{X}(\pi(M))$ denote the module of smooth sections of $\pi^{-1}(TM)$. The sections in $\mathfrak{X}(\pi(M))$ are referred to as π-vector fields and denoted by barred letters such as $\overline{X}$.

The *Liouville vector field* (or fundamental π-vector field) $\mathcal{C}$ is defined by

$$\mathcal{C} := \gamma\,\overline{\eta}, \quad \text{where } \overline{\eta}(u) = (u, u) \text{ for all } u \in \mathcal{T}M.$$

Let us recall some foundational concepts in the Klein-Grifone formalism of Finsler geometry, as detailed in [3, 4, 6].

Nonlinear Connections. A nonlinear connection Γ on M is a vector 1-form on TM, which is C^∞ on $\mathcal{T}M$ and continuous on TM, satisfying:

$$J\Gamma = J, \qquad \Gamma J = -J.$$

This yields associated projection operators:

$$h := \frac{1}{2}(I + \Gamma), \qquad v := \frac{1}{2}(I - \Gamma),$$

where h and v are the horizontal and vertical projectors, respectively. The torsion and curvature of Γ are given by:

$$t := \frac{1}{2}[J, \Gamma], \qquad \mathfrak{R} := -\frac{1}{2}[h, h].$$

Regular Connections. For a linear connection D on $\pi^{-1}(TM)$, the associated *connection map* K is defined by:

$$K : TTM \to \pi^{-1}(TM), \quad K(X) := D_X\overline{\eta}.$$

The horizontal subspace at $u \in TM$ is given by:

$$H_u(TM) := \{X \in T_u(TM) \mid K(X) = 0\}.$$

The connection D is called regular if:

$$T_u(TM) = V_u(TM) \oplus H_u(TM), \quad \forall u \in TM.$$

In such a case, the maps $\rho|_{H(TM)}$ and $K|_{V(TM)}$ are isomorphisms, and we define the horizontal lift $\beta := (\rho|_{H(TM)})^{-1}$.
Covariant Derivatives and Tensors. Let D be a regular connection on $\pi^{-1}(TM)$ with horizontal lift β. For a π-tensor field A of type $(0,p)$, we define the h- and v-covariant derivatives as:

$$(\overset{h}{D} A)(\overline{X},\overline{X}_1,...,\overline{X}_p) := (D_{\beta\overline{X}}A)(\overline{X}_1,...,\overline{X}_p),$$
$$(\overset{v}{D} A)(\overline{X},\overline{X}_1,...,\overline{X}_p) := (D_{\gamma\overline{X}}A)(\overline{X}_1,...,\overline{X}_p).$$

The torsion tensor $\mathbf{T}$ induces:

- (h)h-torsion: $Q(\overline{X},\overline{Y}) := \mathbf{T}(\beta\overline{X},\beta\overline{Y})$,
- (h)hv-torsion: $T(\overline{X},\overline{Y}) := \mathbf{T}(\gamma\overline{X},\beta\overline{Y})$,
- (h)v-torsion: $V(\overline{X},\overline{Y}) := \mathbf{T}(\gamma\overline{X},\gamma\overline{Y})$.

The curvature tensor $\mathbf{K}$ gives rise to:

$$R(\overline{X},\overline{Y})\overline{Z} := \mathbf{K}(\beta\overline{X},\beta\overline{Y})\overline{Z},$$
$$P(\overline{X},\overline{Y})\overline{Z} := \mathbf{K}(\beta\overline{X},\gamma\overline{Y})\overline{Z},$$
$$S(\overline{X},\overline{Y})\overline{Z} := \mathbf{K}(\gamma\overline{X},\gamma\overline{Y})\overline{Z}.$$

The associated (v)-torsions are defined by:

$$\widehat{R}(\overline{X},\overline{Y}) := R(\overline{X},\overline{Y})\overline{\eta}, \quad \widehat{P}(\overline{X},\overline{Y}) := P(\overline{X},\overline{Y})\overline{\eta}, \quad \widehat{S}(\overline{X},\overline{Y}) := S(\overline{X},\overline{Y})\overline{\eta}.$$

Finsler Structures and Connections.

Definition 2.1. *A* Finsler manifold (M,L) *consists of a smooth manifold* M *and a function* $L : TM \to \mathbb{R}$ *satisfying:*

(a) $L(u) > 0$ *for* $u \in \mathcal{T}M$, *and* $L(0) = 0$,
(b) L *is smooth on* $\mathcal{T}M$ *and continuous on* TM,
(c) L *is positively homogeneous of degree one:* $\mathcal{L}_{\mathcal{C}}L = L$,
(d) *The* 2*-form* $\Omega := dd_J E$ *is non-degenerate, where* $E := \frac{1}{2}L^2$.

The associated Finsler metric g *is defined by:*

$$g(\rho X,\rho Y) := \Omega(JX,Y), \quad \textit{for all } X,Y \in \mathfrak{X}(TM).$$

In this case (M,L) *is called regular Finsler manifold, keeping in mind that the positivity of* L *together with the non-degeneracy of* g_{ij} *guarantee the positive-definiteness of* g_{ij} *(see* [9]*). If these conditions are satisfied on a conic subset* $U \subset \mathcal{T}M$*, then* (M,L) *is called a conic Finsler manifold. Moreover, for the sake of applications, if the positivity condition is not satisified, then* L *is called pseudo-Finlser structure.*

A *semispray* is a vector field G on TM (smooth on $\mathcal{T}M$ and C^1 on TM) such that $JG = \mathcal{C}$. If $[\mathcal{C},G] = G$, then G is called a *spray*.

Proposition 2.2 ([6, 4])**.** *For a Finsler manifold* (M,L)*:*

(a) *The canonical spray G satisfies:* $i_G dd_J E = -dE$.
(b) *The Barthel connection* Γ *is given by:* $\Gamma = [J, G]$.

Theorem 2.3 ([13])**.** *Let (M, F) be a Finsler manifold with Finsler metric g. Then there exists a unique regular connection ∇ on $\pi^{-1}(TM)$ such that:*

(i) $\nabla g = 0$ *(metric connection),*
(ii) *The (h)h-torsion vanishes:* $Q = 0$,
(iii) *The (h)hv-torsion T is symmetric with respect to g in its last two arguments.*

This unique connection ∇ is referred as the Cartan connection.

Lemma 2.4 ([13])**.** *Let (M, L) be a Finsler manifold and β the horizontal lift associated with the Cartan connection ∇. Then:*

(a) $(D^\circ_{\gamma\overline{X}} g)(\overline{Y}, \overline{Z}) = 2\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z})$*, and* $\nabla_{\gamma\overline{X}} g = 0$.
(b) $(D^\circ_{\beta\overline{X}} g)(\overline{Y}, \overline{Z}) = -2\widehat{\mathbf{P}}(\overline{X}, \overline{Y}, \overline{Z})$*, and* $\nabla_{\beta\overline{X}} g = 0$.

Here, $\widehat{\mathbf{P}}$ *is the* $(v)hv$*-torsion of type* $(0, 3)$ *defined by* $\widehat{\mathbf{P}}(\overline{X}, \overline{Y}, \overline{Z}) := g(\widehat{P}(\overline{X}, \overline{Y}), \overline{Z})$.

For a more comprehensive treatment of global Finsler geometry in the pullback setting, see [1, 7, 10, 11, 12, 15, 16].

Lemma 2.5. *For a Finsler manifold (M, L), the following identities hold:*

(a) $d_J L(\gamma\overline{X}) = 0$*, and* $D^\circ_{\gamma\overline{X}} L = dL(\gamma\overline{X}) = d_J L(\beta\overline{X}) = \ell(\overline{X})$,
(b) $d_h L(\beta\overline{X}) = D^\circ_{\beta\overline{X}} L = 0$,
(c) $(D^\circ_{\gamma\overline{X}} \ell)(\overline{Y}) = (\nabla_{\gamma\overline{X}} \ell)(\overline{Y}) = L^{-1} \hbar(\overline{X}, \overline{Y})$,
(d) $dd_J E(\gamma\overline{X}, \beta\overline{Y}) = g(\overline{X}, \overline{Y})$.

Here, g is the Finsler metric and $\ell := L^{-1} i_{\overline{\eta}} g$ is the normalized supporting element.

3. Generalized Silver Finsler Metric

In this section, we undertake an intrinsic analysis of the Silver Finsler metric. Specifically, we generalize the classical construction by replacing the underlying Riemannian metric in the silver Finsler metric with a Finslerian one. The resulting structure will henceforth be referred to as the *generalized silver Finsler metric*.

Definition 3.1. [14] *Let (M, L) be a Finsler manifold and let D° denote the Berwald connection on $\mathcal{P}$. A π-vector field $\overline{Y} \in \mathcal{C}(\pi)$ is said to be independent of the directional argument y if and only if*

$$D^\circ_{\gamma\overline{X}} \overline{Y} = 0, \quad \forall\, \overline{X} \in \mathcal{C}(\pi).$$

Similarly, a scalar (or vector) π-form ω is independent of the directional argument y if and only if

$$D^\circ_{\gamma\overline{X}}\, \omega = 0, \quad \forall\, \overline{X} \in \mathcal{C}(\pi).$$

Definition 3.2. *Let (M, L) be a Finsler manifold. Consider the deformation of the Finsler structure L defined by*

$$\widetilde{L}(x,y) = \frac{\mathfrak{B}^2(x,y)}{L(x,y)} - 2\mathfrak{B}(x,y) - L(x,y), \tag{3.1}$$

where $\mathfrak{B}(x,y) := \mathbf{B}(\overline{\eta})$, and $\mathbf{B}$ is a scalar π-form independent of the directional argument y. If $\widetilde{L}$ defines a Finsler structure on M, it is called the generalized silver Finsler metric *(GSFM).*

To study the geometric properties associated with $\widetilde{L}$, we begin with the following auxiliary results:

Lemma 3.3. *Under the transformation $L \mapsto \widetilde{L}$, the vertical component of the Berwald connection remains invariant; that is,*

$$\widetilde{D}^{\circ}_{\gamma\overline{X}}\overline{Y} = D^{\circ}_{\gamma\overline{X}}\overline{Y}.$$

Proof. The result follows from the observation that the change in the horizontal lift, from β to $\widetilde{\beta}$, involves only a vertical vector field. Using the identity $D^{\circ}_{\gamma\overline{X}}\overline{Y} = \rho[\gamma\overline{X}, \beta\overline{Y}]$ (see [13]), and noting that $\rho \circ \gamma = 0$ and the vertical distribution is integrable, the vertical component remains unaffected. $\square$

Lemma 3.4. *Let (M, L) be a Finsler manifold and let $\mathbf{B}$ be a scalar π-form independent of y, with associated π-vector field $\overline{p}$ defined by $i_{\overline{p}}\, g = \mathbf{B}$. Define $\mathfrak{B}(x,y) := \mathbf{B}(\overline{\eta})$. Then, the following properties hold:*

(a): *$d_J\mathfrak{B}(\gamma\overline{X}) = 0$, and*

$$D^{\circ}_{\gamma\overline{X}}\mathfrak{B} = d\mathfrak{B}(\gamma\overline{X}) = d_J\mathfrak{B}(\beta\overline{X}) = \mathbf{B}(\overline{X}).$$

(b): *$d_h\mathfrak{B}(\beta\overline{X}) = D^{\circ}_{\beta\overline{X}}\mathfrak{B} = d\mathfrak{B}(\beta\overline{X}) = L\,\ell(D^{\circ}_{\beta\overline{X}}\overline{p})$, and*

$$d\mathfrak{B}(G) = L\,\ell(D^{\circ}_{G}\overline{p}).$$

(c): *$D^{\circ}_{\gamma\overline{X}}\overline{p} = -2T(\overline{X}, \overline{p})$.*

Proof. The proof follows from the facts that $\rho \circ \gamma$ and $K \circ \beta$ vanish identically, $\rho \circ \beta = id_{\mathfrak{X}(\pi(M))}$, $K \circ \gamma = id_{\mathfrak{X}(\pi(M))}$, $i_{\overline{\eta}}\widehat{\mathbf{P}} = 0$, $D^{\circ}_{\gamma\overline{X}}\mathbf{B} = 0$ and taking into account Lemma 2.4 and Definition 3.1. In more details, we have the following.
(a) For $d_J\mathfrak{B}(\gamma\overline{X})$, we have

$$d_J\mathfrak{B}(\gamma\overline{X}) = (J \circ \gamma\overline{X}) \cdot \mathfrak{B} = \gamma\,(\rho \circ \gamma)\overline{X} \cdot \mathfrak{B} = 0.$$

Moreover, $d_J\mathfrak{B}(\gamma\overline{X})$ can be obtained as follows:

$$\begin{aligned} d_J\mathfrak{B}(\beta\overline{X}) &= J\,(\beta\overline{X} \cdot \mathfrak{B}) = \gamma\,(\rho \circ \beta)\overline{X} \cdot \mathfrak{B} = \gamma\overline{X} \cdot \mathfrak{B} \\ &= (D^{\circ}_{\gamma\overline{X}}\mathbf{B})(\overline{\eta}) + \mathbf{B}(D^{\circ}_{\gamma\overline{X}}\overline{\eta}) \\ &= \mathbf{B}(\overline{X}). \end{aligned}$$

(b) One can see that

$$\begin{aligned} d_h\mathfrak{B}(\beta\overline{X}) &= (\beta\circ\rho\circ\beta\overline{X})\cdot\mathfrak{B} = \beta\overline{X}\cdot\mathfrak{B} = d\mathfrak{B}(\beta\overline{X}) \\ &= \beta\overline{X}\cdot g(\overline{p},\overline{\eta}) = (D^{\circ}_{\beta\overline{X}}\, g)(\overline{p},\overline{\eta}) + g(D^{\circ}_{\beta\overline{X}}\,\overline{p},\overline{\eta}) + g(\overline{p}, D^{\circ}_{\beta\overline{X}}\overline{\eta}) \\ &= -2\widehat{\mathbf{P}}(\overline{X},\overline{p},\overline{\eta}) + g(D^{\circ}_{\beta\overline{X}}\,\overline{p},\overline{\eta}) + 0 \\ &= L\,\ell(D^{\circ}_{\beta\overline{X}}\,\overline{p}). \end{aligned}$$

That is, we get that $d\mathfrak{B}(G) = L\,\ell(D^{\circ}_{G}\,\overline{p})$.
(c) We have

$$\begin{aligned} 0 &= (D^{\circ}_{\gamma\overline{X}}\,\mathbf{B})(\overline{Y}) \\ &= D^{\circ}_{\gamma\overline{X}}\,\mathbf{B}(\overline{Y}) - \mathbf{B}(D^{\circ}_{\gamma\overline{X}}\,\overline{Y}) \\ &= D^{\circ}_{\gamma\overline{X}}\, g(\overline{p},\overline{Y}) - g(\overline{p}, D^{\circ}_{\gamma\overline{X}}\,\overline{Y}) \\ &= (D^{\circ}_{\gamma\overline{X}}\, g)(\overline{p},\overline{Y}) + g(D^{\circ}_{\gamma\overline{X}}\,\overline{p},\overline{Y}) \\ &= 2\mathbf{T}(\overline{p},\overline{X},\overline{Y}) + g(D^{\circ}_{\gamma\overline{X}}\,\overline{p},\overline{Y}). \end{aligned}$$

Thus, we conclude that $D^{\circ}_{\gamma\overline{X}}\,\overline{p} = -2T(\overline{X},\overline{p})$. □

Adopting to the normalized supporting element ℓ and the angular metric tensor $\hbar$, we have the following proposition.

Proposition 3.5. *Under the generalized silver Finsler metric (3.1), we have*

(1) *The supporting form $\widetilde{\ell}$ and ℓ are related by*

$$\widetilde{\ell}(\overline{X}) = \frac{-\mathfrak{B}^2 - L^2}{L^2}\,\ell(\overline{X}) + \frac{2(\mathfrak{B}-L)}{L}\,\mathbf{B}(\overline{X}).$$

(2) *The angular metric tensors $\widetilde{\hbar}$ and $\hbar$ are related by*

$$\begin{aligned} \widetilde{\hbar}(\overline{X},\overline{Y}) = & \frac{\left(\mathfrak{B}^2 + L^2\right)\left(-\mathfrak{B}^2 + 2\mathfrak{B}L + L^2\right)}{L^4}\hbar(\overline{X},\overline{Y}) \\ & + \frac{2\left(\mathfrak{B}^2 - 2\mathfrak{B}L - L^2\right)}{L^2}\,\mathbf{B}(\overline{X})\mathbf{B}(\overline{Y}) \\ & + \frac{2\mathfrak{B}^2\left(\mathfrak{B}^2 - 2\mathfrak{B}L - L^2\right)}{L^4}\,\ell(\overline{X})\,\ell(\overline{Y}) \\ & - \frac{2\mathfrak{B}\left(\mathfrak{B}^2 - 2\mathfrak{B}L - L^2\right)}{L^3}\,\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}. \end{aligned}$$

Proof. Under the generalized silver Finsler metric (3.1), taking Lemma 3.4 into account, we have the following:

(1) Due to the facts that $\rho\circ\gamma=0$ and that $\rho\circ\beta=\rho\circ\widetilde{\beta}=id_{\mathfrak{X}(\pi(M))}$, it follows that

$$\begin{aligned}\widetilde{\ell}(\overline{X}) &= d_J\widetilde{L}(\widetilde{\beta}\overline{X})=d_J\widetilde{L}(\beta\overline{X})\\ &= \frac{\partial\widetilde{L}}{\partial L}\,d_JL(\beta\overline{X})+\frac{\partial\widetilde{L}}{\partial\mathfrak{B}}\,d_J\mathfrak{B}(\beta\overline{X})\\ &= \frac{-\mathfrak{B}^2-L^2}{L^2}\,\ell(\overline{X})+\frac{2(\mathfrak{B}-L)}{L}\,\mathbf{B}(\overline{X})\end{aligned}$$

(2) Applying the previous item, Lemma 2.5, Lemma 3.3, together with Lemma 3.4, one can show that

$$\begin{aligned}\widetilde{\hbar}(\overline{X},\overline{Y}) &= \widetilde{L}(\widetilde{D}^{\circ}_{\gamma\overline{X}}\widetilde{\ell})(\overline{Y})=\widetilde{L}(D^{\circ}_{\gamma\overline{X}}\widetilde{\ell})(\overline{Y})\\ &= \widetilde{L}\,D^{\circ}_{\gamma\overline{X}}\left\{\frac{-\mathfrak{B}^2-L^2}{L^2}\,\ell(\overline{Y})+\frac{2(\mathfrak{B}-L)}{L}\,\mathbf{B}(\overline{Y})\right\}\\ &= \widetilde{L}\left\{\left(D^{\circ}_{\gamma\overline{X}}\left(\frac{-\mathfrak{B}^2-L^2}{L^2}\right)\right)\ell(\overline{Y})+\left(D^{\circ}_{\gamma\overline{X}}\left(\frac{2(\mathfrak{B}-L)}{L}\right)\right)\mathbf{B}(\overline{Y})\right\}\\ &\quad+\widetilde{L}\left\{\left(\frac{2(\mathfrak{B}-L)}{L}\right)(D^{\circ}_{\gamma\overline{X}}\ell)(\overline{Y})+\left(\frac{2\mathfrak{B}-L}{L}\right)(D^{\circ}_{\gamma\overline{X}}\mathbf{B})(\overline{Y})\right\}\\ &= \left(\frac{\mathfrak{B}^2-2\mathfrak{B}L-L^2}{L}\right)\left\{\left(\frac{2\mathfrak{B}^2}{L^3}\,\ell(\overline{X})-\frac{2\mathfrak{B}}{L^2}\,\mathbf{B}(\overline{X})\right)\ell(\overline{Y})\right.\\ &\quad\left.+\left(-\frac{2\mathfrak{B}}{L^2}\,\ell(\overline{X})+\frac{2}{L}\,\mathbf{B}(\overline{X})\right)\mathbf{B}(\overline{Y})\right\}\\ &\quad+\left(\frac{\mathfrak{B}^2-2\mathfrak{B}L-L^2}{L}\right)\left\{\left(\frac{-\mathfrak{B}^2-L^2}{L^2}\right)(L^{-1}\,\hbar(\overline{X},\overline{Y})+0)\right\}.\end{aligned}$$

Hence, the result follows. □

4. The Metric and Cartan Tensors

In this section, we calculate some geometric objects associated to $\widetilde{L}(x,y)$ in terms of the objects associated with L. The following proposition shows the relationship between g and $\widetilde{g}$.

Proposition 4.1. *The Finsler metric $\widetilde{g}$ associated with the generalized silver Finsler metric (3.1) is given by the following relation:*

$$\begin{aligned}\widetilde{g}(\overline{X},\overline{Y}) &= \frac{\left(\mathfrak{B}^2+L^2\right)\left(-\mathfrak{B}^2+2\mathfrak{B}L+L^2\right)}{L^4}\,g(\overline{X},\overline{Y})\\ &\quad+\frac{2\left(3\mathfrak{B}^2-6\mathfrak{B}L+L^2\right)}{L^2}\,\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y})\\ &\quad+\frac{4\mathfrak{B}^4-6\mathfrak{B}^3L-2\mathfrak{B}L^3}{L^4}\,\ell(\overline{X})\,\ell(\overline{Y})\\ &\quad-\frac{2\left(2\mathfrak{B}^3-3\mathfrak{B}^2L-L^3\right)}{L^3}\,\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y})+\mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.\end{aligned}$$

Consequently, the Cartan torsion $\widetilde{\mathbf{T}}$ *of the generalized silver Finsler metric has the form*

$$\begin{aligned}
2\widetilde{\mathbf{T}}(\overline{X},\overline{Y},\overline{Z}) &= 2\frac{\left(\mathfrak{B}^2+L^2\right)\left(-\mathfrak{B}^2+2\mathfrak{B}L+L^2\right)}{L^4}T(\overline{X},\overline{Y},\overline{Z}) \\
&+\frac{4\mathfrak{B}^4-6\mathfrak{B}^3L-2\mathfrak{B}L^3}{L^5}\{\hbar(\overline{X},\overline{Z})\,\ell(\overline{Y})+\hbar(\overline{Y},\overline{Z})\,\ell(\overline{X})\} \\
&-\frac{2\left(2\mathfrak{B}^3-3\mathfrak{B}^2L-L^3\right)}{L^4}\left\{\mathbf{B}(\overline{X})\,\hbar(\overline{Y},\overline{Z})+\mathbf{B}(\overline{Y})\,\hbar(\overline{X},\overline{Z})\right\} \\
&+\left(D^\circ_{\gamma\overline{Z}}\frac{\left(\mathfrak{B}^2+L^2\right)\left(-\mathfrak{B}^2+2\mathfrak{B}L+L^2\right)}{L^4}\right)g(\overline{X},\overline{Y}) \\
&+\left(D^\circ_{\gamma\overline{Z}}\frac{2\left(3\mathfrak{B}^2-6\mathfrak{B}L+L^2\right)}{L^2}\right)\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y}) \\
&+\left(D^\circ_{\gamma\overline{Z}}\frac{4\mathfrak{B}^4-6\mathfrak{B}^3L-2\mathfrak{B}L^3}{L^4}\right)\ell(\overline{X})\,\ell(\overline{Y}) \\
&+\left(D^\circ_{\gamma\overline{Z}}\frac{-2\left(2\mathfrak{B}^3-3\mathfrak{B}^2L-L^3\right)}{L^3}\right)\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y})+\mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.
\end{aligned}$$

Proof. In view of the generalized silver Finsler metric (3.1), using Proposition 3.5, we have the following:

$$\begin{aligned}
\widetilde{\ell}(\overline{X}) &= \frac{-\mathfrak{B}^2-L^2}{L^2}\,\ell(\overline{X})+\frac{2(\mathfrak{B}-L)}{L}\,\mathbf{B}(\overline{X}). \\
\widetilde{\hbar}(\overline{X},\overline{Y}) &= \frac{\left(\mathfrak{B}^2+L^2\right)\left(-\mathfrak{B}^2+2\mathfrak{B}L+L^2\right)}{L^4}\hbar(\overline{X},\overline{Y}) \\
&+\frac{2\left(\mathfrak{B}^2-2\mathfrak{B}L-L^2\right)}{L^2}\,\mathbf{B}(\overline{X})\mathbf{B}(\overline{Y}) \\
&+\frac{2\mathfrak{B}^2\left(\mathfrak{B}^2-2\mathfrak{B}L-L^2\right)}{L^4}\,\ell(\overline{X})\,\ell(\overline{Y}) \\
&-\frac{2\mathfrak{B}\left(\mathfrak{B}^2-2\mathfrak{B}L-L^2\right)}{L^3}\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y})+\mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.
\end{aligned}$$

Hence, using the definition of the angular metric tensor $\widetilde{\hbar} := \widetilde{g}-\widetilde{\ell}\otimes\widetilde{\ell}$, the expression of $\widetilde{g}$ is obtained. Moreover, using the expression of the metric $\widetilde{g}$, taking into account the fact that $(D^\circ_{\gamma\overline{Z}}g)(\overline{X},\overline{Y}) = 2\mathbf{T}(\overline{X},\overline{Y},\overline{Z})$ (Lemma 2.4), it follows the expression of the Cartan torsion $\widetilde{\mathbf{T}}$ of the generalized silver Finsler metric. □

Theorem 4.2. *The metric tensor* $\widetilde{g}$ *of* $\widetilde{L}$ *is non-degenerate if and only if*

$$L^2(1-2p^2)+3\mathfrak{B}^2\neq 0. \tag{4.1}$$

That is, the generalized Silver Finsler metric is a Finsler structure (or, conic Finsler structure) if and only if the condition (4.1) *is satisfied.*

Before going through the proof, we have the following remark.

Remark 4.3. *The generalized silver Finsler metric $\widetilde{L}(x,y) = L\,\phi(s)$ with the condition (4.1) together with the inequality $1-\sqrt{2} > s > 1+\sqrt{2}$ represents a regular Finsler structure. That is, if s does not satisfy this inequality, then $\widetilde{L}(x,y)$ can be negative and hence it represents a pseudo-Finsler structure.*

Proof. Assume that $\widetilde{g}$ be the Finsler metric associated with the generalized Silver Finsler metric, defined by (3.1). To prove the non-degenerate property of $\widetilde{g}$. Suppose that $\widetilde{g}(\overline{X},\overline{Y}) = 0$ for all $\overline{X} \in \mathfrak{X}(\pi(M))$. By using Proposition 4.1, we obtain

$$\begin{aligned}
0 =& \frac{\left(\mathfrak{B}^2+L^2\right)\left(-\mathfrak{B}^2+2\mathfrak{B}L+L^2\right)}{L^4}\, g(\overline{X},\overline{Y}) \\
&+\frac{2\left(3\mathfrak{B}^2-6\mathfrak{B}L+L^2\right)}{L^2}\,\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y}) \\
&+\frac{4\mathfrak{B}^4-6\mathfrak{B}^3L-2\mathfrak{B}L^3}{L^4}\,\ell(\overline{X})\,\ell(\overline{Y}) \\
&-\frac{2\left(2\mathfrak{B}^3-3\mathfrak{B}^2L-L^3\right)}{L^3}\,\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y})+\mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.
\end{aligned} \tag{4.2}$$

From which, by substituting $\overline{X} = \overline{p}$, noting that $\ell(\overline{p}) = \frac{\mathfrak{B}}{L}$ and $\mathbf{B}(\overline{p}) = g(\overline{p},\overline{p}) =: p^2$, one can show that

$$\zeta_1\,\ell(\overline{Y}) + \zeta_2\,\mathbf{B}(\overline{Y}) = 0, \tag{4.3}$$

where

$$\begin{aligned}
\zeta_1 &:= \frac{2\left(-2\mathfrak{B}^3+3\mathfrak{B}^2L+L^3\right)\left(L^2p^2-\mathfrak{B}^2\right)}{L^5}, \\
\zeta_2 &:= \frac{-5\mathfrak{B}^4+8\mathfrak{B}^3L+6\mathfrak{B}^2L^2p^2-12\mathfrak{B}L^3p^2+4\mathfrak{B}L^3+2L^4p^2+L^4}{L^4}.
\end{aligned}$$

Similarly, by substituting $\overline{X} = \overline{\eta}$, taking into account the facts that $\ell(\overline{\eta}) = L$ and $\mathbf{B}(\overline{\eta}) = \mathfrak{B}$, we obtain

$$\zeta_3\,\ell(\overline{Y}) + \zeta_4\,\mathbf{B}(\overline{Y}) = 0, \tag{4.4}$$

with

$$\begin{aligned}
\zeta_3 &:= \frac{-\mathfrak{B}^4+2\mathfrak{B}^3L+2\mathfrak{B}L^3+L^4}{L^3}, \\
\zeta_4 &:= \frac{2\left(\mathfrak{B}^3-3\mathfrak{B}^2L+\mathfrak{B}L^2+L^3\right)}{L^2}.
\end{aligned}$$

Now, the system of the algebraic equations (4.3) and (4.4) has non-trivial solution (i.e. $\ell(\overline{Y}) \neq 0$ or $\mathbf{B}(\overline{Y}) \neq 0$) if and only if

$$\frac{\left(\mathfrak{B}^2-2\mathfrak{B}L-L^2\right)^3\left(3\mathfrak{B}^2+L^2\left(1-2p^2\right)\right)}{L^7} = 0.$$

Now, keeping in mind that $\widetilde{L} \neq 0$ over $\mathfrak{T}M$, then if $3\mathfrak{B}^2 + L^2\left(1-2p^2\right) \neq 0$ then $\ell(\overline{Y}) = \mathbf{B}(\overline{Y}) = 0$. By substituting by $\ell(\overline{Y}) = 0$ and $\mathbf{B}(\overline{Y}) = 0$ into (4.2), we obtain

$$g(\overline{X}, \overline{Y}) = 0.$$

Making use of the fact that g is non-degenerate, then we get $\overline{Y} = 0$. This means that $\widetilde{g}$ is non-degenerate. Conversely, if $\widetilde{g}$ is non-degenerate, then by the same way we get $\ell(\overline{Y}) = 0$ and $\mathbf{B}(\overline{Y}) = 0$. That is, $3\mathfrak{B}^2 + L^2\left(1-2p^2\right) \neq 0$. □

Form now on, we consider that the generalized Silver Finsler metric $\widetilde{L}$ satisfies the condition (4.1).

5. **Geodesic Spray and Berwald Connection**

To avoid the complications of the formalae and to be able to find the geodesic spray of the generalized Silver Finsler metric, we restrict ourselves to a special 1-form. Precisely, we have the following definition.

Definition 5.1. [14] *Assume (M, L) is a Finsler manifold. A π-vector field $\overline{p} \in \mathfrak{X}(\pi(M))$ is called a concurrent π-vector field if it satisfies the following conditions*

$$\nabla_{\beta\overline{X}}\,\overline{p} = -\overline{X} = D^\circ_{\beta\overline{X}}\,\overline{p}, \quad \nabla_{\gamma\overline{X}}\,\overline{p} = 0 = D^\circ_{\gamma\overline{X}}\,\overline{p}. \tag{5.1}$$

Moreover, if $\mathbf{B}$ is the π-form associated with $\overline{p}$ under the duality defined by the metric g: $\mathbf{B} = i_{\overline{p}}\, g$, then the π-form $\mathbf{B}$ has the properties

$$(\nabla_{\beta\overline{X}}\mathbf{B})(\overline{Y}) = -g(\overline{X}, \overline{Y}) = (D^\circ_{\beta\overline{X}}\mathbf{B})(\overline{Y}), \quad (\nabla_{\gamma\overline{X}}\mathbf{B})(\overline{Y}) = 0 = (D^\circ_{\gamma\overline{X}}\mathbf{B})(\overline{Y}).$$

If the π-vector field $\overline{p}(x, y)$ associated with the given scalar π-form $\mathbf{B}$, means that $\mathbf{B}$ is a concurrent vector field over (M, L), then $\widetilde{L}(x, y)$ will be called a special generalized Silver Finsler metric.

In [14], Nabil et al. investigated an intrinsic study of concurrent π-vector fields in Finsler geometry. Moreover, they characterized the concurrent π-vector fields. That is, we have the following.

Lemma 5.2. *Let (M, L) be a Finsler manifold equipping a scalar π-form $\mathbf{B}$ which is independent of the directional argument y, and $\overline{p}$ its the associated π-vector field is concurrent π-vector field. Then $d\mathfrak{B}(G) = -L^2$.*

Proof. The proof follows by applying Lemma 3.4 (b) and taking Definition 5.1 into account. □

Theorem 5.3. [14] *A concurrent π-vector field $\overline{p}$ and its associated π-form **B** are independent of the directional argument y.*

In this section, we find the relationship between the canonical (geodesic) spray $\widetilde{G}$ corresponding to the special generalized silver Finsler metric $\widetilde{L}$, in terms of the geodesic spray G of L. Precisely, we have the following theorem.

Theorem 5.4. *The canonical spray $\widetilde{G}$ associated with the special generalized silver Finsler metric* (3.1)*, is given by*

$$\widetilde{G} = G - \frac{2L^2\left(-2\mathfrak{B}^3 + 3\mathfrak{B}^2 L + L^3\right)}{\left(\mathfrak{B}^2 - 2\mathfrak{B}L - L^2\right)\left(3\mathfrak{B}^2 + L^2\left(1 - 2p^2\right)\right)}\,\mathcal{C} - \frac{2L^4}{3\mathfrak{B}^2 + L^2\left(1 - 2p^2\right)}\,\gamma\overline{p},$$

where $\mathcal{C}$ is the Liouville vector field defined by $\mathcal{C} := \gamma\,\overline{\eta}$ and $p^2 := \mathbf{B}(\overline{p}) = g(\overline{p}, \overline{p})$.

Proof. Due to the special generalized silver Finsler metric *(3.1)*, taking into account the expression of the exterior π-form $\widetilde{\Omega} := \frac{1}{2} dd_J\,\widetilde{L}^2$, the fact that the difference between two sprays is a vertical vector field (i.e. $\widetilde{G} = G + \gamma\overline{\mu}$, for some π-vector field $\overline{\mu}$) and using Proposition 2.2, one can show that

$$\begin{aligned} -d\widetilde{E}(X) &= i_{\widetilde{G}}\,\widetilde{\Omega}(X) = i_{G+\gamma\overline{\mu}}\,(\tfrac{1}{2}\,dd_J\widetilde{L}^2)(X) \\ &= \tfrac{1}{2}i_G\,dd_J\widetilde{L}^2(X) + \tfrac{1}{2}i_{\gamma\overline{\mu}}\,dd_J\widetilde{L}^2(X). \end{aligned} \tag{5.2}$$

Therefore, after some computation together the fact thats $\beta\overline{\eta} = G$ and $X = hX + vX = \beta\rho X + \gamma K X$, together with Lemmas 2.5, 3.4 and 5.2, we have

$$\begin{aligned} & d\widetilde{E}(X) = \\ = \ & \frac{1}{2}d\widetilde{L}^2(X) = \widetilde{L}\,d\widetilde{L}(X) \\ = \ & \left(\frac{\mathfrak{B}^2 - 2\mathfrak{B}L - L^2}{L}\right)\left\{\left(\frac{-\mathfrak{B}^2 - L^2}{L^2}\right)dL(X) + \left(\frac{2(\mathfrak{B} - L)}{L}\right)d\mathfrak{B}(X)\right\} \\ = \ & \frac{-\mathfrak{B}^4 + 2\mathfrak{B}^3 L + 2\mathfrak{B}L^3 + L^4}{L^3}\,dL(X) + \frac{2\left(\mathfrak{B}^3 - 3\mathfrak{B}^2 L + \mathfrak{B}L^2 + L^3\right)}{L^2}\,d\mathfrak{B}(X). \end{aligned}$$

Also, we have

$$\begin{aligned} \frac{1}{2}\,i_G\,dd_J\widetilde{L}^2(X) &= \frac{1}{2}\{dd_J\widetilde{L}^2(\beta\overline{\eta}, X)\} \\ &= \frac{1}{2}\left\{G\cdot d_J\widetilde{L}^2(X) - X\cdot d_J\widetilde{L}^2(G) - d_J\widetilde{L}^2[G, X]\right\} \\ &= \frac{1}{2}\left\{G\cdot(2\widetilde{L}\widetilde{\ell}(\rho X)) - X\cdot(2\widetilde{L}\widetilde{\ell}(\overline{\eta})) - 2\widetilde{L}\widetilde{\ell}(\rho[G, X])\right\} \\ &= ((G\cdot\widetilde{L})\,\widetilde{\ell}(\rho X) + \widetilde{L}\,G\cdot\widetilde{\ell}(\rho X)) - (X\cdot\widetilde{L}^2) - \widetilde{L}\,\widetilde{\ell}(\rho[G, X]). \end{aligned}$$

From which taking into account Lemmas 3.4, 5.2 and the following facts

$$\begin{aligned}
G\cdot\widetilde{L} &= d\widetilde{L}(G)=2L^2-2\mathfrak{B}L\\
X\cdot\widetilde{L} &= d\widetilde{L}(X)=\left(\frac{-\mathfrak{B}^2-L^2}{L^2}\right)dL(X)+\left(\frac{2(\mathfrak{B}-L)}{L}\right)d\mathfrak{B}(X),\\
\widetilde{\ell}(\overline{X}) &= \left(\frac{-\mathfrak{B}^2-L^2}{L^2}\right)\ell(\overline{X})+\left(\frac{2(\mathfrak{B}-L)}{L}\right)\mathbf{B}(\overline{X}),\\
\rho[G,X] &= \rho[G,hX+vX]=D^\circ_G\rho X-KX,\\
(D^\circ_G\,\mathbf{B})(\overline{X}) &= -g(\overline{X},\overline{\eta})=-L\,\ell(\overline{X}),\\
(D^\circ_G\,\ell)(\overline{X}) &= (\nabla_G\,\ell)(\overline{X})=0,\\
d\mathfrak{B}(X) &= \mathbf{B}(KX)-L\ell(\rho X),\\
dL(X) &= dL(\gamma KX)=\ell(KX).
\end{aligned}$$

Using the above identities, then straightforward calculations imply the above relation reduces to

$$\begin{aligned}
&\frac{1}{2}\,i_G\,dd_J\widetilde{L}^2(X)=\\
=\;&(2L^2-2\mathfrak{B}L)\left(\left(\frac{-\mathfrak{B}^2-L^2}{L^2}\right)\ell(\rho X)+\left(\frac{2(\mathfrak{B}-L)}{L}\right)\mathbf{B}(\rho X)\right)\\
&+\left(\frac{\mathfrak{B}^2-2\mathfrak{B}L-L^2}{L}\right)G\cdot\left(\left(\frac{-\mathfrak{B}^2-L^2}{L^2}\right)\ell(\rho X)+\left(\frac{2(\mathfrak{B}-L)}{L}\right)\mathbf{B}(\rho X)\right)\\
&-2\left(\frac{\mathfrak{B}^2-2\mathfrak{B}L-L^2}{L}\right)\left(\left(\frac{-\mathfrak{B}^2-L^2}{L^2}\right)dL(X)+\left(\frac{2(\mathfrak{B}-L)}{L}\right)d\mathfrak{B}(X)\right)\\
&-\left(\frac{\mathfrak{B}^2-2\mathfrak{B}L-L^2}{L}\right)\left(\left(\frac{-\mathfrak{B}^2-L^2}{L^2}\right)\ell(\rho[G,X])+\left(\frac{2(\mathfrak{B}-L)}{L}\right)\mathbf{B}(\rho[G,X])\right)\\
=\;&\left(\frac{2\left(2\mathfrak{B}^3-3\mathfrak{B}^2L-L^3\right)}{L}\right)\ell(\rho X)-\left(2\left(3\mathfrak{B}^2-6\mathfrak{B}L+L^2\right)\right)\mathbf{B}(\rho X)\\
&+\frac{\mathfrak{B}^4-2\mathfrak{B}^3L-2\mathfrak{B}L^3-L^4}{L^3}\,dL(X)-\frac{2\left(\mathfrak{B}^3-3\mathfrak{B}^2L+\mathfrak{B}L^2+L^3\right)}{L^2}\,d\mathfrak{B}(X).
\end{aligned}$$

On the other hand, using Proposition 4.1, we have

$$\begin{aligned}
&\frac{1}{2}\,i_{\gamma\overline{\mu}}\,dd_J\widetilde{L}^2(X)=\widetilde{g}(\overline{\mu},\rho X)\\
&=\frac{\left(\mathfrak{B}^2+L^2\right)\left(-\mathfrak{B}^2+2\mathfrak{B}L+L^2\right)}{L^4}\,g(\overline{\mu},\rho X)+\frac{2\left(3\mathfrak{B}^2-6\mathfrak{B}L+L^2\right)}{L^2}\,\mathbf{B}(\overline{\mu})\,\mathbf{B}(\rho X)\\
&-\frac{2\left(2\mathfrak{B}^3-3\mathfrak{B}^2L-L^3\right)}{L^3}\,\{\mathbf{B}(\overline{\mu})\,\ell(\rho X)+\mathbf{B}(\rho X)\,\ell(\overline{\mu})\}\\
&+\frac{4\mathfrak{B}^4-6\mathfrak{B}^3L-2\mathfrak{B}L^3}{L^4}\,\ell(\overline{\mu})\,\ell(\rho X).
\end{aligned}$$

Plugging the last two relations into Equation (5.2), after some calculation, it follows that

$$\begin{aligned}
&\frac{\mathfrak{B}^4-2\mathfrak{B}^3L-2\mathfrak{B}L^3-L^4}{L^3}\,dL(X)-\frac{2\left(\mathfrak{B}^3-3\mathfrak{B}^2L+\mathfrak{B}L^2+L^3\right)}{L^2}\,d\mathfrak{B}(X)\\
=\;&\left(\frac{2\left(2\mathfrak{B}^3-3\mathfrak{B}^2L-L^3\right)}{L}\right)\ell(\rho X)-\left(2\left(3\mathfrak{B}^2-6\mathfrak{B}L+L^2\right)\right)\,\mathbf{B}(\rho X)\\
&+\frac{\mathfrak{B}^4-2\mathfrak{B}^3L-2\mathfrak{B}L^3-L^4}{L^3}\,dL(X)-\frac{2\left(\mathfrak{B}^3-3\mathfrak{B}^2L+\mathfrak{B}L^2+L^3\right)}{L^2}\,d\mathfrak{B}(X)\\
&+\frac{\left(\mathfrak{B}^2+L^2\right)\left(-\mathfrak{B}^2+2\mathfrak{B}L+L^2\right)}{L^4}\,g(\overline{\mu},\rho X)+\frac{2\left(3\mathfrak{B}^2-6\mathfrak{B}L+L^2\right)}{L^2}\,\mathbf{B}(\overline{\mu})\,\mathbf{B}(\rho X)\\
&-\frac{2\left(2\mathfrak{B}^3-3\mathfrak{B}^2L-L^3\right)}{L^3}\,\left\{\mathbf{B}(\overline{\mu})\,\ell(\rho X)+\mathbf{B}(\rho X)\,\ell(\overline{\mu})\right\}\\
&+\frac{4\mathfrak{B}^4-6\mathfrak{B}^3L-2\mathfrak{B}L^3}{L^4}\,\ell(\overline{\mu})\,\ell(\rho X).
\end{aligned}$$

Adopting to the non-degenerate property of the Finsler metric g, the above relation reduces to

$$\begin{aligned}
&\frac{\left(\mathfrak{B}^2+L^2\right)\left(-\mathfrak{B}^2+2\mathfrak{B}L+L^2\right)}{L^4}\,\overline{\mu}\\
&=\left\{\frac{-2\left(2\mathfrak{B}^3-3\mathfrak{B}^2L-L^3\right)}{L^2}+\frac{2\left(2\mathfrak{B}^3-3\mathfrak{B}^2L-L^3\right)}{L^4}\,\mathbf{B}(\overline{\mu})\right.\\
&\left.-\frac{4\mathfrak{B}^4-6\mathfrak{B}^3L-2\mathfrak{B}L^3}{L^5}\,\ell(\overline{\mu})\right\}\overline{\eta}+\left\{2\left(3\mathfrak{B}^2-6\mathfrak{B}L+L^2\right)\right.\\
&\left.-\frac{2\left(3\mathfrak{B}^2-6\mathfrak{B}L+L^2\right)}{L^2}\,\mathbf{B}(\overline{\mu})+\frac{2\left(2\mathfrak{B}^3-3\mathfrak{B}^2L-L^3\right)}{L^3}\,\ell(\overline{\mu})\right\}\overline{p},
\end{aligned}\tag{5.3}$$

where $\ell(\overline{\mu})$ and $\mathbf{B}(\overline{\mu})$ are geometric quantities given by the following system

$$\begin{aligned}
A_1\,\ell(\overline{\mu})+B_1\,\mathbf{B}(\overline{\mu}) &= C_1,\\
A_2\,\ell(\overline{\mu})+B_2\,\mathbf{B}(\overline{\mu}) &= C_2,
\end{aligned}\tag{5.4}$$

with coefficients determined by

$$\begin{aligned}
A_1 &:= \frac{-\mathfrak{B}^4+2\mathfrak{B}^3L+2\mathfrak{B}L^3+L^4}{L^4},\\
B_1 &:= \frac{2\left(\mathfrak{B}^3-3\mathfrak{B}^2L+\mathfrak{B}L^2+L^3\right)}{L^3},\\
C_1 &:= \frac{2\left(\mathfrak{B}^3-3\mathfrak{B}^2L+\mathfrak{B}L^2+L^3\right)}{L},
\end{aligned}$$

$$
\begin{aligned}
A_2 &:= \frac{2\left(-2\mathfrak{B}^3+3\mathfrak{B}^2L+L^3\right)\left(L^2p^2-\mathfrak{B}^2\right)}{L^5},\\
B_2 &:= \frac{-5\mathfrak{B}^4+8\mathfrak{B}^3L+6\mathfrak{B}^2L^2p^2+4\mathfrak{B}L^3\left(1-3p^2\right)+L^4\left(2p^2+1\right)}{L^4},\\
C_2 &:= \frac{2\left(-2\mathfrak{B}^4+3\mathfrak{B}^3L+3\mathfrak{B}^2L^2p^2-6\mathfrak{B}L^3p^2+\mathfrak{B}L^3+L^4p^2\right)}{L^2},\\
p^2 &:= \mathbf{B}(\overline{p}).
\end{aligned}
$$

Making use of the condition (4.1), the system (5.4) has the following solution

$$
\begin{aligned}
\ell(\overline{\mu}) &= \frac{-2L^3\left(-\mathfrak{B}^3+\mathfrak{B}^2L-\mathfrak{B}L^2+L^3\right)}{\left(\mathfrak{B}^2-2\mathfrak{B}L-L^2\right)\left(3\mathfrak{B}^2+L^2\left(1-2p^2\right)\right)},\\
\mathbf{B}(\overline{\mu}) &= \frac{2L^2\left(2\mathfrak{B}^4-3\mathfrak{B}^3L-\mathfrak{B}^2L^2p^2+\mathfrak{B}L^3\left(2p^2-1\right)+L^4p^2\right)}{\left(\mathfrak{B}^2-2\mathfrak{B}L-L^2\right)\left(3\mathfrak{B}^2+L^2\left(1-2p^2\right)\right)}.
\end{aligned}
$$

Consequently, in view of Equation (5.3) taking into account the assumption $\widetilde{G}=G+\gamma\overline{\mu}$, it follows that the canonical sprays G and $\widetilde{G}$, are related by

$$
\widetilde{G}=G-\frac{2L^2\left(-2\mathfrak{B}^3+3\mathfrak{B}^2L+L^3\right)}{\left(\mathfrak{B}^2-2\mathfrak{B}L-L^2\right)\left(3\mathfrak{B}^2+L^2\left(1-2p^2\right)\right)}\,\mathcal{C}-\frac{2L^4}{3\mathfrak{B}^2+L^2\left(1-2p^2\right)}\,\gamma\overline{p}.
$$

This completes the proof. □

Theorem 5.5. *The Barthel connection $\widetilde{\Gamma}$ associated with the special Silver Finsler metric (3.1), is given by*

$$
\widetilde{\Gamma}=\Gamma-\lambda_1\,J-d_J\lambda_1\otimes\gamma\overline{\eta}+d_J\lambda_2\otimes\gamma\overline{p},
$$

where λ_1 and λ_2 are scalar functions determined by

$$
\begin{aligned}
\lambda_1 &:= \frac{2L^2\left(-2\mathfrak{B}^3+3\mathfrak{B}^2L+L^3\right)}{\left(\mathfrak{B}^2-2\mathfrak{B}L-L^2\right)\left(3\mathfrak{B}^2+L^2\left(1-2p^2\right)\right)},\\
\lambda_2 &:= \frac{-2L^4}{3\mathfrak{B}^2+L^2\left(1-2p^2\right)}.
\end{aligned}
$$

Consequently, the horizontal map $\widetilde{\beta}$ associated with the special generalized first approximation Matsumoto metric has the form

$$
\widetilde{\beta}\overline{X}=\beta\overline{X}-\frac{1}{2}\left\{\lambda_1\,\gamma\overline{X}+d_J\lambda_1(\beta\overline{X})\,\gamma\overline{\eta}-d_J\lambda_2(\beta\overline{X})\,\gamma\overline{p}\right\}.
$$

Proof. The proof follows from $\widetilde{\Gamma}=[J,\widetilde{G}]$, Theorem 5.4, the formula [2]:

$$
[fX,J]=f[X,J]+df\wedge i_XJ-d_Jf\otimes X,
$$

taking into account the given expressions of λ_1 and λ_2, and the facts that

$$
d_Jp^2 = 0,\;\; i_{\gamma\overline{\eta}}\,J=0=i_{\gamma\overline{p}}\,J,\;\; [\gamma\overline{p},J]X=0.
$$

□

Theorem 5.6. *The Barthel curvature tensor* $\widetilde{\mathfrak{R}}$ *associated with the special generalized second approximation Matsumoto metric* (3.1) *is determined by*

$$\widetilde{\mathfrak{R}} = \mathfrak{R} - [h, \mathbb{L}] - N_{\mathbb{L}},$$

where $N_{\mathbb{L}} := \frac{1}{2}[\mathbb{L}, \mathbb{L}]$ *is the Nijenhuis torsion of a vector 1-form* $\mathbb{L}$ *defined by*

$$\mathbb{L} := -\frac{1}{2}\left\{\lambda_1\, J + d_J\lambda_1 \otimes \gamma\overline{\eta} - d_J\lambda_2 \otimes \gamma\overline{p}\right\}. \tag{5.5}$$

Proof. The proof follows from Theorem 5.5, together with the fact that $\widetilde{\mathfrak{R}} = -\frac{1}{2}[\widetilde{h}, \widetilde{h}]$, and taking into account the proprties of the Frölicher-Nijenhuis bracket. □

The Berwald vertical counterpart is given by Lemma 3.3 and the Berwald horizontal counterpart is given by the following result.

Proposition 5.7. *For the special generalized silver Finsler metric* (3.1)*, the Berwald horizontal counterpart is given by*

$$\begin{aligned}\widetilde{D^\circ}_{\widetilde{\beta}\overline{X}}\,\overline{Y} \;=\;& D^\circ{}_{\beta\overline{X}}\overline{Y} - \frac{1}{2}\{\lambda_1\, D^\circ_{\gamma\overline{X}}\,\overline{Y} + d_J\lambda_1(\beta\overline{X})\, D^\circ_{\gamma\overline{\eta}}\,\overline{Y} \\ & -d_J\lambda_1(\beta\overline{X})\,\overline{Y} - d_J\lambda_1(\beta\overline{Y})\,\overline{X} - d_J\lambda_2(\beta\overline{X})\, D^\circ_{\gamma\overline{p}}\,\overline{Y}\} \\ & +\frac{1}{2}\left\{dd_J\lambda_1(\gamma\overline{Y}, \beta\overline{X})\,\overline{\eta} - dd_J\lambda_2(\gamma\overline{Y}, \beta\overline{X}))\,\overline{p}\right\}.\end{aligned}$$

Proof. The proof follows from the fact that $v := \gamma\circ K$, $h := \beta\circ\rho$, $\gamma D^\circ{}_{hX}\,\overline{Y} := v[hX, JY]$ and $D^\circ{}_{\gamma\overline{X}}\,\rho Y := \rho[\gamma\overline{X}, \beta\overline{Y}]$ ([13, Proposition 4.4]), taking into account Theorem 5.6, and the facts that the map $\gamma : \pi^{-1}(TM) \to VTM$ is an isomorphism, the Berwlad (v)v-curvature $\widetilde{S}^\circ = 0$, $[JX, JY] = J[X, JY] + J[JX, Y]$, $vJ = J$ and $Jv = 0$. □

We now introduce an example of a conic Finsler metric that admits a concurrent π-vector field. Following this, we derive the associated π-form corresponding to the given structure. Comprehensive computations related to this example are provided in the supplementary PDF and Maple files based on the Finsler package [17], available at:
`https://github.com/salahelgendi/Silver_Finsler_metrics`

Example 5.8. Let $M = \{(x^1, x^2, x^3, x^4) \in U \subset \mathbb{R}^4 \mid x^1,\, x^2 \neq 0\}$ and L be a conic Finsler metric given by

$$L = \sqrt{(x^1)^2\left(\frac{(x^2)^2(y^4)^2 + 2\,y^2\,y^4}{y^2}\right)^2 + (y^1)^2 + (y^3)^2},$$

where $(x^1, x^2, x^3, x^4;\, y^1, y^2, y^3, y^4) \in \mathfrak{T}U \subset \mathbb{R}^4 \times \mathbb{R}^4$.

The metric tensor has the following non-vanishing components g_{ij}:

$$g_{11} = 1, \qquad g_{22} = \frac{(x^1)^2\,(x^2)^2\,(y^4)^3(3(x^2)^2 y^4 + 4y^2)}{(y^2)^4},$$

$$g_{24} = -\frac{2\,(x^1)^2\,(x^2)^2 (y^4)^2 (2(x^2)^2 y^4 + 3y^2)}{(y^2)^3},$$

$$g_{33} = 1, \quad g_{44} = \frac{2\,(x^1)^2\,(3(x^2)^4 (y^4)^2 + 6(x^2)^2 y^2 y^4 + 2(y^2)^2)}{(y^2)^2}.$$

The inverse metric tensor has the following non-vanishing components g^{ij}:

$$g^{11} = 1, \quad g^{33} = 1,$$

$$g^{22} = \frac{(y^2)^4\,(3(x^2)^4 (y^4)^2 + 6(x^2)^2 y^2 y^4 + 2(y^2)^2)}{(x^1)^2 (x^2)^2 (y^4)^3\,((x^2)^6 (y^4)^3 + 6(x^2)^4 y^2 (y^4)^2 + 12(x^2)^2 (y^2)^2 y^4 + 8(y^2)^3)},$$

$$g^{24} = \frac{\left(2(x^2)^2 y^4 + 3y^2\right)(y^2)^3}{(x^1)^2 y^4\,((x^2)^6 (y^4)^3 + 6(x^2)^4 y^2 (y^4)^2 + 12(x^2)^2 (y^2)^2 y^4 + 8(y^2)^3)},$$

$$g^{44} = \frac{1}{2}\frac{\left(3(x^2)^2 y^4 + 4y^2\right)(y^2)^2}{(x^1)^2\,((x^2)^6 (y^4)^3 + 6(x^2)^4 y^2 (y^4)^2 + 12(x^2)^2 (y^2)^2 y^4 + 8(y^2)^3)}.$$

The Cartan tensor has the following non-vanishing components C_{ijk}:

$$C_{222} = -\frac{6\,(x^1)^2\,(x^2)^2\,(y^4)^3 ((x^2)^2 y^4 + y^2)}{(y^2)^5},$$

$$C_{224} = \frac{6\,(x^1)^2\,(x^2)^2\,(y^4)^2 ((x^2)^2 y^4 + y^2)}{(y^2)^4},$$

$$C_{244} = -\frac{6\,(x^1)^2\,(x^2)^2\,y^4 ((x^2)^2 y^4 + y^2)}{(y^2)^3},$$

$$C_{444} = \frac{6\,(x^1)^2\,(x^2)^2\,((x^2)^2 y^4 + y^2)}{(y^2)^2}.$$

The coefficients G^i of the geodesic spray are given by

$$G^1 = -\frac{x^1\,(y^4)^2\left((x^2)^4 (y^4)^2 + 4(x^2)^2 y^2 y^4 + 4(y^2)^2\right)}{2(y^2)^2}, \quad G^2 = \frac{(x^2\,y^1 - x^1\,y^2)\,y^2}{x^1\,x^2},$$

$$G^4 = \frac{y^1\,y^4}{x^1}, \quad G^3 = 0.$$

By performing direct computations, or alternatively using the Finsler package [17], one can obtain the coefficients of the Cartan connection. For instance,

$$\Gamma^2_{12} = \frac{1}{x^1}, \quad \Gamma^4_{14} = \frac{1}{x^1}, \quad \Gamma^1_{11} = 0, \quad \Gamma^i_{3j} = 0.$$

It is evident that this metric supports a concurrent π-vector field of the form $\overline{p} = p^i\,\overline{\partial_i}$, where $\overline{\partial_i}$ are the local basis vectors of the fibers of $\pi^{-1}(TM)$, with components defined by $p^2(x) = p^4(x) = 0$ and $p^1(x) = x^1, p^3(x) = x^3$. In this case, it follows that $p^i C_{ijk} = 0$, and for instance, we compute

$$p^i_{|j} = \delta_j p^i + p^h \Gamma^i_{hj} = \delta_j p^i + p^1 \Gamma^i_{1j} + p^3 \Gamma^i_{3j}.$$

$$p^1_{|1} = \delta_1 p^1 + p^1 \Gamma^1_{11} = 1, \quad p^2_{|2} = \delta_2 p^2 + p^1 \Gamma^2_{12} = 1,$$

$$p^3_{|3} = \delta_3 p^3 + p^1 \Gamma^3_{13} = 1, \quad p^4_{|4} = \delta_4 p^4 + p^1 \Gamma^4_{14} = 1.$$

While all other components of $p^i_{|j}$ vanish. In addition, the corresponding π-form $\mathbf{B}$ has components $\mathbf{B}^2 = \mathbf{B}^4 = 0$, $\mathbf{B}^1 = x^1, \mathbf{B}^3 = x^3$, which implies the associated 1-form is given by $\mathfrak{B} = x^1y^1 + x^3y^3$.

Therefore, we have

$$\begin{aligned}
\widetilde{L}(x,y) &= L\,\phi(s) \\
&= L\,(s^2 - 2s - 1) \\
&= \frac{\mathfrak{B}^2(x,y)}{L(x,y)} - 2\mathfrak{B}(x,y) - L(x,y) \\
&= \frac{(x^1y^1 + x^3y^3)^2}{\sqrt{(x^1)^2\left(\frac{(x^2)^2(y^4)^2 + 2\,y^2\,y^4}{y^2}\right)^2 + (y^1)^2 + (y^3)^2}} - 2(x^1y^1 + x^3y^3) \\
&\quad - \sqrt{(x^1)^2\left(\frac{(x^2)^2(y^4)^2 + 2\,y^2\,y^4}{y^2}\right)^2 + (y^1)^2 + (y^3)^2}.
\end{aligned}$$

Hence, the given Finsler structure $\widetilde{L}$ defines a conic Finsler structure over M, under the condition of non-degeneracy given by (4.1).

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References

1. S. G. Elgendi and A. Soleiman, *An intrinsic proof of Numata's theorem on Landsberg spaces*, J. Korean Math. Soc., **61**, 1 (2024), 149–160. arXiv: 2304.07925 [math. DG].
2. A. Frölicher and A. Nijenhuis, *Theory of vector-valued differential forms*, I, Ann. Proc. Kon. Ned. Akad., A, **59** (1956), 338–359.
3. J. Grifone, *Structure présque-tangente et connexions,* I, Ann. Inst. Fourier, Grenoble, **22**, 1 (1972), 287-334.
4. J. Grifone, *Structure presque-tangente et connexions,* II, Ann. Inst. Fourier, Grenoble, **22**, 3 (1972), 291-338.
5. C.E. Hretcanu, *Submanifolds in Riemannian manifold with Golden structure*, Workshop on Finsler geometry and its applications, Hungary, 2007.
6. J. Klein and A. Voutier, *Formes extérieures génératrices de sprays*, Ann. Inst. Fourier, Grenoble, **18**, 1 (1968), 241-260.
7. R. Miron and M. Anastasiei, *The geometry of Lagrange spaces: Theory and applications*, Kluwer Acad. Publ., **59**, 1994.
8. M. Ozkan and B. Peltek, *A new structure on manifolds: Silver structure*, Int. Electron. J. Geom., **9**(2)(2016), 59–69.
9. L. Rezso, *A note on Finsler-Minkowski norms*, Houston J. Math., **33** 3 (2007), 701–707.
10. A. Soleiman, *Recurrent Finsler manifolds under projective change*, Int. J. Geom. Meth. Mod. Phys., **13** (2016).
11. A. Soleiman and S. G. Elgendi, *Intrinsic study of generalized horizontal recurrent Finsler connection*, Publ. Math. Debrecen, **105**/3-4 (2024), 321–339.

12. J. Szilasi, R.L. Lovas and D.Cs. Kertész, *Connections, Sprays and Finsler Structures*, World Scientific, 2014.
13. Nabil L. Youssef, S. H. Abed and A. Soleiman, *Cartan and Berwald connections in the pullback formalism*, Algebras, Groups and Geometries, **25** (2008), 363-384. arXiv: 0707.1320 [math. DG].
14. Nabil L. Youssef, S. H. Abed and A. Soleiman, *Concurrent π-vector fields and energy β-change*, Int. J. Geom. Meth. Mod. Phys., **60** (2011), 1003–1031.
15. Nabil L. Youssef, S. H. Abed and A. Soleiman, *A global approach to the theory of connections in Finsler geometry*, Tensor, N. S., **71** (2009), 187-208. arXiv: 0801.3220 [math.DG].
16. Nabil L. Youssef, S. H. Abed and A. Soleiman, *Geometric objects associated with the fundumental connections in Finsler geometry*, J. Egypt. Math. Soc., **18** (2010), 67-90. arXiv: 0805.2489 [math.DG].
17. Nabil L. Youssef and S. G. Elgendi, *New Finsler package*, Comput. Phys. Commun., **185**, 3 (2014), 986–997.

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