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# Projectively flat Finsler spaces with some special transformed metrics

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**Abstract.** In this research paper, we have considered the various type of $\beta$-change in Finsler metric $F$ such as square change Finsler metric, cubic change Finsler metric, quartic change Finsler metric and obtained fundamental metric tensor, Cartan's tensor of these metrics. Further, we obtained the necessary and sufficient conditions under which said metrics are projectively flat and also given a example to support our results.

**Keywords:** Cartan's tensor, Fundamental metric tensor, $\beta$ change in Finsler metric, projective flatness.

## 1. Introduction

An smooth manifold $M$ together with Finsler structure $F$, i,e., $(M, F)$ is called Finsler space and corresponding geometry is called Finsler geometry. Then, Shibata in [15] introduced the notion of $\beta$-change for general Finsler

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metric $F$, i.e., $F^\star = f(F,\beta)$. A Finsler metric $F$ on an open subset $\mathcal{U}$ of $\Re^n$ is called projectively flat if and only if all the geodesics are straight lines in $\mathcal{U}$ (see [3]).

The idea of $(\alpha,\beta)$-metrics was given by Matsumoto [5]. An $(\alpha,\beta)$-metric on a connected smooth manifold $M$ is a Finsler metric $F$ constructed from a Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and one form $\beta = b_i(x)y^i$ on $M$ and is of the form $F = \alpha\phi(\frac{\beta}{\alpha})$, where $\phi$ is a smooth function on $M$. $(\alpha,\beta)$-metrics are generalization of Randers metrics. Many researchers such as [1], [2], [8], [9],[12], [13],[16], [17] and [19] have been carried out various results on $(\alpha,\beta)$-metrics. There are so many classical examples of $(\alpha,\beta)$-metrics such as Randers metric, Kropina metric, Matsumoto metric, square metric, cubic metric, exponential metric, infinite series metric.

1.1. **Definition.** : Let $(M,F)$ be an $n$-dimensional Finsler space and $\beta = b_i(x)y^i$ be a 1-form on $M$, then the metric

$$F^\star = F + \beta, \tag{1.1}$$

is called Randers changed Finsler metric, and the change given by (1.1) is called Randers change [6].

1.2. **Definition.** :Let $(M,F)$ be an $n$-dimensional Finsler space and $\beta = b_i(x)y^i$ be a one form on $M$. Then the change $F \to F^\star = f(F,\beta)$ is called $\beta$-change of Finsler metric $F$, where $f(F,\beta)$ is a positively homogeneous function of $F$ and $\beta$ of degree one. [15]

1.3. **Definition.** : Let $(M,F)$ be an $n$-dimensional Finsler space. The function

$$g_{ij} = (\frac{F^2}{2})_{y^iy^j} = FF_{y^iy^j} + F_{y^i}F_{y^j} = h_{ij} + l_il_j$$

is called fundamental metric tensor of the metric $F$ [7].

1.4. **Definition.** : Let $(M,F)$ be an $n$- dimensional Finsler space, then its Cartan's tensor is defined as

$$C_{ijk}(y) = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4}(F^2)_{y^iy^jy^k},$$

which is symmetric in all the three indices $i,j,k$ [7].

From above definitions, we propose some special Finsler metrics via $\beta$-change of Finsler metric. Our further studies will be based on these metrics. Let $(M,F)$ be an $n$- dimensional Finsler space and $\beta = b_i(x)y^i$ be a one form on $M$. Then, we propose the following

$$F^\star = \frac{(F+\beta)^2}{F}, \quad F^\star = \frac{(F+\beta)^3}{F^2}, F^\star = \frac{(F+\beta)^4}{F^3},$$

which are called square change, cubic change and quartic change of the Finsler metric $F$.

Here, we shall try to obtain the fundamental metric tensor, Cartan's tensor and further necessary and sufficient condition for projectively flatness of the said metrics.

## 2. Fundamental metric tensor and Cartan's tensor for square $\beta$-change Finsler metric

The square $\beta$-change of Finsler metric $F$ is

$$F^\star = \frac{(F+\beta)^2}{F}. \tag{2.1}$$

Differentiating (2.1) with respect to $y^i$, we have

$$F^\star_{y^i} = \frac{2(F+\beta)(F_{y^i}+b_i)}{F} + \frac{(F+\beta)^2 F_{y^i}}{F^2}. \tag{2.2}$$

Again differentiating (2.2) with respect to $y^j$, we have

$$F^\star_{y^i y^j} = (1-\frac{\beta^2}{F^2})F_{y^i y^j} - \frac{2\beta}{F^2}(b_i F_{y^j} + b_j F_{y^j}) + \frac{2}{F}b_i b_j + \frac{2\beta^2}{F^3}F_{y^i}F_{y^j}, \tag{2.3}$$

Now,

$$\begin{aligned} g^\star_{ij} &= F^\star F^\star_{y^i y^j} + F^\star_{y^i}F^\star_{y^j} \\ &= \Big\{\frac{(F+\beta)^2}{F}\Big\}\Big\{(1-\frac{\beta^2}{F^2})F_{y^i y^j} - \frac{2\beta}{F^2}(b_i F_{y^j} + b_j F_{y^j}) + \frac{2}{F}b_i b_j + \frac{2\beta^2}{F^3}F_{y^i}F_{y^j}\Big\} \\ &+ \Big\{\frac{2(F+\beta)(F_{y^i}+b_i)}{F} + \frac{(F+\beta)^2F_{y^i}}{F^2}\Big\}\Big\{\frac{2(F+\beta)(F_{y^j}+b_j)}{F} \\ &+ \frac{(F+\beta)^2F_{y^j}}{F^2}\Big\}. \end{aligned} \tag{2.4}$$

After simplifying, we get

$$\begin{aligned} g^\star_{ij} &= \Big(\frac{F^4+2\beta F^3-\beta^4-2\beta^3F}{F^4}\Big)g_{ij} + \Big(\frac{4\beta^4-2\beta F^3+6\beta^3F}{F^4}\Big)F_{y^i}F_{y^j} \\ &+ \Big(\frac{2F^3-6\beta^2F-4\beta^3}{F^3}\Big)\Big(b_iF_{y^j}+b_jF_{y^i}\Big) + \Big(\frac{6F^2+8\beta F+6\beta^2}{F^2}\Big)b_ib_j, \end{aligned} \tag{2.5}$$

Hence, we have the following

**Theorem 2.1.** *Let $(M, F^\star)$ be an $n$-dimensional Finsler space with $F^\star = (F+\beta)^2/F$ as a square change of $F$. Then its fundamental metric tensor is given by (2.5).*

Next, we find Cartan's tensor for square $\beta$-change of Finsler metric $F$. By definition, we have

$$2C_{ijk}(y) = \frac{\partial g_{ij}}{\partial y^k}. \tag{2.6}$$

From equation (2.5), we get

$$
\begin{aligned}
&2C^{\star}_{ijk} = \frac{(F^4+2\beta F^3-\beta^4-2\beta^3F)}{F^4}(2C_{ijk}) + (h_{ij}+\frac{y_iy_j}{F^2})\\
&\Big[(\frac{4F^3+6F^2\beta-2\beta^3-4F^4-8\beta F^3+4\beta^3+8\beta^3F}{F^6})y_k+\\
&(\frac{2F^4-4\beta^3-6\beta^2F}{F^5})b_k\Big] + \Big[(\frac{-6\beta F^3+6\beta^3F-16\beta^4+8\beta F^3-24\beta^3F}{F^6})y_k+\\
&(\frac{16\beta^3-2F^4+18\beta^2F^2}{F^5})b_k\Big]\frac{y_iy_j}{F^2} + (\frac{4F^3-2\beta F^3+6\beta^3F}{F^4})(\frac{y_ih_{jk}+y_jh_{ik}}{F^2})+\\
&\Big[(\frac{6F^3-6\beta^2F-6F^3+18\beta^2F+12\beta^3}{F^5})y_k - (\frac{12\beta^2F+12\beta^2}{F^3})b_k](\frac{b_iy_j+b_jy_i}{F})+\\
&(\frac{2F^3-6\beta^2F-4\beta^3}{F^3})(\frac{b_ih_{jk}+b_jh_{ik}}{F})\Big] + \Big[(\frac{12F^2+18\beta F-12F^3-16\beta F^2-12\beta^2F}{F^4})\\
&y_k + (\frac{8F^2+12\beta F}{F^3})b_k\Big]b_ib_j.
\end{aligned}
\tag{2.7}
$$

After simplification, we get

$$
\begin{aligned}
2C^{\star}_{ijk} =& \frac{2(F^4+2\beta F^3-\beta^4-2\beta^3F)}{F^4}(C_{ijk}) + h_{ij}[(\frac{6F^2\beta+2\beta^3}{F^6})y_k+\\
&(\frac{2F^3-4\beta^3-6\beta^2F}{F^5})b_k] + h_{jk}[(\frac{6\beta^4-2\beta F^3+6\beta^3F}{F^6})y_i+\\
&(\frac{2F^3-6\beta^2F-4\beta^3}{F^4})b_i] + h_{ik}[(\frac{4\beta^4-2\beta F^3+6\beta^3F}{F^6})y_j+\\
&(\frac{2F^3-6\beta^2F-4\beta^3}{F^4})b_j] - \frac{y_iy_jy_k}{F^5}(\frac{14\beta F^3+16\beta^4+2\beta F^2+24\beta^3F}{F^3})+\\
&(\frac{12\beta^3+18\beta^2F-6\beta^2F}{F^6})(b_iy_jy_k+b_jy_ky_i+b_ky_iy_j) + (\frac{8F+2\beta}{F^2})b_ib_jb_k+\\
&(\frac{18\beta-16F\beta-12\beta^2}{F^3})(b_ib_jy_k+b_jb_ky_i+b_ib_ky_j).
\end{aligned}
\tag{2.8}
$$

Hence, we have the following

**Theorem 2.2.** *Let $(M, F^\star)$ be an $n$-dimensional Finsler space with $F^\star = (F+\beta)^2/F$ as a square change of $F$, then its Cartan's tensor is given by (2.8).*

## 3. Fundamental metric tensor and Cartan's tensor for cubic $\beta$-change Finsler metric

The cubic $\beta$-change of Finsler metric $F$ is

$$F^\star = \frac{(F+\beta)^3}{F^2}. \tag{3.1}$$

Differentiating (3.1) with respect to $y^i$, we have

$$F^{\star}_{y^i} = \frac{(F+\beta)^2(F-2\beta)F_{y^i}}{F^3} + \frac{3(F+\beta)^2 b_i}{F^2}, \tag{3.2}$$

Again differentiating (3.2) with respect to $y^j$, we have

$$\begin{aligned} F^{\star}_{y^i y^j} &= (\frac{F^3 - 3F\beta^2 - 3\beta^3}{F^3})F_{y^i y^j} - \frac{6\beta(\beta+F)}{F^3}(b_i F_{y^j} + b_j F_{y^j}) \\ &+ \frac{6(F+\beta)}{F^2} b_i b_j + \frac{6\beta(\beta F + \beta^2)}{F^4} F_{y^i} F_{y^j}. \end{aligned} \tag{3.3}$$

Now, we have

$$\begin{aligned} g^{\star}_{ij} &= \{\frac{(F+\beta)^3}{F^2}\}\{(\frac{F^3 - 3F\beta^2 - 3\beta^3}{F^3})F_{y^i y^j} - \frac{6\beta(\beta+F)}{F^3}(b_i F_{y^j} + b_j F_{y^j}) \\ &+ \frac{6(F+\beta)}{F^2} b_i b_j + \frac{6\beta(\beta F + \beta^2)}{F^4} F_{y^i} F_{y^j}\} + \{\frac{(F+\beta)^2(F-2\beta)F_{y^i}}{F^3} \\ &+ \frac{3(F+\beta)^2 b_i}{F^2}\}\{\frac{(F+\beta)^2(F-2\beta)F_{y^j}}{F^3} + \frac{3(F+\beta)^2 b_j}{F^2}\}. \end{aligned} \tag{3.4}$$

After simplifying, we get

$$\begin{aligned} g^{\star}_{ij} =& (\frac{F^6 - 3\beta^6 - 11\beta^3F^3 - 12\beta^5F - 11\beta^4F^2 + 3\beta F^5}{F^6})g_{ij} + \\ & (\frac{48F\beta^5 - 3\beta F^5 + 55\beta^3F^3 + 13\beta^6 - 10\beta^4F^2}{F^6})F_{y^i}F_{y^j} + \\ & (\frac{3F^5 - 12\beta^5 - 33\beta^4F - 30\beta^2F^3 - 64\beta^3F^3}{F^5})(b_i F_{y^j} + b_j F_{y^i}) \\ & + (\frac{15F^4 + 15\beta^4 + 90\beta^2F^2 + 60\beta F^3 + 30\beta^3 F}{F^4})b_i b_j, \end{aligned} \tag{3.5}$$

Hence, we have

**Theorem 3.1.** *Let $(M, F^{\star})$ be an $n$-dimensional Finsler space with $F^{\star} = (F+\beta)^3/F^2$ as a cubic change of $F$, then its fundamental metric tensor is given by (3.5).*

From equation (3.5), we get

$$2C^{\star}_{ijk} = (\frac{F^6 - 3\beta^6 - 11\beta^3F^3 - 12\beta^5F - 11\beta^4F^2 + 3\beta F^5}{F^6})(2C_{ijk}) + (h_{ij} + \frac{y_iy_j}{F^2})$$
$$[(\frac{18\beta^6 - 5F^6 + 33F^3\beta^3 + 60\beta^5F - 3F^5\beta + 44\beta^4F^2}{F^8})y_k + (\frac{3F^6 - 18\beta^5F}{F^7})b_k$$
$$-(\frac{33\beta^2F^4 + 60\beta^4F^2 + 44F\beta^3}{F^7})b_k] + [(\frac{-240F\beta^5 + 3F^5\beta - 165\beta^3F^3}{F^8})y_k -$$
$$(\frac{78\beta^6 - 40F^2\beta^4}{F^8})y_k + (\frac{240\beta^4F - 3F^6 + 165\beta^2F^4 + 78\beta^5F - 40\beta^3F^3}{F^7})b_k]\frac{y_iy_j}{F^2} +$$
$$(\frac{48F\beta^5 - 3\beta F^5 + 55\beta^3F^3 + 13\beta^6 - 10\beta^4F^2}{F^5})(\frac{y_ih_{jk} + y_jh_{ik}}{F^2}) + [(\frac{60\beta^2}{F^4})y_k +$$
$$(\frac{132\beta^4F + 60\beta^5 + 128\beta^3F^3}{F^7})y_k - (\frac{60\beta^4 + 132\beta^3F^2 + 60\beta F^4 + 192\beta^2F^3}{F^6})b_k]$$
$$(\frac{b_iy_j + b_jy_i}{F}) + (\frac{3F^5 - 12\beta^5 - 33\beta^4 - 30\beta^2F^3 - 64\beta^3F^2}{F^5})(\frac{b_ih_{jk} + b_jh_{ik}}{F}) +$$
$$[(\frac{30\beta^3F - 180F^2\beta^2 - 6F^3\beta - 60\beta^4}{F^6})y_k + (\frac{60\beta^3F - 180\beta F^3 + 60F^4 + 90\beta^2F^2}{F^5})b_k]b_ib_j. \tag{3.6}$$

After simplification, we get

$$2C^{\star}_{ijk} = (\frac{2(F^6 - 3\beta^6 - 11\beta^3F^3 - 12\beta^5F - 11\beta^4F^2 + 3\beta F^5)}{F^6})(C_{ijk}) + h_{ij}$$
$$[(\frac{18\beta^6 - 5F^6 + 33F^3\beta^3 + 60\beta^5F - 3F^5\beta + 44\beta^4F^2}{F^8})y_k + (\frac{3F^6 - 18\beta^5F}{F^7})b_k] -$$
$$h_{ij}[\frac{33\beta^2F^4 + 60\beta^4F^2 + 44F\beta^3}{F^7})b_k] + h_{jk}[(\frac{3F^5 - 12\beta^5 - 33\beta^4F - 30\beta^2F^3 - 64\beta^3F^2}{F^6})$$
$$y_i + (\frac{48F\beta^5 - 3\beta F^5 + 55\beta^3F^3 + 13\beta^6 - 10\beta^4F^2}{F^7})b_i] + h_{ik}[(\frac{3F^5 - 12\beta^5 - 33\beta^4F}{F^6})b_j -$$
$$(\frac{30\beta^2F^3 - 64\beta^3F^2}{F^6})b_j + (\frac{48F\beta^5 - 3\beta F^5 + 55\beta^3F^3 + 13\beta^6 - 10\beta^4F^2}{F^6})y_j] - \frac{y_iy_jy_k}{F^2}$$
$$(\frac{18\beta^6 - 5F^6 + 33F^3\beta^3 + 60\beta^5F - 3F^5\beta}{F^8}) + \frac{y_iy_jy_k}{F^2}(\frac{44\beta^4F - 240\beta^5 - 165\beta^3F^2}{F^7}) -$$
$$(\frac{78\beta^6 - 40F^2\beta^4}{F^8}) + (\frac{60\beta^2F^3 + 132\beta^4F + 60\beta^5 + 128\beta^3F^3}{F^8})(b_iy_jy_k + b_jy_ky_i + b_ky_iy_j) +$$
$$(\frac{60\beta^3F - 180\beta F^3 + 60F^4 + 90\beta^2F^2}{F^5})b_ib_jb_k - (\frac{60\beta^4 + 132\beta^3F^2 + 60\beta F^4 + 192\beta^2F^3}{F^6})$$
$$(b_ib_jy_k + b_jb_ky_i + b_ib_ky_j). \tag{3.7}$$

Hence, we have

**Theorem 3.2.** *Let $(M, F^{\star})$ be an $n$-dimensional Finsler space with $F^{\star} = (F+\beta)^3/F^2$ as a cubic change of $F$, then its Cartan's tensor is given by (3.7).*

## 4. Fundamental metric tensor and Cartan's tensor for quartic $\beta$-change Finsler metric

The quartic $\beta$-change of Finsler metric $F$ is

$$F^\star = \frac{(F+\beta)^4}{F^3}, \tag{4.1}$$

Differentiating (4.1) with respect to $y^i$, we have

$$F^\star_{y^i} = \frac{(F+\beta)^2(F-2\beta)F_{y^i}}{F^3} + \frac{3(F+\beta)^2 b_i}{F^2}, \tag{4.2}$$

Again differentiating (4.2) with respect to $y^j$, we have

$$\begin{aligned} F^\star_{y^i y^j} &= \Big(\frac{F^3 - 3F\beta^2 - 3\beta^3}{F^3}\Big)F_{y^i y^j} - \frac{6\beta(\beta+F)}{F^3}(b_i F_{y^j} + b_j F_{y^j}) + \\ &\frac{6(F+\beta)}{F^2} b_i b_j + \frac{6\beta(\beta F + \beta^2)}{F^4} F_{y^i} F_{y^j}. \end{aligned} \tag{4.3}$$

Now,

$$\begin{aligned} g^\star_{ij} &= \{\frac{(F+\beta)^3}{F^2}\}\{(\frac{F^3 - 3F\beta^2 - 3\beta^3}{F^3})F_{y^i y^j} - \frac{6\beta(\beta+F)}{F^3}(b_i F_{y^j} + \\ & b_j F_{y^j}) + \frac{6(F+\beta)}{F^2} b_i b_j + \frac{6\beta(\beta F+\beta^2)}{F^4} F_{y^i} F_{y^j}\} + \{\frac{(F+\beta)^2(F-2\beta)F_{y^i}}{F^3} + \\ & \frac{3(F+\beta)^2 b_i}{F^2}\}\{\frac{(F+\beta)^2(F-2\beta)F_{y^j}}{F^3} + \frac{3(F+\beta)^2 b_j}{F^2}\}. \end{aligned} \tag{4.4}$$

After simplifying, we get

$$\begin{aligned} g^\star_{ij} &= \Big(\frac{F^6 - 3\beta^6 - 11\beta^3F^3 - 12\beta^5F - 11\beta^4F^2 + 3\beta F^5}{F^6}\Big)g_{ij} + \Big(\frac{48F\beta^5 - 3\beta F^5}{F^6}\Big)F_{y^i}F_{y^j} \\ &+ \frac{55\beta^3F^3 + 13\beta^6 - 10\beta^4F^2}{F^6}\Big)F_{y^i}F_{y^j} + \Big(\frac{3F^5 - 12\beta^5 - 33\beta^4F - 30\beta^2F^3 - 64\beta^3F^3}{F^5}\Big) \\ &(b_iF_{y^j} + b_jF_{y^i}) + \Big(\frac{15F^4 + 15\beta^4 + 90\beta^2F^2 + 60\beta F^3 + 30\beta^3F}{F^4}\Big)b_ib_j. \end{aligned} \tag{4.5}$$

Hence, we have

**Theorem 4.1.** *Let $(M, F^\star)$ be an $n$-dimensional Finsler space with $F^\star = (F+\beta)^4/F^3$ as a quartic change of $F$, then its fundamental metric tensor is given by (4.5).*

Next, we find Cartan's tensor for quartic $\beta$-change of Finsler metric $F$. From equation (4.5), we get

$$\begin{aligned}
2C^{\star}_{ijk} &= \frac{2(F^7-28\beta^3F^4-67\beta^4F^3-38\beta^6-8\beta^7-72\beta^5F^2-4\beta F^6)}{F^7}(C_{ijk})\\
&+ h_{ij}[(\frac{84F^4\beta^3+268\beta^4F^3+228\beta^6F+360\beta^5F^2+56\beta^7+4\beta F^6}{F^9})y_k\\
&+ (\frac{84\beta^2F^5+268\beta^3F^4+228\beta^5F^2+56\beta^6F+360\beta F^6+360\beta^4F^3+4F^7}{F^8})b_k]\\
&+ h_{jk}[(\frac{4F^7-12\beta^7-272\beta^3F^4-84\beta^2F^5-276\beta^5F^2-384\beta^4F^3-96\beta^6F}{F^7})b_i\\
&+ (\frac{4\beta F^7-48\beta^2F^6-104\beta^3F^5-5\beta^4F^4+336\beta^5F^3+206\beta^6F^2+80\beta^7F+12\beta^8}{F^8})y_i]\\
&+ h_{ik}[(\frac{4F^7-12\beta^7-272\beta^3F^4-84\beta^2F^5-276\beta^5F^2-384\beta^4F^3-96\beta^6F}{F^7})b_j\\
&+ (\frac{4\beta F^7-48\beta^2F^6-104\beta^3F^5-5\beta^4F^4+336\beta^5F^3+206\beta^6F^2+80\beta^7F+12\beta^8}{F^8})y_j]\\
&- \frac{y_iy_jy_k}{F^2}(\frac{396\beta^3F^5+288\beta^4F^4-1008\beta^6F^2-1320\beta^5F^3-504\beta^7F-96\beta^8+96F^6\beta^2}{F^{10}})\\
&+ (\frac{816\beta^3F^4+84\beta^7+168F^5\beta^2+1380F^2\beta^5+1536\beta^4F^3+576\beta^6F}{F^9})(b_iy_jy_k+b_jy_ky_i\\
&+ b_ky_iy_j)+(\frac{168\beta^5F+1680\beta^3F^3+840\beta F^5+168F^6+1680\beta^2F^4+840F^2\beta^4}{F^7})b_ib_jb_k\\
&- (\frac{84\beta^6F+816\beta^2F^5-168\beta F^6+1380\beta^4F^3+1536\beta^3F^4+576\beta^5F^2}{F^9})(b_ib_jy_k\\
&+ b_jb_ky_i+b_ib_ky_j).
\end{aligned} \tag{4.6}$$

Hence, we have the following

**Theorem 4.2.** *Let $(M,F^{\star})$ be an $n$-dimensional Finsler space with $F^{\star}=\frac{(F+\beta)^4}{F^3}$ as a quartic change of $F$, then its Cartan's tensor is given by (4.6).*

## 5. **Projective flatness of square, cubic and quartic Finsler metrics**

In [4], Hamel found the necessary and sufficient conditions in order that a space satisfying a system of axioms, be projectively flat. After Hamel, many researchers such as [2], [10] ,[11], [14] and [18] have worked on this topic. Here we find necessary and sufficient conditions for $\beta$-change of $(\alpha,\beta)$-metrics, namely square change, cubic change and quartic change Finsler metrics to be projectively flat. For this we have used some definitions and results related to be projectively flatness.

5.1. **Definition.** : Two Finsler metric $F$ and $F^{\star}$ on a manifold $M$ are called projectively equivalent if they have same geodesics as point sets [7].

**Theorem 5.1.** *Let $F$ and $F^\star$ be two Finsler metrics on a manifold $M$. Then $F$ is projectively equivalent to $F^\star$ if and only if*

$$F_{x^iy^j}y^i - F_{x^j} = 0.$$

*Here, the spray's coefficients of both the metrics are related by $G^i = G^{\star i} + Py^i$, where $P = \frac{F_{x^k}y^k}{2F}$ is called projective factor of $F$ [3], [4].*

5.2. **Definition.** : For a Finsler metric $F$ on an open subset $\mathcal{U}$ of $\Re^n$, the geodesic of $F$ are straight lines if and only if the spray's coefficients satisfy

$$G^i = Py^i,$$

where $P$ is same as defined in theorem (5.1) [7].

5.3. **Definition.** : A Finsler metric $F$ on an open subset $\mathcal{U}$ of $\Re^n$ is called projectively flat iff all the geodesics are straight lines in $\mathcal{U}$, and a Finsler metric $F$ on a manifold $M$ is called locally projectively flat, if at any point, there is a local co-ordinate system $(x^i)$ in which $F$ is projectively flat [3]. Therefore, by theorem 5.1 and definition 5.3, we have

**Theorem 5.2.** *A Finsler metric $F$ on an open subset $\mathcal{U}$ of $\Re^n$ is projectively flat iff it satisfies the following system of differential equations [7]*

$$F_{x^iy^j}y^i - F_{x^j} = 0. \tag{5.1}$$

Now, we find necessary and sufficient conditions for $F^\star$, square change of square root of Finsler metric $F$, i.e.,

$$F^\star = \frac{(F+\beta)^2}{F}$$

to be projectively flat. Let us put $F^2 = \mathcal{A}$ in $F^\star$. Then

$$F^\star = \mathcal{A}^{\frac{1}{2}} + \frac{\beta^2}{\mathcal{A}^{\frac{1}{2}}} + 2\beta. \tag{5.2}$$

Differentiating (5.2) w.r.t $x^i$, we get

$$F^\star_{x^i} = \frac{1}{2}\mathcal{A}^{\frac{-1}{2}}\mathcal{A}_{x^i} + 2\beta\mathcal{A}^{\frac{-1}{2}}\beta_{x^i} - \frac{1}{2}\beta^2\mathcal{A}^{\frac{-3}{2}}\mathcal{A}_{x^i} + 2\beta_{x^i}. \tag{5.3}$$

Again differentiating (5.3) w.r.t $y^j$, we get

$$\begin{aligned} F^\star_{x^iy^j} &= \frac{1}{2}\mathcal{A}^{\frac{-1}{2}}\mathcal{A}_{x^iy^j} - \frac{1}{4}\mathcal{A}^{\frac{-3}{2}}\mathcal{A}_{x^i}\mathcal{A}_j + 2\beta\mathcal{A}^{\frac{-1}{2}}\beta_{x^iy^j} + 2\beta_j\mathcal{A}^{\frac{-1}{2}}\beta_{x^i} - \\ &\beta\mathcal{A}^{\frac{-3}{2}}\mathcal{A}_j\beta_{x^i} - \frac{1}{2}\beta^2\mathcal{A}^{\frac{-3}{2}}\mathcal{A}_{x^iy^j} - \beta\beta_j\mathcal{A}^{\frac{-1}{2}}\mathcal{A}_{x^i} + \frac{3}{4}\beta^2\mathcal{A}^{\frac{-5}{2}}\mathcal{A}_j\mathcal{A}_{x^i} + 2\beta_{x^iy^j}. \end{aligned} \tag{5.4}$$

Contracting (5.4) with $y^i$, we get

$$F^\star_{x^i y^j} y^i = \frac{1}{4}\mathcal{A}^{\frac{-5}{2}}[2\mathcal{A}^2\mathcal{A}_{0j} - \mathcal{A}\mathcal{A}_j\mathcal{A}_0 + 8\beta\mathcal{A}^2\beta_{0j} + 8\beta_j\mathcal{A}^2\beta_0 - 4\beta\mathcal{A}\mathcal{A}_j\beta_0$$
$$-2\beta^2\mathcal{A}\mathcal{A}_{0j} - 4\beta\beta_j\mathcal{A}\mathcal{A}_0 + 3\beta^2\mathcal{A}_j\mathcal{A}_0 + 8\mathcal{A}^{\frac{5}{2}}\beta_{0j}].$$

From equation (5.3), we get

$$F^\star_{x^j} = \frac{1}{4}\mathcal{A}^{\frac{-5}{2}}[2\mathcal{A}^2\mathcal{A}_{x^j} + 8\beta\mathcal{A}^2\beta_{x^j} - 2\beta^2\mathcal{A}\mathcal{A}_{x^j} + 8\mathcal{A}^{\frac{5}{2}}\beta_{x^j}].$$

Substituting the values of $F^\star_{x^i y^j} y^i$ and $F^\star_{x^j}$ in equation (5.1) and simplifying, we get

$$\begin{aligned}&3\beta^2\mathcal{A}_j\mathcal{A}_0 + 2\mathcal{A}\beta^2(\mathcal{A}_{x^j} - \mathcal{A}_{0j}) - 2\beta(\mathcal{A}_j\beta_0 + \beta_j\mathcal{A}_0) - \mathcal{A}_j\mathcal{A}_0 + 2\mathcal{A}^2\mathcal{A}_{0j}\\ &- \mathcal{A}_{x^j} + 4\beta_j\beta_0 + 4\beta(\beta_{0j} - \beta_{x^j}) + 8A^{\frac{5}{2}}\{\beta_{0j} - \beta_{x^j}\} = 0.\end{aligned} \tag{5.5}$$

From the above equation, we conclude that $F^\star$ is projectively flat if and only if following equations are satisfied

$$\mathcal{A}_j\mathcal{A}_0 = 0. \tag{5.6}$$

$$\beta_{0j} = \beta_{x^j}. \tag{5.7}$$

$$\beta^2(\mathcal{A}_{x^j} - \mathcal{A}_{0j}) - 2\beta(\mathcal{A}_j\beta_0 + \beta_j\mathcal{A}_0) = 0. \tag{5.8}$$

$$\mathcal{A}_{0j} - \mathcal{A}_{x^j} + 4\beta_j\beta_0 = 0. \tag{5.9}$$

Hence, we have the following

**Theorem 5.3.** *Let $(M, F^\star)$ be an n-dimensional Finsler space with $F^\star = \frac{(F+\beta)^2}{F}$ as a square change of square root Finsler metric $F = \sqrt{\mathcal{A}}$, then $F$ is projectively flat if and only if the equations (5.6), (5.7), (5.8) and (5.9) are satisfied.*

Further, we shall find necessary and sufficient conditions for $F^\star$, cubic change of square root of Finsler metric $F$, i.e.,

$$F^\star = \frac{(F+\beta)^3}{F^2}.$$

to be projectively flat. Let us put $F^2 = \mathcal{A}$ in $F^\star$. Then

$$F^\star = \mathcal{A}^{\frac{1}{2}} + \frac{\beta^3}{\mathcal{A}} + 3\beta^2\mathcal{A}^{\frac{-1}{2}} + 3\beta. \tag{5.10}$$

Differentiating (5.10) w.r.t $x^i$, we get

$$F^\star_{x^i} = \frac{1}{2}\mathcal{A}^{\frac{-1}{2}}\mathcal{A}_{x^i} + 3\beta^2\mathcal{A}^{-1}\beta_{x^i} - \beta^3\mathcal{A}^{-2}\mathcal{A}_{x^i} + 6\beta\mathcal{A}^{\frac{-1}{2}}\beta_{x^i} - \frac{3}{2}\beta^2\mathcal{A}^{\frac{-3}{2}}\mathcal{A}_{x^i} + 3\beta_{x^i}. \tag{5.11}$$

Again differentiating (5.11) w.r.t $y^j$, we get

$$\begin{aligned}F^{\star}_{x^iy^j} &= \frac{1}{2}\mathcal{A}^{\frac{-1}{2}}\mathcal{A}_{x^iy^j} - \frac{1}{4}\mathcal{A}^{\frac{-3}{2}}\mathcal{A}_{x^i}\mathcal{A}_j + 6\beta\beta_j\mathcal{A}^{-1}\beta_{x^i} - 3\beta^2\mathcal{A}^{-2}\beta_{x^i}\mathcal{A}_j\\
&+ 3\beta^2\mathcal{A}^{-1}\beta_{x^iy^j} - 3\beta^2\beta_j\mathcal{A}^{-2}\mathcal{A}_{x^i} + 2\beta^3\mathcal{A}^{-3}\mathcal{A}_{x^i}\mathcal{A}_j - \beta^3\mathcal{A}^{-2}\mathcal{A}_{x^iy^j}\\
&+ 6\beta_j\mathcal{A}^{\frac{-1}{2}}\beta_{x^i} - 3\beta\mathcal{A}^{\frac{-3}{2}}\mathcal{A}_j\beta_{x^i} + 6\beta\mathcal{A}^{\frac{-1}{2}}beta_{x^iy^j}\\
&- 3\beta\beta_j\mathcal{A}^{\frac{-3}{2}}\mathcal{A}_{x^i} + \frac{9}{4}\beta^2\mathcal{A}^{\frac{-5}{2}}\mathcal{A}_{x^i}\mathcal{A}_j - \frac{3}{2}\beta^2\mathcal{A}^{\frac{-3}{2}}\mathcal{A}_{x^iy^j} + 3\beta_{x^iy^j}.\end{aligned} \tag{5.12}$$

Contracting the above equation with $y^i$, we get

$$\begin{aligned}F^{\star}_{x^iy^j}y^i = \frac{1}{4}\mathcal{A}^{\frac{-5}{2}}\Big[&4\mathcal{A}^2\mathcal{A}_{0j} - \mathcal{A}\mathcal{A}_j\mathcal{A}_0 + 24\beta\beta_j\beta_0\mathcal{A}^{\frac{3}{2}} - 12\beta^2\mathcal{A}^{\frac{1}{2}}\beta_0\mathcal{A}_j\\
&+12\beta^2\beta_{0j}\mathcal{A}^{\frac{3}{2}} - 12\beta^2\beta_j\mathcal{A}_0\mathcal{A}^{\frac{1}{2}} + 8\beta^3\mathcal{A}_j\mathcal{A}_0\mathcal{A}^{\frac{-1}{2}}\\
&-4\beta^3\mathcal{A}_{0j}\mathcal{A}^{\frac{1}{2}} + 24\beta_j\beta_0\mathcal{A}^2 - 12\beta\mathcal{A}_j\beta_0\mathcal{A} + 24\beta_{0j}\mathcal{A}^2\\
&-12\beta_j\mathcal{A}_0\mathcal{A} + 9\beta^2\mathcal{A}_0\mathcal{A}_j - 6\beta^2\mathcal{A}\mathcal{A}_{0j} + 12\mathcal{A}^{\frac{5}{2}}\beta_{0j}\Big].\end{aligned}$$

From equation (5.11), we get

$$\begin{aligned}F^{\star}_{x^j} = \frac{1}{4}\mathcal{A}^{\frac{-5}{2}}\Big[&2\mathcal{A}^2\mathcal{A}_{x^j} + 12\beta^2\mathcal{A}^{\frac{3}{2}}\beta_{x^j} - 4\beta^3\mathcal{A}_{x^j}\mathcal{A}^{\frac{1}{2}} + 24\beta\mathcal{A}^2\beta_{x^j}\\
&-6\beta^2\mathcal{A}\mathcal{A}_{x^j} + 12\mathcal{A}^{\frac{5}{2}}\beta_{x^j}\Big].\end{aligned}$$

Substituting the values of $F^{\star}_{x^iy^j}y^i$ and $F^{\star}_{x^j}$ in equation (5.1) and simplifying, we get

$$\begin{aligned}&9\beta^2\mathcal{A}_j\mathcal{A}_0 - 6\mathcal{A}\beta^2(\mathcal{A}_{x^j} + \mathcal{A}_{0j}) - 6\beta(\mathcal{A}_j\beta_0 + \beta_j\mathcal{A}_0) - \mathcal{A}_j\mathcal{A}_0 + 4\mathcal{A}^2\mathcal{A}_{0j} - \mathcal{A}_{x^j}\\
&+ 12\beta_j\beta_0 + 12\beta(\beta_{0j} - \beta_{x^j}) + 12\mathcal{A}^{\frac{5}{2}}(\beta_{0j} - \beta_{x^j}) + 2\mathcal{A}^{\frac{3}{2}}\beta(12\beta_j\beta_0)\\
&+ 6\beta^2(\beta_{0j} - \beta_{x^j}) - 2\mathcal{A}^{\frac{1}{2}}6\beta^2(\beta_0\mathcal{A}_j + \beta_j\mathcal{A}_0) + 2\beta^3(\mathcal{A}_{0j} + \mathcal{A}_{x^j}) = 0.\end{aligned} \tag{5.13}$$

From the above equation, we conclude that $F^{\star}$ is projectively flat if and only if following equations are satisfied:

$$\mathcal{A}_j\mathcal{A}_0 = 0. \tag{5.14}$$

$$\beta_{0j} = \beta_{x^j}. \tag{5.15}$$

$$\beta\beta_j\beta_0 = 0. \tag{5.16}$$

$$3\beta^2(\mathcal{A}_{x^j} + \mathcal{A}_{0j}) - 6\beta(\mathcal{A}_j\beta_0 + \beta_j\mathcal{A}_0) = 0. \tag{5.17}$$

$$2\mathcal{A}_{0j} - \mathcal{A}_{x^j} + 12\beta(\beta_{0j} + \beta_{x^j}) = 0. \tag{5.18}$$

$$6\beta^2(\beta_0\mathcal{A}_j + \beta_j\mathcal{A}_0) + 2\beta^3(\mathcal{A}_{oj} + \mathcal{A}_{x^j}) = 0. \tag{5.19}$$

Hence, we have the following

**Theorem 5.4.** *Let $(M, F^\star)$ be an $n$-dimensional Finsler space with $F^\star = \frac{(F+\beta)^3}{F^2}$ as a cubic change of square root Finsler metric $F = \sqrt{\mathcal{A}}$, then $F$ is projectively flat if and only if the equations (5.14), (5.15), (5.16), (5.17), (5.18) and (5.19) are satisfied.*

Next, we find necessary and sufficient conditions for$F^\star$, quartic change of square root of Finsler metric $F$, i.e.,

$$F^\star = \frac{(F+\beta)^4}{F^3}$$

to be projectively flat. Let us put $F^2 = \mathcal{A}$ in $F^\star$. Then

$$F^\star = \mathcal{A}^{\frac{1}{2}} + \beta^4\mathcal{A}^{\frac{-3}{2}} + 6\beta^2\mathcal{A}^{\frac{-1}{2}} + 4\beta^3\mathcal{A}^{-1} + 4\beta, \tag{5.20}$$

Differentiating (5.20) w.r.t $x^i$, we get

$$\begin{aligned}F^\star_{x^i} &= \frac{1}{2}\mathcal{A}^{\frac{-1}{2}}\mathcal{A}_{x^i} + 4\beta^3\mathcal{A}^{\frac{-3}{2}}\beta_{x^i} - \frac{3}{2}\beta^4\mathcal{A}^{\frac{-5}{2}}\mathcal{A}_{x^i} + 12\beta\mathcal{A}^{\frac{-1}{2}}\beta_{x^i} - 3\beta^2\mathcal{A}^{\frac{-3}{2}}\mathcal{A}_{x^i}\\ &+ 12\beta^2\mathcal{A}^{-1}\beta_{x^i} + 4\beta_{x^i} - 4\beta^3\mathcal{A}^{-2}\mathcal{A}_{x^i}.\end{aligned} \tag{5.21}$$

Again differentiating (5.21) w.r.t $y^j$, we get

$$\begin{aligned}F^\star_{x^iy^j} &= \frac{1}{2}\mathcal{A}^{\frac{-1}{2}}\mathcal{A}_{x^iy^j} - \frac{1}{4}\mathcal{A}^{\frac{-3}{2}}\mathcal{A}_{x^i}\mathcal{A}_j + 12\beta^2\mathcal{A}^{\frac{-3}{2}}\beta_j\beta_{x^i} + 4\beta^3\mathcal{A}^{\frac{-3}{2}}\beta_{x^iy^j}\\ &- 6\beta^3\mathcal{A}^{\frac{-5}{2}}\beta_{x^i}\mathcal{A}_j + \frac{15}{4}\mathcal{A}^{\frac{-7}{2}}\beta^4\mathcal{A}_j\mathcal{A}_{x^i} - \frac{3}{2}\mathcal{A}^{\frac{-5}{2}}\beta^4\mathcal{A}_{x^iy^j} -\\ &6\beta^3\beta_j\mathcal{A}^{\frac{-5}{2}}\mathcal{A}_{x^i} + 12\beta_j\mathcal{A}^{\frac{-1}{2}}\beta_{x^i} - 6\beta\mathcal{A}^{\frac{-3}{2}}\mathcal{A}_j\beta_{x^i} + 12\beta\mathcal{A}^{\frac{-1}{2}}\beta_{x^iy^j} - 6\beta\beta_j\mathcal{A}_{x^i}\mathcal{A}^{\frac{-3}{2}}\\ &+ \frac{9}{2}\beta^2\mathcal{A}^{\frac{-5}{2}}\mathcal{A}_j\mathcal{A}_{x^i} - 3\beta^2\mathcal{A}^{\frac{-3}{2}}\mathcal{A}_{x^iy^j} + 12\beta^2\mathcal{A}^{-1}\beta_{x^iy^j} + 24\beta\mathcal{A}^{-1}\beta_j\mathcal{A}_{x^i} - 12\beta^2\mathcal{A}^{-2}\mathcal{A}_j\beta_{x^i}\\ &- 12\beta^2\beta_j\mathcal{A}^{-2}\mathcal{A}_{x^i} + 8\beta^3\mathcal{A}^{-3}\mathcal{A}_{x^i}\mathcal{A}_j - 4\beta^3\mathcal{A}^{-2}\mathcal{A}_{x^iy^j} + 4\beta_{x^iy^j},\end{aligned} \tag{5.22}$$

Contracting the above equation with $y^i$, we get

$$\begin{aligned}F^\star_{x^iy^j}y^i = \frac{1}{4}\mathcal{A}^{\frac{-7}{2}}\Big[&2\mathcal{A}^3\mathcal{A}_{0j} - \mathcal{A}^2\mathcal{A}_j\mathcal{A}_0 + 48\beta^2\mathcal{A}^2\beta_j\beta_0 + 16\beta^3\mathcal{A}^2\beta_{0j}\\ &-24\beta^3\mathcal{A}\mathcal{A}_j\beta_0 + 15\beta^4\mathcal{A}_j\mathcal{A}_0 - 6\mathcal{A}\beta^4\mathcal{A}_{0j} - 24\beta^3\mathcal{A}_j\beta_0\\ &+48\beta_j\beta_0\mathcal{A}^3 - 24\beta\mathcal{A}_j\beta_0\mathcal{A}^2 + 48\beta\mathcal{A}^3\beta_{0j} - 24\beta\beta_j\mathcal{A}^2\mathcal{A}_0\\ &+18\beta^2\mathcal{A}\mathcal{A}_0\mathcal{A}_j - 12\beta^2\mathcal{A}^2\mathcal{A}_{0j} + 48\beta^2\mathcal{A}^{\frac{5}{2}}\beta_{0j} + 96\beta\beta_j\mathcal{A}^{\frac{5}{2}}\beta_0\\ &-48\beta^2\mathcal{A}^{\frac{3}{2}}\mathcal{A}_j\beta_0 - 48\beta^2\beta_j\mathcal{A}^{\frac{3}{2}}\mathcal{A}_0 + 32\beta^3\mathcal{A}^{\frac{1}{2}}\mathcal{A}_j\mathcal{A}_0 - 16\beta^3\mathcal{A}^{\frac{3}{2}}\mathcal{A}_{0j} + 16\beta_{0j}\Big].\end{aligned}$$

From equation (5.21), we get

$$F^{\star}_{x^j} = \frac{1}{4}\mathcal{A}^{\frac{-7}{2}}[2\mathcal{A}^3\mathcal{A}_{x^j} + 16\beta^3\mathcal{A}^2\beta_{x^j} - 6\mathcal{A}\beta^4\mathcal{A}_{x^j} + 48\beta\mathcal{A}^3\beta_{x^j} - 12\beta^2\mathcal{A}^2\mathcal{A}_{x^j}$$
$$+48\beta^2\mathcal{A}^{\frac{5}{2}}\beta_{x^j} - 16\beta^3\mathcal{A}^{\frac{3}{2}}\mathcal{A}_{x^j} + 16\beta_{x^j}\mathcal{A}^{\frac{7}{2}}].$$

Substituting the values of $F^{\star}_{x^i y^j}y^i$ and $F^{\star}_{x^j}$ in equation (5.1) and simplifying, we get

$$15\beta^4\mathcal{A}_j\mathcal{A}_0 - 2\mathcal{A}\Big\{3\beta^4(\mathcal{A}_{0j} - \mathcal{A}_{x^j}) + 12\beta^3(\mathcal{A}_j\beta_0 + \beta_j\mathcal{A}_0)$$
$$+\beta^2(\mathcal{A}_j\mathcal{A}_0 - 6\mathcal{A}_{0j})\Big\} + 2\mathcal{A}^2\Big\{6\beta^2(4\beta_0\beta_j - \mathcal{A}_{0j} + \mathcal{A}_{x^j}) - \mathcal{A}_j\mathcal{A}_0$$
$$+8\beta^3(\beta_{0j} - \beta_{x^j}) - 12\beta(\mathcal{A}_j\beta_0 - \beta_j\mathcal{A}_0)\Big\} + 2\mathcal{A}^3\Big\{24\beta_j\beta_0$$
$$+24\beta(\beta_{0j} - \beta_{x^i}) - \mathcal{A}_{x^j} + \mathcal{A}_{0j}\Big\} + 2\mathcal{A}^{\frac{5}{2}}\Big\{24\beta^2(\beta_{0j}$$
$$-\beta_{x^i}) + 48\beta(\beta_j\beta_0)\Big\} + 2\mathcal{A}^{\frac{3}{2}}\Big\{24\beta^2(\mathcal{A}_j\beta_0 + \beta_j\mathcal{A}_0$$
$$+8\beta^3(\mathcal{A}_{0j} - \mathcal{A}_{x^j})\Big\} + 16\mathcal{A}^{\frac{7}{2}}\{\beta_{0j} - \beta_{x^j}\} + 2\mathcal{A}^{\frac{1}{2}}\{16\beta^3\mathcal{A}_j\mathcal{A}_0\} = 0.$$

From the above equation, we conclude that $F^{\star}$ is projectively flat if and only if following equations are satisfied:

$$\mathcal{A}_j\mathcal{A}_0 = 0, \tag{5.23}$$

$$\mathcal{A}_{x^j} - \mathcal{A}_{0j} = 0, \tag{5.24}$$

$$\beta_{0j} - \beta_{x^j} = 0, \tag{5.25}$$

$$\beta_j\beta_0 = 0, \tag{5.26}$$

$$12\beta(\mathcal{A}_j\beta_0 - \beta_j\mathcal{A}_0) = 0, \tag{5.27}$$

$$12\beta^3(\mathcal{A}_j\beta_0 + \beta_j\mathcal{A}_0) - 6\beta^2\mathcal{A}_{0j} = 0. \tag{5.28}$$

Hence, we have the following

**Theorem 5.5.** *Let $(M, F^{\star})$ be an $n$-dimensional Finsler space with $F^{\star} = (F+\beta)^4/F^3$ as a quartic change of square root Finsler metric $F = \sqrt{\mathcal{A}}$, then $F$ is projectively flat if and only if the equations (5.23), (5.24), (5.25), (5.26), (5.27) and (5.28) are satisfied.*

## 6. Examples for support to the said metrics

In this section, we have given examples for support to given results. From [7] we have the following example:

6.1. **Example.** Let $M$ be a plane embedded in the Euclidean space $\Re^3$, i.e., $M \to \Re^3$ by

$$x = (x^1, x^2) \to (x^1, x^2, z = f(x^1, x^2)), \tag{6.1}$$

where $f : \Re^2 \to \Re$ is a smooth function given by

$$f(x^1, x^2) = Px^1 + Qx^2 + R.$$

Here, $P, Q$ and $R$ are real constants. $M$ is the graph of $z = f(x^1, x^2)$ with Riemann metric [20] $\alpha = \sqrt{\mathcal{A}} = \sqrt{a_{ij}(x)y^i y^j}$ induced from $\Re^3$ defined as follows

$$(a_{ij}) = \begin{bmatrix} 1 + f^2_{x^1} & f_{x^1} f_{x^2} \\ 1 + f^2_{x^2} & f_{x^1} f_{x^2} \end{bmatrix}$$

Choose a 1-form $\beta = f_{x^1} y^1 + f_{x^2} y^2$ on $M$.
where
$\alpha = \sqrt{(1 + P^2)(y^1)^2 + 2PQy^1y^2 + (1 + Q^2)(y^2)^2}$,
$\beta = Py^1 + Qy^2$
From above, the entities required to check projective flatness of $F^\star$ is given by

$$\begin{aligned} &\mathcal{A} = \alpha^2 = F^2 = (1 + P^2)(y^1)^2 + 2PQy^1y^2 + (1 + Q^2)(y^2)^2, \\ &\beta = Py^1 + Qy^2, \mathcal{A}_{x^j} = 0, \mathcal{A}_0 = \mathcal{A}_{x^i} y^i = 0, \\ &\mathcal{A}_{0j} = \mathcal{A}_{x^i y^j} y^i = 0, \\ &\beta_{x^j} = 0, \beta_0 = \beta_{x^i} y^i = 0, \beta_{0j} = \mathcal{A}_{x^i y^j} y^i = 0, \\ &\mathcal{A}_j = \mathcal{A}_{y^j}, \beta_j = \beta_{y^j}, \beta_1 = \beta_{y^1} = P, \\ &\beta_2 = \beta_{y^2} = Q, \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} \mathcal{A}_1 = \mathcal{A}_{y^i} = 2(1 + Q^2)y^1 + 2PQy^2, \\ \mathcal{A}_2 = \mathcal{A}_{y^2} = 2(1 + Q^2)y^2 + 2PQy^1. \end{aligned} \tag{6.3}$$

Now, take surface $M$, Riemannian metric $\alpha$ and 1-form $\beta$ as given in example (6.1). We obtain square change of square root of Finsler metric $F = \sqrt{\mathcal{A}}$, as follows:

$$F^\star = \frac{(F + \beta)^2}{F} = \frac{(\mathcal{A} + \beta^2 + 2\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}} = \frac{\alpha^2 + \beta^2 + 2\alpha\beta}{\alpha}$$

This is square $(\alpha, \beta)$-metric obtained from Finsler metric $F$, the conditions (5.6), (5.7), (5.8) and (5.9) are clearly satisfied by the equation (6.2) and (6.3). Therefore, the surface $M$ with square change of Finsler square root metric is projectively flat.

Further, we consider surface $M$, Riemannian metric $\alpha$ and 1-form $\beta$ as given in example (6.1). We obtain cubic change of square root of Finsler metric

$F = \sqrt{\mathcal{A}}$, as follows:

$$\begin{aligned} F^{\star} &= \frac{(F+\beta)^3}{F^2} \\ &= \frac{(\mathcal{A}^{\frac{3}{2}} + \beta^3 + 3\mathcal{A}^{\frac{1}{2}}\beta^2 + 3\mathcal{A}\beta)}{\mathcal{A}} \\ &= \frac{((\alpha^2)^{\frac{3}{2}} + \beta^3 + 3(\alpha^2)^{\frac{1}{2}}\beta^2 + 3\alpha^2\beta)}{\alpha^2}. \end{aligned}$$

After solving, we get

$$F^{\star} = \frac{(\alpha+\beta)^3}{\alpha^2}.$$

This is cubic $(\alpha, \beta)$-metric obtained from Finsler metric $F$ and it is seen that conditions (5.14), (5.15), (5.16), (5.17), (5.18) and (5.19) are clearly satisfied by the equation (6.2) and (6.3). Therefore, the surface $M$ with cubic change of Finsler square root metric is projectively flat.

Next, we take surface $M$, Riemannian metric $\alpha$ and 1-form $\beta$ as given in example (6.1). We obtain quartic change of square root of Finsler metric $F = \sqrt{\mathcal{A}}$, as follows:

$$\begin{aligned} F^{\star} = \frac{(F+\beta)^4}{F^3} &= \frac{(F^4 + \beta^4 + 6F^2\beta^2 + 4F\beta^3 + 4F^3\beta)}{F^3} \\ &= \frac{(\mathcal{A}^2 + \beta^4 + 6\mathcal{A}\beta^2 + 4\mathcal{A}^{\frac{1}{2}}\beta^3 + 4\mathcal{A}^{\frac{3}{2}}\beta)}{\mathcal{A}^{\frac{3}{2}}}. \end{aligned}$$

After solving, we get

$$F^{\star} = \frac{(\alpha+\beta)^4}{\alpha^3}.$$

This is quartic $(\alpha, \beta)$-metric obtained from Finsler metric $F$ and it is easily seen that conditions (5.23), (5.24), (5.25), (5.26), (5.27) and (5.28) are clearly satisfied by the equation (6.2) and (6.3). Therefore, the surface $M$ with quartic change of Finsler square root metric is projectively flat.

## References

1. S.I. Amari and H. Nagaoka, *Methods of Information Geometry*, Translation of Mathematical Monographs, AMS, Oxford Univ. Press **191**(2000).
2. S. A. Baby and G. Shankar, *On Randers- conformal change of Finsler space with special $(\alpha, \beta)$- metrics*, Int. J. Cont. Math. Sci. **11**(9)(2016), 415-423.
3. S. S. Chern and Z. Shen, *Riemann Finsler Geometry, Nankai Tracts in Mathematics* World Scientific, Singapore **6**(2005).
4. G. Hamel, *Über die Geometrieen in denen die Gerden die Kurzesten sind*, Math. Ann. **57**(2)(1903), 231-264.
5. M. Matsumoto, *On C-reducible Finsler spaaces*, Tensor, N. S. **24**(1972), 29-37.
6. M. Matsumoto, *On Finsler spaces with Randers metric and special forms of important tensors*, J. Math. Kyoto Univ. **14**(3)(1974), 477-498.
7. Gauree Shankar, Sarita Rani and Kirandeep Kaur, *Projectively flat Finsler spaces with transformed metrics*, Iran. J. Math. Sci. Inform. **18**(2) (2023), 31-50.

8. P.K. Dwivedi, Sachin Kumar and C.K. Mishra, *On projectively flat Finsler space with $n$-power $(\alpha, \beta)$-metric*, J. Finsler Geom. Appl. **5**(2) (2024), 14-24.
9. S. Rani and G. Shankar, *On CW-translations of a homogeneous Finsler space with $(\alpha, \beta)$-metric*, Balkan. J. Geom. Appl. **26**(1) (2021).
10. G. Shankar and S. A. Baby, *On the Riemannian and Ricci curvature of Finsler spaces with special $(\alpha, \beta)$-metric*, Int. J. Math. Comb. **4**(2016).
11. G. Shankar and P. Gupta, *Projectively flat Finsler spaces with special $(\alpha, \beta)$-metric*, J. Int. Acad. Phys. Sci. **17**(4) (2013), 369-376.
12. G. Shankar and S. Rani, *On $S$-curvature of a homogeneous Finsler space with square metric*, Int. J. Geom. Meth. Mod. Phys. **17**(2) (2020), 2050019 (16 pages).
13. G. Shankar, S. Rani and K. Kaur, *Dully flat Finsler spaces associated with Randers-$\beta$-change*, J. Raj. Acad. Phys. Sci. **18**(1-2) (2019), 95-106.
14. G. Shankar, R.K. Sharma and R.D.S. Kushwaha, *On the Matsumoto change of a Finsler space with $M^{th}$-root metric*, Acta. Math. Acad. Paedag. Nyireg. **32**(2) (2017), 255-264.
15. C. Shibata, *On invariant tensor of $\beta$-change of Finsler metrics*, J. Math. Kyoto. Univ. **24**(1) (1984), 163-188.
16. A. Tayebi and E. Payghan, *On $C_3$-like Finsler metrics*, Iran. J. Math. Sci. Inform. **7**(1) (2012), 1-6.
17. B. Tiwari and M. Kumar, *On Randers change of a Finsler space with $m^t h$-root metric*, Int. J. Geom. Meth. Mod. Phys. **11**(10) (2014), 1450087 (13 pages)
18. R. Yadav and G. Shankar, *On some projectively flat $(\alpha, \beta)$-metrices*, Gulf. J. Math. **1**(2013), 72-77.
19. S. Rani and Gauree Shankar, *On $S$-curvatureof a homogeneous Finsler space with Randers change square metric*, Facta. Univ. Ser. Math. Inf. **35**(3) (2020) 673-691.
20. M. Matsumoto, *A slop of mountain is a Finsler surface with respect to a time measure*, J. Math. Kyoto. Univ. **29**(1) (1989), 17-25.

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