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# Finslerian metrics locally conformally $R$-Einstein

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**Abstract.** Let $R$ be the $hh$-curvature associated with the Chern connection or the Cartan connection. In this paper, an intrinsic characterization of $R$-Einstein metrics is given and a theory on Finslerian warped product metrics is developped. Finslerian metrics which are locally conformally $R$-Einstein are classified.

**Keywords:** Einstein metrics, Conformal deformations, Finslerian connections, Warped product metric.

## 1. **Introduction**

Finslerian metrics are of considerable interest due to their rich structure including Riemann, Randers, Minkowski and Berwald type metrics. Some areas in which they have significant impacts are Differential Geometry, Einstein's

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theory of General Relativity and Biology [1, 2]. A natural and important problem is the classification of metrics conformally Einstein. In 1923, Brinkmann obtained in [5] the necessary and sufficient conditions for an $n$-dimensional Riemannian manifold to be conformally Einstein. Later, Szekeres [14] in 1963, Kozameh-Newmann-Tod [6] in 1985, Listing [7] in 2001 as well as Kühnel-Rademacher [12] in 2016 studied this problem from a different point of view, both for (pseudo-)Riemannian metrics. This motivates us to study the above problem for a general Finslerian metric.

In the present paper, we study and characterize Finslerian metrics which are locally conformal to $R$-Einstein metrics. Unfortunately, the specificity of the Finslerian metric and his associated fundamental tensor do not allow us to use the same technics and tools as in the Riemannian case to obtain general classifications of (locally or globally) conformally Finslerian $R$-Einstein metrics. Hence, we exploit the pulled-back bundle approach used in [2] and introduce a globally theory on conformal Finslerian $R$-Einstein geometry. Let $M$ be an $n$-dimensional $C^\infty$ connected manifold and $\mathring{T}M := TM\backslash\{0\}$ its slit tangent bundle. The submersion $\pi : \mathring{T}M \longrightarrow M$ pulls back the tangent bundle $TM$ to a vector bundle $\pi^*TM$ over $\mathring{T}M$. Given a Finslerian metric $F$ on $M$ and $g$ its fundamental tensor, we have introduced in [10], the following tensor. The trace-free horizontal Ricci tensor of a Finslerian manifold $(M, F)$ is the application

$$\mathbf{E}_F^H : \begin{array}{ccc} \Gamma(\pi^*TM)\times\chi(\mathring{T}M) & \to & C^\infty(\mathring{T}M,\mathbb{R}) \\ (\xi, X) & \mapsto & (\mathbf{Ric}_F^H - \frac{1}{n}\mathbf{Scal}_F^H \underline{g})(\xi, X) \end{array}$$

where $\mathbf{Ric}_F^H$ is the horizontal Ricci tensor, $\mathbf{Scal}_F^H$ is the horizontal scalar curvature and $\underline{g} := \pi^*g$ is the pullback of $g$ by the submersion $\pi : \mathring{T}M \longrightarrow M$. One of advantage of the tensor $\mathbf{E}_F^H$, it vanishes when $F$ is an $R$-Einstein metric. Furthermore, it is insensitive to whether we use the Chern connection or the Cartan connection. Our main results in this work are given by the following.

**Proposition 1.1.** *Let $F$ be a Finslerian metric on an $n$-dimensional manifold. $F$ is locally conformal to an $R$-Einstein metric $\widetilde{F}$, with $\widetilde{F} = e^u F$, if and only if the conformal factor $e^u$ is a solution of the equation*

$$\boldsymbol{E}_F(\partial_i, \hat{\partial}_j) - (n-2)\left(\nabla_j\nabla_i u - \nabla_i u\nabla_j u\right) + \frac{n-2}{n}\left(\nabla^d\nabla_d u - \nabla^d u\nabla_d u\right) g_{ij} + \frac{n-1}{2nF}\left(\nabla_r u\nabla^q u\right)\frac{\partial(F^2 g^{rs} - 2y^r y^s)}{\partial y^q} g^{kl}\mathcal{A}_{skl} g_{ij} = 0. \quad (1.1)$$

To determine the solution(s) of the equation (1.1), we consider it as a system of partial differential equations in the conformal factor $e^u$ and curvatures associated with $F$ on a neighborhood of the given manifold. The explicit solution $u$ can tell us how $F$ is constructed. Hence, we prove the following Theorem.

**Theorem 1.2.** *A Finslerian metric $F$ on a 2-dimensional manifold is locally conformally $R$-Einstein if and only if one of the following two cases holds :*

(i) *the conformal factor is constant and $F$ is $R$-Einstein.*
(ii) *$F$ is a Riemannian metric.*

Note that, the warped product of two $R$-Einstein metrics with different horizontal scalar curvatures is not $R$-Einstein. It is studied in [4] the special case where the conformal factor only depends on the base of a warped product Riemannian manifold. Thus we have the Theorem 1.3

**Theorem 1.3.** *Let $F$ be a Finslerian metric on a cylinder $\mathbb{R}\times \overset{2}{M}$ of dimension $n \geq 3$ and $\overset{2}{F}$ a Finslerian metric on $\overset{2}{M}$. Let $u$ be a $C^\infty$ function on $\mathbb{R}\times \overset{2}{M}$ such that $u(t,x) = u(t)$ for every $t \in \mathbb{R}$ and $x \in \overset{2}{M}$. Then $F$ is locally horizontally conformal to an Einstein metric $\widetilde{F}$, with $\widetilde{F} = e^u F$, if and only if the conformal factor $e^u$ and $\overset{2}{F}$ satisfy one of the following cases :*

(i) *$e^{u(t,x)} = \alpha t + \beta$ for some real constants $\alpha, \beta$ and $\overset{2}{F}$ is horizontally Ricci-flat. In particular, if $\alpha = 0$ then $\beta$ must be positive and hence $u$ is constant.*
(ii) *$e^{u(t,x)} = cosh^{-1}\left(\sqrt{\frac{\boldsymbol{Scal}^H_F}{(n-1)(n-2)}}t + \gamma\right)$ for some real constant $\gamma$ and $\overset{2}{F}$ is horizontally Ricci-constant with positive horizontal scalar curvature $\boldsymbol{Scal}^H_F$.*
(iii) *$e^{u(t,x)} = \mu cos^{-1}\left(\sqrt{\frac{-\boldsymbol{Scal}^H_F}{(n-1)(n-2)}}t + \theta\right)$ for some real constants $\mu, \theta$ and $\overset{2}{F}$ is horizontally Ricci-constant with negative horizontal scalar curvature $\boldsymbol{Scal}^H_F$.*

For non-warped product Finslerian metrics, we obtain the Theorem 1.4

**Theorem 1.4.** *If the conformal factor on a 3-dimensional (respectively 4-dimensional) Finslerian manifold is locally conformally $R$-Einstein then the horizontal Cotton-York (respectively the horizontal Bach) tensor vanishes.*

The rest of this paper is organised as follows. In Section 2, we give some basic notions on Finslerian manifolds. The Section 3 is devoted to study the Finslerian $R$-Einstein metrics. In the Section 4, we derive Finslerian locally conformal $R$-Einstein equation. The Theorem 1.2 is proved in the Section 5. An intrinsic theory on Finslerian warped product metrics is developped in the Section 6 and the Theorem 1.3. Finally the Theorem 1.4 is proved in the Section 7.

## 2. **Preliminaries**

Throughout this paper, all manifolds are assumed to be connected and, all manifolds and mappings are supposed to be differentiable of classe $C^\infty$. However, our results presented hold under the differentiability of class $C^4$. Let $M$

be an $n-$dimensional manifold. We denote by $T_xM$ the tangent space at $x \in M$ and by $TM := \bigcup_{x\in M} T_xM$ the tangent bundle of $M$. Set $\mathring{T}M = TM\backslash\{0\}$ and $\pi : TM \longrightarrow M, \pi(x, y) \longmapsto x$ the natural projection. Let $(x^1, ..., x^n)$ be a local coordinate on an open subset $U$ of $M$ and $(x^1, ..., x^n, y^1, ..., y^n)$ be the local coordinate on $\pi^{-1}(U) \subset TM$. The local coordinate system $(x^i)_{i=1,...,n}$ produces the local coordinate bases $\{\frac{\partial}{\partial x^i}\}_{i=1,...,n}$ and $\{dx^i\}_{i=1,...,n}$ respectively, for $TM$ and cotangent bundle $T^*M$. We use Einstein summation convention : repeated upper and lower indices will automatically be summed unless otherwise will be noted.

**Definition 2.1.** *A function $F : TM \longrightarrow [0, \infty)$ is called a Finslerian metric on $M$ if :*

(1) *$F$ is $C^\infty$ on the entire slit tangent bundle $\mathring{T}M$,*

(2) *$F$ is positively 1-homogeneous on the fibers of $TM$, that is* $\forall c > 0,\ F(x, cy) = cF(x, y)$,

(3) *the Hessian matrix $(g_{ij}(x, y))_{1\le i,j\le n}$ with elements*

$$g_{ij}(x, y) := \frac{1}{2}\frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} \tag{2.1}$$

*is positive definite at every point $(x, y)$ of $\mathring{T}M$.*

**Remark 2.2.** *$F(x, y) \neq 0$ for all $x \in M$ and for every $y \in T_xM\backslash\{0\}$.*

Consider the differential map $\pi_*$ of the submersion $\pi : \mathring{T}M \longrightarrow M$. The vertical subspace of $T\mathring{T}M$ is defined by $\mathcal{V} := ker(\pi_*)$ and is locally spanned by the set $\{F\frac{\partial}{\partial y^i}, 1 \le i \le n\}$, on each $\pi^{-1}(U) \subset \mathring{T}M$.

An horizontal subspace $\mathcal{H}$ of $T\mathring{T}M$ is by definition any complementary to $\mathcal{V}$. The bundles $\mathcal{H}$ and $\mathcal{V}$ give a smooth splitting

$$T\mathring{T}M = \mathcal{H} \oplus \mathcal{V}. \tag{2.2}$$

An Ehresmann connection is a selection of a horizontal subspace $\mathcal{H}$ of $T\mathring{T}M$. As explain in [8], $\mathcal{H}$ can be canonically defined from the geodesics equation.

**Definition 2.3.** *Let $\pi : \mathring{T}M \longrightarrow M$ be the restricted projection.*

(1) *An Ehresmann-Finsler connection of $\pi$ is the subbundle $\mathcal{H}$ of $T\mathring{T}M$ given by*

$$\mathcal{H} := ker\theta, \tag{2.3}$$

*where $\theta : T\mathring{T}M \longrightarrow \pi^*TM$ is the bundle morphism defined by*

$$\theta = \frac{\partial}{\partial x^i} \otimes \frac{1}{F}(dy^i + N^i_j dx^j) \tag{2.4}$$

*with $N^i_j(x, y) := \frac{\partial G^i(x,y)}{\partial y^j}$ for*

$$G^i(x, y) := \frac{1}{4}g^{il}(x, y)\Big[\frac{\partial g_{jl}}{\partial x^k}(x, y) + \frac{\partial g_{kl}}{\partial x^j}(x, y) - \frac{\partial g_{jk}}{\partial x^l}(x, y)\Big]y^j y^k. \tag{2.5}$$

(2) *The form* $\theta : T\overset{\circ}{T}M \longrightarrow \pi^*TM$ *induces a linear map*

$$\theta|_{(x,y)} : T_{(x,y)}\overset{\circ}{T}M \longrightarrow T_xM, \tag{2.6}$$

*for each point* $(x,y) \in \overset{\circ}{T}M$*; where* $x = \pi(x,y)$.
*The vertical lift of a section* $\xi$ *of* $\pi^*TM$ *is a unique section* $\boldsymbol{v}(\xi)$ *of* $T\overset{\circ}{T}M$ *such that for every* $(x,y) \in \overset{\circ}{T}M$,

$$\pi_*(\boldsymbol{v}(\xi))|_{(x,y)} = 0_{(x,y)} \text{ and } \theta(\boldsymbol{v}(\xi))|_{(x,y)} = \xi_{(x,y)}. \tag{2.7}$$

(3) *The differential projection* $\pi_* : T\overset{\circ}{T}M \longrightarrow \pi^*TM$ *induces a linear map*

$$\pi_*|_{(x,y)} : T_{(x,y)}\overset{\circ}{T}M \longrightarrow T_xM, \tag{2.8}$$

*for each point* $(x,y) \in \overset{\circ}{T}M$*; where* $x = \pi(x,y)$.
*The horizontal lift of a section* $\xi$ *of* $\pi^*TM$ *is a unique section* $\boldsymbol{h}(\xi)$ *of* $T\overset{\circ}{T}M$ *such that for every* $(x,y) \in \overset{\circ}{T}M$,

$$\pi_*(\boldsymbol{h}(\xi))|_{(x,y)} = \xi_{(x,y)} \text{ and } \theta(\boldsymbol{h}(\xi))|_{(x,y)} = 0_{(x,y)}. \tag{2.9}$$

We have the following.

**Definition 2.4.** *Let* $p_1, p_2, q_1$ *and* $q_2$ *be nonnegative integers, non both zero. A tensor field* $T$ *of type* $(p_1, p_2; q_1, q_2)$ *on* $(M,F)$ *is a mapping*

$$T : \Gamma^{p_1}(\pi^*T^*M) \times \Gamma^{p_2}(T^*\overset{\circ}{T}M) \times \Gamma^{q_1}(\pi^*TM) \times \Gamma^{q_2}(T\overset{\circ}{T}M) \longrightarrow C^\infty(\overset{\circ}{T}M, \mathbb{R})$$

*which is* $C^\infty(\overset{\circ}{T}M, \mathbb{R})$*-linear in each argument.*

**Remark 2.5.** *In a local chart,*

$$T = T^{k_1 \ldots k_{p_1} l_1 \ldots l_{p_2}}_{i_1 \ldots i_{q_1} j_1 \ldots j_{q_2}} \partial_{k_1} \otimes \ldots \otimes \partial_{k_{p_1}} \otimes \delta_{l_1} \otimes \ldots \otimes \delta_{l_{p_2}} \otimes dx^{i_1} \otimes \ldots \otimes dx^{i_{q_1}} \otimes \varepsilon^{j_1} \otimes \ldots \otimes \varepsilon^{j_{q_2}}$$

*where* $\{\partial_{k_r} := \frac{\partial}{\partial x^{k_r}}\}_{r=1,\ldots,p_1}$, $\{\delta_{l_m}\}_{m=1,\ldots,p_2}$ *and* $\{\varepsilon^{j_s}\}_{s=1,\ldots,q_2}$ *are respectively the basis sections for* $\pi^*TM$, $\pi^*T^*M$ *(dual of* $\pi^*TM$*) and* $T^*\overset{\circ}{T}M$ *(dual of* $T\overset{\circ}{T}M$*).*

**Example 2.6.** (1) *A vector field* $X$ *on* $\overset{\circ}{T}M$ *is of type* $(0,1;0,0)$.
(2) *The fundamental tensor* $g$ *is of type* $(0,0;2,0)$.
(3) *A section* $\xi$ *of* $\pi^*TM$ *is a tensor of type* $(1,0;0,0)$.

The following lemma defines the Chern connection on $\pi^*TM$.

**Lemma 2.7.** [9] *Let* $(M,F)$ *be a Finslerian manifold and* $g$ *its fundamental tensor. There exists a unique linear connection* $\nabla$ *on the vector bundle* $\pi^*TM$ *such that, for all* $X, Y \in \chi(\overset{\circ}{T}M)$ *and for every* $\xi, \eta \in \Gamma(\pi^*TM)$*, one has the following properties :*

(i) $\nabla_X \pi_* Y - \nabla_Y \pi_* X = \pi_*[X,Y]$,
(ii) $X(g(\xi,\eta)) = g(\nabla_X \xi, \eta) + g(\xi, \nabla_X \eta) + 2\mathcal{A}(\theta(X), \xi, \eta)$
*where* $\mathcal{A} := \frac{F}{2}\frac{\partial g_{ij}}{\partial y^k} dx^i \otimes dx^j \otimes dx^k$ *is the Cartan tensor.*

One has

$$\nabla_{\frac{\delta}{\delta x^j}}\frac{\partial}{\partial x^k} = \Gamma^i_{jk}\frac{\partial}{\partial x^i},\ \Gamma^i_{jk} := \frac{1}{2}g^{il}\left(\frac{\delta g_{jl}}{\delta x^k} + \frac{\delta g_{lk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^l}\right) \tag{2.10}$$

where

$$\left\{\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i\frac{\partial}{\partial y^j} = \mathbf{h}(\frac{\partial}{\partial x^i})\right\}_{i=1,\dots,n} \text{ with } N^i_j = \Gamma^i_{jk}y^k. \tag{2.11}$$

The generalized Cartan connection on $\pi^*TM$ is given as follows.

**Lemma 2.8.** [9] *Let $(M,F)$ be a Finslerian manifold and $g$ its fundamental tensor. There exists a unique linear connection ${}^c\nabla$ on the vector bundle $\pi^*TM$ such that, for all $X,Y \in \chi(\mathring{T}M)$ and for every $\xi,\eta,\nu \in \Gamma(\pi^*TM)$, one has the following properties :*

(i) ${}^c\nabla_X\pi_*Y - {}^c\nabla_Y\pi_*X = \pi_*[X,Y] + (\mathcal{A}(\theta(X),\pi_*Y,\bullet))^\sharp - (\mathcal{A}(\pi_*X,\theta(Y),\bullet))^\sharp$,

(ii) $X(g(\xi,\eta)) = g({}^c\nabla_X\xi,\eta) + g(\xi,{}^c\nabla_X\eta)$ *where $\mathcal{A}$ is the Cartan tensor and $(\ )^\sharp$ the section of $\pi^*TM$ dual to $\mathcal{A}$ defined by $g\left(\mathcal{A}(\xi,\eta,\bullet)^\sharp,\nu\right) = \mathcal{A}(\xi,\eta,\nu)$.*

## 3. Finslerian $R$-Einstein metrics

### 3.1. First curvature $R$ associated with the Chern connection or the Cartan connection.

**Definition 3.1.** *The full curvature of a linear connection $\nabla$ on the vector bundle $\pi^*TM$ over the manifold $\mathring{T}M$ is the application*

$$\phi : \begin{array}{ccc} \chi(\mathring{T}M)\times\chi(\mathring{T}M)\times\Gamma(\pi^*TM) & \to & \Gamma(\pi^*TM) \\ (X,Y,\xi) & \mapsto & \phi(X,Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi. \end{array}$$

By the relation (2.2), we have

$$\nabla_X = \nabla_{\hat{X}} + \nabla_{\check{X}},$$

where $X = \hat{X} + \check{X}$ with $\hat{X} \in \Gamma(\mathcal{H})$ and $\check{X} \in \Gamma(\mathcal{V})$.

Using the metric $F$, one can define the full curvature of $\nabla$ as :

$$\begin{aligned} \Phi(\xi,\eta,X,Y) &= g(\phi(X,Y)\xi,\eta) \\ &= g(\phi(\hat{X},\hat{Y})\xi + \phi(\hat{X},\check{Y})\xi + \phi(\check{X},\hat{Y})\xi + \phi(\check{X},\check{Y})\xi,\eta) \\ &= \mathbf{R}(\xi,\eta,X,Y) + \mathbf{P}(\xi,\eta,X,Y) + \mathbf{Q}(\xi,\eta,X,Y), \end{aligned}$$

where

$$\mathbf{R}(\xi,\eta,X,Y) = g(\phi(\hat{X},\hat{Y})\xi,\eta),\ \ \mathbf{P}(\xi,\eta,X,Y) = g(\phi(\hat{X},\check{Y})\xi,\eta) + g(\phi(\check{X},\hat{Y})\xi,\eta)$$

and

$$\mathbf{Q}(\xi,\eta,X,Y) = g(\phi(\check{X},\check{Y})\xi,\eta)$$

are respectively the *first (horizontal) curvature*, *mixed curvature* and *vertical* curvature.

In particular, if $\nabla$ is the Chern connection, the $\mathbf{Q}$-curvature vanishes.

**Proposition 3.2.** *Let $^c\Phi$ be the full curvature tensor associated with the Cartan connection. Then in the horizontal direction, $^c\Phi = \Phi$.*

*Proof.* If $X, Y \in \mathcal{H}$ then $X = \hat{X} = \hat{X}^k \frac{\delta}{\delta x^k}$ and $Y = \hat{Y} = \hat{Y}^r \frac{\delta}{\delta x^r}$. By the relation (2.4), we get

$$\begin{aligned}
\theta(\hat{X}) &= \left[\frac{\partial}{\partial x^i} \otimes \frac{1}{F}(dy^i + N^i_j dx^j)\right](\hat{X}^k \frac{\delta}{\delta x^k}) \\
&= -\frac{\hat{X}^k}{F} N^s_k \delta^i_s \frac{\partial}{\partial x^i} + \frac{\hat{X}^k}{F} N^i_j \delta^j_k \frac{\partial}{\partial x^i} \\
&= 0. \qquad (3.1)
\end{aligned}$$

Using the relation (3.1) together with the Lemma 2.7 and the Lemma 2.8, we get $^c\nabla = \nabla$ for horizontal vectors fields on $\mathring{T}M$. Thus, the curvatures of $^c\nabla$ and $\nabla$ are equal. □

3.2. **Horizantal Ricci tensor and horizantal scalar curvatures.** With respect to the Chern connection or the Cartan connection, we have the following.

**Definition 3.3.** *The horizontal Ricci tensor $\boldsymbol{Ric}^H_F$ and the horizontal scalar curvature $\boldsymbol{Scal}^H_F$ of $(M, F)$ are respectively defined by*

$$\boldsymbol{Ric}^H_F(\xi, X) := g^{ij}\boldsymbol{R}(\xi, \partial_i, X, \hat{\partial}_j), \qquad (3.2)$$

$$\boldsymbol{Scal}^H_F := trace_{\underline{g}}\Big(\boldsymbol{Ric}^H_F\Big), \ \underline{g} := \pi^* g. \qquad (3.3)$$

**Remark 3.4.** *Let $l := \frac{y^i}{F}\frac{\partial}{\partial x^i}$ be the distinguish section for $\pi^*TM$. The tensor $\boldsymbol{Ric}^H_F$ can be expressed in term of the classical Akbar-Zadeh Ricci curvatures [13] $\mathcal{R}ic$ and $\boldsymbol{Ric}_{ij}$ as follows.*

$$\begin{aligned}
\boldsymbol{Ric}^H_F(l, \boldsymbol{h}(l)) &\overset{(3.2)}{=} g^{ij}\boldsymbol{R}(l, \partial_i, \boldsymbol{h}(l), \hat{\partial}_j) \\
&= g^{ij} l^l \boldsymbol{R}(\partial_l, \partial_i, , \hat{\partial}_k, \hat{\partial}_j) l^k \\
&= \mathcal{R}ic \\
&= l^i l^j \boldsymbol{Ric}_{ij}.
\end{aligned}$$

3.3. **Finslerian $R$-Einstein metric.** It is known [3], $F$ is Einstein if there exists a $C^\infty$ function $k$ on $M$ such that

$$\mathcal{R}ic = (n-1)k. \qquad (3.4)$$

Now, we introduce the following.

**Definition 3.5.** *A Finslerian metric $F$ on an $n$-dimensional manifold is $R$-Einstein if*

$$\boldsymbol{Ric}^H_F = \frac{1}{n}\boldsymbol{Scal}^H_F \underline{g}. \qquad (3.5)$$

**Remark 3.6.** *If $F$ satisfies (3.5) for a constant function $\boldsymbol{Scal}_F^H$ (respectively for $\boldsymbol{Scal}_F^H \equiv 0$) then $F$ is said to be horizontally Ricci-constant (respectively, $F$ is called horizontally Ricci-flat metric).*

**Remark 3.7.** *If $F$ is a Finslerian R-Einstein metric on an $n$-dimensional manifold $M$ then its associated horizontal scalar curvature is a function on $M$. That is, for any $(x, y)$ of $\mathring{T}M$, $\boldsymbol{Scal}_F^H(x, y) = n(n-1)k(x)$.*

**Definition 3.8.** *Let $\boldsymbol{T}$ be a $(p_1, p_2; 0, 0)$-tensor on $(M, F)$ and $X \in T\mathring{T}M$. The covariant derivative of $\boldsymbol{T}$ in the direction of $X$ is given by the following formula :*

$$
\begin{aligned}
(\nabla_X \boldsymbol{T})(\xi_1, ..., \xi_{p_1}, X_1, ..., X_{p_2}) \quad := \quad & X\left(\boldsymbol{T}(\xi_1, ..., \xi_{p_1}, X_1, ..., X_{p_2})\right) \\
- & \sum_{i=1}^{p_1} \left[\boldsymbol{T}(\xi_1, ..., \nabla_X \xi_i, ..., \xi_{p_1}, X_1, ..., X_{p_2})\right] \\
- & \sum_{j=1}^{p_2} \left[\boldsymbol{T}(\xi_1, ..., \xi_{p_1}, X_1, ..., \boldsymbol{h}(\nabla_X \pi_* X_j), ..., X_{p_2})\right] \\
- & \sum_{j=1}^{p_2} \left[\boldsymbol{T}(\xi_1, ..., \xi_{p_1}, X_1, ..., \boldsymbol{h}(\nabla_X \theta(X_j)), ..., X_{p_2})\right] \\
- & \sum_{j=1}^{p_2} \left[\boldsymbol{T}(\xi_1, ..., \xi_{p_1}, X_1, ..., \boldsymbol{v}(\nabla_X \pi_* X_j), ..., X_{p_2})\right] \\
- & \sum_{j=1}^{p_2} \left[\boldsymbol{T}(\xi_1, ..., \xi_{p_1}, X_1, ..., \boldsymbol{v}(\nabla_X \theta(X_j)), ..., X_{p_2})\right].
\end{aligned}
$$

We obtain the Finslerian horizantal Bianchi identity given in the following.

**Lemma 3.9.** *If $\xi, \eta \in \Gamma(\pi_* TM)$ and $X, Y, Z \in \chi(\mathring{T}M)$ then*

$$(\nabla_Z \boldsymbol{R})(\xi, \eta, X, Y) + (\nabla_X \boldsymbol{R})(\xi, \eta, Y, Z) + (\nabla_Y \boldsymbol{R})(\xi, \eta, Z, X) = 0. \tag{3.6}$$

*Proof.* The Lemma 3.9 is obtained from the symmetry of $\nabla$ and the Jacobi identity and by the Definition 3.8 applied to the first curvature $\mathbf{R}$. □

3.4. **Schur's type lemma.** We prove a Schur lemma for $\mathbf{Scal}_F^H$.

**Lemma 3.10.** *If $F$ is horizontally an Einstein metric on a connected manifold of dimension $n \geq 3$ then its horizontal scalar curvature is constant.*

*Proof.* If $F$ is horizontally an Einstein metric then the relation (3.5) holds.

Applying the horizontal covariant derivative on each side of the relation (3.5), we obtain

$$\nabla_k \mathbf{Ric}_F^H(\partial_i, \hat{\partial}_j) \quad = \quad \frac{1}{n}(\nabla_k \mathbf{Scal}_F^H) g_{ij}$$

where $\nabla_i = \nabla_{\hat{\partial}_i}$. Multiplying this last equation by $g^{ik}$ and setting $g^{ik}\nabla_i = \nabla^k$ we get

$$\nabla^i \mathbf{Ric}^H_F(\partial_i, \hat{\partial}_j) \quad = \quad \frac{1}{n}\nabla_j \mathbf{Scal}^H_F. \tag{3.7}$$

By contracting twice on equation (3.6) written in a local coordinate, we have

$$\begin{aligned} \frac{1}{2}\nabla_j \mathbf{Scal}^H_F \quad &= \quad \nabla^i \mathbf{Ric}^H_F(\partial_i, \hat{\partial}_j) \\ &\overset{(3.7)}{=} \quad \frac{1}{n}\nabla_j \mathbf{Scal}^H_F. \end{aligned} \tag{3.8}$$

When $n > 2$, the equations (3.7) and (3.8) together with the Lemma 3.7 imply

$$\begin{aligned} 0 \quad &= \quad \nabla_j \mathbf{Scal}^H_F \\ &= \quad \frac{\partial \mathbf{Scal}^H_F}{\partial x^j}. \end{aligned}$$

Hence, $\mathbf{Scal}^H_F$ must be constant. □

## 4. Finslerian locally conformal $R$-Einstein equation

### 4.1. Conformal change of Finslerian trace-free horizontal Ricci tensors.

**Definition 4.1.** *A Finslerian metric $F$ on a manifold $M$ is locally conformally $R$-Einstein if each point $x \in M$ has a neighborhood $U$ on which there exists a $C^\infty$-function $u$ such that the conformal deformation $\widetilde{F}$ of $F$, with $\widetilde{F} = e^u F$, is an $R$-Einstein metric on $U$.*

**Lemma 4.2.** [10] *Let $F$ and $\widetilde{F}$ be two Finslerian metrics on an $n$-dimensional manifold $M$. If $F$ is conformal to $\widetilde{F}$, with $\widetilde{F} = e^u F$, then the trace-free horizontal Ricci tensors $\boldsymbol{E}^H_F$ and $\widetilde{\boldsymbol{E}}^H_{\widetilde{F}}$ are related by*

$$\widetilde{\boldsymbol{E}}^H_{\widetilde{F}} \quad = \quad \boldsymbol{E}^H_F - (n-2)\,(H_u - du \circ du) - \frac{(n-2)}{n}\left(\Delta^H u + ||\nabla u||^2_g\right)\underline{g} + \Psi_u^{\boldsymbol{E}^H_F}$$

*where* $\Psi_u^{\boldsymbol{E}_F^H}$ *is the* $(1,1;0,0)$*-tensor on* $(M,F)$ *given by*

$$\begin{aligned}
\Psi_u^{\boldsymbol{E}_F^H}(\xi,X) \;:=\; & (2-n)\left[\mathcal{A}(\nabla u,\mathcal{B}(X),\xi)+\mathcal{A}(\nabla u,\pi_*X,\mathcal{B}(\boldsymbol{h}(\xi)))\right] \\
& +(n-4)\mathcal{A}(\mathcal{B}(\boldsymbol{h}(\nabla u),\pi_*X,\xi)) \\
& +\frac{1}{n}g^{ij}\Big[2(n-2)\mathcal{A}(\nabla u,\partial_i,\mathcal{B}(\hat{\partial}_j))) \\
& -3\mathcal{A}(\mathcal{B}(\boldsymbol{h}(\nabla u),\partial_j,\partial_i))\big]\, g(\xi,\pi_*X). \\
& +g^{ij}\Big[g\Big(\Theta(X,\boldsymbol{h}(\Theta(\hat{\partial}_j,\boldsymbol{h}(\xi)))),\partial_i\Big)-g\Big(\Theta(\hat{\partial}_j,\boldsymbol{h}(\Theta(X,\boldsymbol{h}(\xi))),\partial_i\Big)\Big] \\
& +g^{ij}\Big[g\Big((\nabla_X\Theta)(\hat{\partial}_j,\boldsymbol{h}(\xi)),\partial_i\Big)-g\left((\nabla_j\Theta)(\boldsymbol{h}(\xi),X),\partial_i\right)\Big] \\
& -\frac{1}{n}g^{ij}g^{kl}\left[\mathcal{A}(\mathcal{B}(\boldsymbol{h}(\Theta_{jk}),\partial_l,\partial_i))-\mathcal{A}(\mathcal{B}(\boldsymbol{h}(\Theta_{kl}),\partial_j,\partial_i))\right]g(\xi,\pi_*X) \\
& -\frac{1}{n}g^{ij}g^{kl}\left[g\left((\nabla_l\Theta)_{jk},\partial_i\right)-g\left((\nabla_j\Theta)_{kl},\partial_i\right)\right]g(\xi,\pi_*X), \qquad (4.1)
\end{aligned}$$

*for every* $\xi\in\Gamma(\pi^*TM)$ *and* $X\in\chi(\mathring{T}M)$ *with* $\Theta_{ij}=\Theta(\hat{\partial}_i,\hat{\partial}_j)$ *and* $\mathcal{B}$ *is the application which maps* $\pi^*TM$ *to* $\pi^*TM$ *defined by*

$$\mathcal{B}=\mathcal{B}^i_j\partial_i\otimes dx^j \qquad (4.2)$$

*with*

$$\mathcal{B}^i_j=\frac{1}{2F}\left(\nabla_r u\right)\frac{\partial(F^2g^{ir}-2y^iy^r)}{\partial y^j}. \qquad (4.3)$$

### 4.2. **Proof of the Proposition 1.1.**

*Proof.* Let $F$ and $\widetilde{F}$ be two conformal Finslerian metrics on a manifold of dimension $n$. If $F$ is conformally $R$-Einstein then $\widetilde{\mathbf{E}}^H_{\widetilde{F}}$ vanishes. By the Lemma 4.2, in a local chart we have

$$\begin{aligned}
0 \;=\; & \Big[\mathbf{E}^H_F-(n-2)\left(H_u-du\circ du\right)-\frac{(n-2)}{n}\left(\Delta^H u+||\nabla u||^2_g\right)\underline{g}\Big](\partial_i,\hat{\partial}_j) \\
& +\Psi_u^{\mathbf{E}_F^H}(\partial_i,\hat{\partial}_j) \qquad (4.4)
\end{aligned}$$

where

$$\begin{aligned}
\Psi_u^{\mathbf{E}_F^H}(\partial_i,\hat{\partial}_j) \;=\; & (2-n)\Big[\mathcal{A}(\nabla u,\mathcal{B}(\hat{\partial}_j),\partial_i)+\mathcal{A}(\nabla u,\pi_*\hat{\partial}_j,\mathcal{B}(\mathbf{h}(\partial_i)))\Big] \\
& +(n-4)\mathcal{A}(\mathcal{B}(\mathbf{h}(\nabla u),\pi_*\hat{\partial}_j,\partial_i)) \\
& +\frac{1}{n}g^{kl}\Big[2(n-2)\mathcal{A}(\nabla u,\partial_k,\mathcal{B}(\hat{\partial}_l)))-3\mathcal{A}(\mathcal{B}(\mathbf{h}(\nabla u),\partial_l,\partial_k))\Big]g(\partial_i,\pi_*\hat{\partial}_j). \\
& +g^{ij}\Big[g\Big(\Theta(\hat{\partial}_j,\mathbf{h}(\Theta(\hat{\partial}_l,\mathbf{h}(\partial_i)))),\partial_k\Big)-g\Big(\Theta(\hat{\partial}_j,\mathbf{h}(\Theta(\hat{\partial}_l,\mathbf{h}(\partial_i))),\partial_k\Big)\Big] \\
& +g^{kl}\Big[g\Big((\nabla_j\Theta)(\hat{\partial}_l,\mathbf{h}(\partial_i)),\partial_k\Big)-g\Big((\nabla_l\Theta)(\mathbf{h}(\partial_i),\hat{\partial}_j),\partial_k\Big)\Big] \\
& -\frac{1}{n}g^{rs}g^{kl}\Big[\mathcal{A}(\mathcal{B}(\mathbf{h}(\Theta_{sk}),\partial_l,\partial_r))-\mathcal{A}(\mathcal{B}(\mathbf{h}(\Theta_{kl}),\partial_s,\partial_r))\Big]g_{ij} \\
& -\frac{1}{n}g^{rs}g^{kl}\left[g\left((\nabla_l\Theta)_{sk},\partial_r\right)-g\left((\nabla_s\Theta)_{kl},\partial_r\right)\right]g_{ij}. \qquad (4.5)
\end{aligned}$$

Using the relation (4.2), we have $\mathcal{B}(\hat{\partial}_l) = \mathcal{B}^{s_1}_{s_2}\delta^{s_2}_l\partial_{s_1} = \mathcal{B}^{s_1}_l\partial_{s_1}$ and $\mathcal{B}(\mathbf{h}(\nabla u)) = \mathcal{B}^{s_1}_{s_2}\partial_{s_1}\otimes dx^{s_2}(\mathbf{h}(\nabla^l u\partial_l)) = \nabla^l u\mathcal{B}^{s_1}_l\partial_{s_1}$. Thus, from (4.5), we have

$$\begin{aligned}
I_1 &= (2-n)\Big[\mathcal{A}(\nabla u, \mathcal{B}(\hat{\partial}_j), \partial_i) + \mathcal{A}(\nabla u, \pi_*\hat{\partial}_j, \mathcal{B}(\mathbf{h}(\partial_i)))\Big] \\
&\quad +(n-4)\mathcal{A}(\mathcal{B}(\mathbf{h}(\nabla u), \pi_*\hat{\partial}_j, \partial_i)) \\
&= (n-4)\nabla^{s_2}u\mathcal{B}^{s_1}_{s_2}\mathcal{A}_{s_1ij} - (n-2)\Big(\nabla^{s_2}u\mathcal{B}^{s_1}_i\mathcal{A}_{s_1js_2} + \nabla^{s_2}u\mathcal{B}^{s_1}_j\mathcal{A}_{s_1is_2}\Big), \\
I_2 &= \frac{1}{n}g^{kl}\Big[2(n-2)\mathcal{A}(\nabla u, \partial_k, \mathcal{B}(\hat{\partial}_l))) - 3\mathcal{A}(\mathcal{B}(\mathbf{h}(\nabla u), \partial_l, \partial_k))\Big]g_{ij} \\
&= -\frac{1}{n}g^{kl}\nabla^{s_2}u\Big[-3\mathcal{B}_{s_2}s_1\mathcal{A}_{s_1kl} + 3\mathcal{B}_k s_1\mathcal{A}_{s_1ls_2} - (2n-1)\mathcal{B}_k s_1\mathcal{A}_{s_1ls_2}\Big]g_{ij}, \\
I_3 &= -\frac{1}{n}g^{rs}g^{kl}\left[\mathcal{A}(\mathcal{B}(\mathbf{h}(\Theta_{sk}), \partial_l, \partial_r)) - \mathcal{A}(\mathcal{B}(\mathbf{h}(\Theta_{kl}), \partial_s, \partial_r))\right]g_{ij}, \\
I_4 &= g^{kl}\Big[g\Big(\Theta(\hat{\partial}_j, \mathbf{h}(\Theta(\hat{\partial}_l, \mathbf{h}(\partial_i))))), \partial_k\Big) - g\Big(\Theta(\hat{\partial}_l, \mathbf{h}(\Theta(\hat{\partial}_j, \mathbf{h}(\partial_i))), \partial_k\Big)\Big] \\
&= g^{kl}\delta^r_i\delta^s_j\Big[g\Big(\Theta(\hat{\partial}_s, \mathbf{h}(\Theta(\hat{\partial}_l, \mathbf{h}(\partial_r))))), \partial_k\Big) - g\Big(\Theta(\hat{\partial}_l, \mathbf{h}(\Theta(\hat{\partial}_s, \mathbf{h}(\partial_r))), \partial_k\Big)\Big] \\
&= \frac{1}{n}g^{kl}g^{rs}g_{ij}\Big[g\Big(\Theta(\hat{\partial}_s, \mathbf{h}(\Theta(\hat{\partial}_l, \mathbf{h}(\partial_r))))), \partial_k\Big) - g\Big(\Theta(\hat{\partial}_l, \mathbf{h}(\Theta(\hat{\partial}_s, \mathbf{h}(\partial_r))), \partial_k\Big)\Big] \\
&= -I_3, \\
I_5 &= -\frac{1}{n}g^{rs}g^{kl}\left[g\left((\nabla_l\Theta)_{sk}, \partial_r\right) - g\left((\nabla_s\Theta)_{kl}, \partial_r\right)\right]g_{ij}, \\
I_{16} &= g^{ij}\Big[g\Big((\nabla_X\Theta)(\hat{\partial}_j, \mathbf{h}(\xi)), \partial_i\Big) - g\left((\nabla_j\Theta)(\mathbf{h}(\xi), X), \partial_i\right)\Big] \\
&= -I_5.
\end{aligned}$$

Hence, putting the expressions of $I_1, I_2, I_3, I_4, I_5$ and $I_6$ in the right-hand side of (4.4) we obtain the equation 1.1. □

**Remark 4.3.** *The equation (1.1) is called Finslerian locally conformal R-Einstein equation.*

## 5. Locally conformally $R$-Einstein metrics in dimensions 1 and 2

5.1. **For $n = 1$.** Every Finslerian metric is conformally $R$-Einstein.

**Theorem 5.1.** *Let $(M, F)$ be a Finslerian manifold of dimension one. Then $(M, F)$ is always R-flat.*

*Proof.* This follows from the Lemma 2.7 and the skewsymmetry the curvature $\mathbf{R}$. □

5.2. **For $n = 2$: Proof of the Theorem 1.2.**

*Proof.* When $n = 2$, the equation (1.1) reduces to

$$\mathbf{E}_F(\partial_i, \hat{\partial}_j) + \frac{1}{4F}\left(\nabla_r u\nabla^q u\right)\frac{\partial(F^2g^{rs} - 2y^r y^s)}{\partial y^q}g^{kl}\mathcal{A}_{skl}g_{ij} = 0. \quad (5.1)$$

Contracting (5.1) by $g^{ij}$ yields

$$\frac{1}{2F}\left(\nabla_r u \nabla^q u\right) \frac{\partial(F^2 g^{rs} - 2y^r y^s)}{\partial y^q} g^{kl} \mathcal{A}_{skl} = 0. \tag{5.2}$$

Using the famous Euler theorem on homogeneous functions (see [2], page 5), applied on functions $y^s \mathcal{A}_{skl} = y^s \frac{\partial g_{kl}}{\partial y^s} = 0$, we have

$$\frac{\partial(-2y^r y^s)}{\partial y^q} g^{kl} \mathcal{A}_{skl} = -2y^s \mathcal{A}_{skl} g^{kl} \delta^r_q = 0.$$

Since $F$ is a Finslerian metric, $F(x,y) \neq 0$ for every $(x,y) \in \overset{\circ}{T}M$ and since $g$ is positive-definite the $g^{kl}$ functions do not vanish for any $k,l \in \{1,2\}$. Hence, the only solution of the equation (5.2) are $\nabla_r u = 0$ or $\mathcal{A} \equiv 0$.

(i) If $\nabla_r u = 0$, the conformal factor $u$ is constant. Further, if $u$ is constant then by equation (5.1) $\mathbf{E}^H_F$ vanishes.

(ii) If $\mathcal{A} \equiv 0$, by Deicke's theorem, $F$ is Riemannian. Hence, the result follows by the fact that any Riemannian metric on a 2-dimensional manifold is Einstein (see [3]).

Conversely, if the conformal deformation is homothetic and $F$ is horizontally locally Einstein then the relation (5.1) is satisfied. Thus, if $\mathcal{A}$ vanishes it is known that $F$ is Riemannian and, when $n = 2$, every Riemannian metric in conformally Einstein. □

**Example 5.2.** *For a Finsler-Minkowskian metric on $\mathbb{R}^2$, $F(x,y) = F(y)$. The conformal deformations of $F$ are of the form $\widetilde{F} = cF$ for all $c > 0$. Since, the $\boldsymbol{R}$-curvature on $\mathbb{R}^2$ vanishes, the tensor $\boldsymbol{E}^H_F$ vanishes. Then $F$ is globally (and automatically locally) conformally $R$-Einstein.*

## 6. Locally conformally $R$-Einstein metrics on a cylinder of dimension $n \geq 3$

6.1. **Warped product of Finslerian metrics.** Let $\overset{1}{M}$ and $\overset{2}{M}$ be two manifolds. The set of all product coordinate systems in $\overset{1}{M} \times \overset{2}{M}$ is an atlas on $M = \overset{1}{M} \times \overset{2}{M}$ called *product manifold* of $\overset{1}{M}$ and $\overset{2}{M}$.

**Example 6.1.** *The product $\mathbb{R} \times \overset{1}{M}$ is called an infinite cylinder over $\overset{1}{M}$.*

**Example 6.2.** *In the Example 6.1, if we replace $\mathbb{R}$ by an open interval $(1, \varepsilon)$, we obtain a finite cylinder $(1,\varepsilon) \times \overset{1}{M}$ over $\overset{1}{M}$.*

**Remark 6.3.** *In general, the product manifold of $k$ manifolds $\overset{1}{M}, \ldots, \overset{k-1}{M}$ and $\overset{k}{M}$ is the cartesian product $M = \overset{1}{M} \times \ldots \times \overset{k}{M}$.*

Let $\overset{1}{M}$ and $\overset{2}{M}$ be two $C^\infty$ manifolds. For every $(x_1, x_2) \in \overset{1}{M} \times \overset{2}{M}$, we have the following properties derived from $\overset{1}{M}$ and $\overset{2}{M}$.

(1) The projections

$$\overset{1}{p} \; : \; \overset{1}{M} \times \overset{2}{M} \longrightarrow \overset{1}{M} \text{ such that } \overset{1}{p}(x_1, x_2) = x_1$$
$$\overset{2}{p} \; : \; \overset{1}{M} \times \overset{2}{M} \longrightarrow \overset{1}{M} \text{ such that } \overset{2}{p}(x_1, x_2) = x_2$$

are $C^\infty$ submersions.

(2) $dim(\overset{1}{M} \times \overset{2}{M}) = dim \overset{1}{M} + dim \overset{2}{M}$.

The warped product manifold of two Finslerian manifolds is defined as follows.

**Definition 6.4.** *Let $(\overset{1}{M}, \overset{1}{F})$ and $(\overset{2}{M}, \overset{2}{F})$ be two Finslerian manifolds. Let $f$ be a positive $C^\infty$ function on $\overset{1}{M}$. The warped product of $(\overset{1}{M}, \overset{1}{F})$ and $(\overset{2}{M}, \overset{2}{F})$ is a manifold $M = \overset{1}{M} \times_f \overset{2}{M}$ equipped with the Finslerian metric*

$$F : \overset{\circ}{T} \overset{1}{M} \times \overset{\circ}{T} \overset{2}{M} \longrightarrow \mathbb{R}^+ \tag{6.1}$$

*such that for any vector tangent $y \in T_xM$, with $x = (x_1, x_2) \in M$ and $y = (y_1, y_2)$,*

$$F(x, y) = \sqrt{\overset{1}{F}{}^2 (x_1, \overset{1}{p}_* y) + f^2(\overset{1}{p}(x_1, x_2)) \overset{2}{F}{}^2 (x_2, \overset{2}{p}_* y)} \tag{6.2}$$

*where $\overset{1}{p}$ and $\overset{2}{p}$ are respectively the projections of $\overset{1}{M} \times \overset{2}{M}$ onto $\overset{1}{M}$ and $\overset{2}{M}$.*

**Remark 6.5.** *Let $F$ be a Finsler metric on a warped product manifold $\overset{1}{M} \times_f \overset{2}{M}$.*

(1) *$F$ is not $C^\infty$ on the tangent vectors of the form $(y_1, 0)$ nor $(0, y_2)$ at a point $(x_1, x_2) \in \overset{1}{M} \times_f \overset{2}{M}$.*

(2) *$\overset{1}{M}$ is called the base manifold while $\overset{2}{M}$ is the fiber manifold and $f$ is called the warping function.*
*If $f \equiv 1$ then $(\overset{1}{M} \times_f \overset{2}{M}, \sqrt{\overset{1}{F}{}^2 (x_1, \overset{1}{p}_* y) + f^2(\overset{1}{p}(x_1, x_2)) \overset{2}{F}{}^2 (x_2, \overset{2}{p}_* y)}$ reduces to a Finslerian product manifold $(\overset{1}{M} \times \overset{2}{M}, \sqrt{\overset{1}{F}{}^2 (x_1, \overset{1}{p}_* y) + \overset{2}{F}{}^2 (x_2, \overset{2}{p}_* y)}$.*

The function $F$ defined in (6.1) and (6.2) is a Finslerian manifold. More precisely,

(i) $F$ is $C^\infty$ on $\overset{\circ}{T} \overset{1}{M} \times \overset{\circ}{T} \overset{2}{M}$ since $\overset{1}{F}$ and $\overset{2}{F}$ are respectively $C^\infty$ on $\overset{\circ}{T} \overset{1}{M}$ and $\overset{\circ}{T} \overset{2}{M}$.

(ii) $F$ is homogeneous of degree 1 in $y = (y_1, y_2) \in T_xM$. Namely, for any $c > 0$,

$$\begin{aligned} F(x, cy) &\overset{(6.2)}{=} \sqrt{\overset{1}{F}{}^2 (x_1, (cy_1)) + f^2(x_1) \overset{2}{F}{}^2 (x_2, (cy_2))} \\ &= c\sqrt{\overset{1}{F}{}^2 (x_1, y_1) + f^2(x_1) \overset{2}{F}{}^2 (x_2, y_2)} \\ &= cF(x, y). \end{aligned}$$

(iii) If $n_1$ and $n_2$ are respectively the dimensions of $(\overset{1}{M}, \overset{1}{F})$ and $(\overset{2}{M}, \overset{2}{F})$, each element of the Hessian matrix $(g_{ij}(x,y))_{1\leq i,j\leq n_1+n_2}$ of $\frac{1}{2}F^2$, has the form :

$$\begin{aligned} g_{ij}(x,y) &:= \frac{\partial^2\left[\frac{1}{2}F^2(x,y)\right]}{\partial y^i \partial y^j} \\ &= \frac{1}{2}\frac{\partial^2\left[\overset{1}{F}{}^2(x_1,y_1)+f^2(x_1)\overset{2}{F}{}^2(x_2,y_2)\right]}{\partial y^i \partial y^j} \\ &= \frac{1}{2}\frac{\partial^2 \overset{1}{F}{}^2(x_1,y_1)}{\partial y_1^i \partial y_1^j}+\frac{1}{2}f^2(x_1)\frac{\partial^2 \overset{2}{F}{}^2(x_2,y_2)}{\partial y_2^i \partial y_2^j}. \end{aligned}$$

for every point $(x,y)=(x_1,x_2,y_1,y_2)\in \overset{\circ}{T}\overset{1}{M}\times\overset{\circ}{T}\overset{2}{M}$. Thus,

$$\big(g_{ij}(x,y)\big)=\begin{pmatrix} \big(\overset{1}{g}_{ij}(x_1,y_1)\big) & 0 \\ 0 & \big(\overset{2}{g}_{ij}(x_2,y_2)\big) \end{pmatrix} \tag{6.3}$$

where $\overset{1}{g}_{ij}(x_1,y_1):=\frac{1}{2}\frac{\partial^2\overset{1}{F}{}^2(x_1,y_1)}{\partial y_1^i\partial y_1^j}$ and $\overset{2}{g}_{ij}(x_2,y_2):=\frac{1}{2}f^2(x_1)\frac{\partial^2\overset{2}{F}{}^2(x_2,y_2)}{\partial y_2^i\partial y_2^j}$. So the fundamental tensor $g$ of $F$ is positive definite at every point $(x_1,x_2,y_1,y_2)\in \overset{\circ}{T}\overset{1}{M}\times\overset{\circ}{T}\overset{2}{M}$ since $\overset{1}{g}$ and $\overset{2}{g}$ are.

6.2. **Curvatures associated with warped product Finslerian metrics.** Given the submersions $\overset{1}{\pi}: \overset{\circ}{T}\overset{1}{M}\longrightarrow\overset{1}{M}$ and $\overset{2}{\pi}: \overset{\circ}{T}\overset{2}{M}\longrightarrow\overset{2}{M}$, the fundamental tensors $\overset{1}{g}$ and $\overset{2}{g}$ associated with $\overset{1}{F}$ and $\overset{2}{F}$ are Riemannian metrics on the respective pulled-back tangent bundles $\overset{1}{\pi}{}^{*}\,T\overset{1}{M}$ and $\overset{2}{\pi}{}^{*}\,T\overset{2}{M}$. Thus, $\overset{1}{\pi}$ gives rise to the Ehresmann-Finsler connection

$$\overset{1}{\mathcal{H}}=ker\theta_1 \text{ where } \theta_1: T\overset{\circ}{T}\overset{1}{M}\longrightarrow\overset{1}{\pi}{}^{*}\,T\overset{1}{M} \tag{6.4}$$

while $\overset{2}{\pi}$ give rise to the Ehresmann-Finsler

$$\overset{2}{\mathcal{H}}=ker\theta_2 \text{ where } \theta_2: T\overset{\circ}{T}\overset{2}{M}\longrightarrow\overset{2}{\pi}{}^{*}\,T\overset{2}{M}\,. \tag{6.5}$$

The Ehresmann-Finslerian product connection $\mathcal{H}$ is given by the product form $\theta$ of $\theta_1$ and $\theta_2$, that is

$$\theta=\theta_1\times\theta_2: T\overset{\circ}{T}\overset{1}{M}\times T\overset{\circ}{T}\overset{2}{M}\equiv T(\overset{\circ}{T}\overset{1}{M}\times\overset{\circ}{T}\overset{2}{M})\longrightarrow\overset{1}{\pi}{}^{*}\,T\overset{1}{M}\times\overset{2}{\pi}{}^{*}\,T\overset{2}{M} \tag{6.6}$$

such that

$$ker\theta=ker(\theta_1\times\theta_2)=ker\theta_1\oplus ker\theta_2. \tag{6.7}$$

Now, let $\overset{1}{\mathcal{V}}$ and $\overset{2}{\mathcal{V}}$ be the vertical subbundle of $T\overset{\circ}{T}\overset{1}{M}$ and $T\overset{\circ}{T}\overset{2}{M}$, respectively. We obtain the following decomposition

$$T\overset{\circ}{T}(\overset{1}{M}\times\overset{1}{M})=\overset{1}{\mathcal{H}}\oplus\overset{1}{\mathcal{V}}\oplus\overset{2}{\mathcal{H}}\oplus\overset{2}{\mathcal{V}}\,. \tag{6.8}$$

**Proposition 6.6.** *Let* $(\overset{1}{M},\overset{1}{F})$ *and* $(\overset{2}{M},\overset{2}{F})$ *be two Finslerian manifolds. On a warped product manifold* $M=\overset{1}{M}\times_f\overset{2}{M}$, *if* $\overset{1}{\xi}\in\Gamma(\overset{1}{\pi^*}\, T\,\overset{1}{M})$, $\overset{2}{\xi}\in\Gamma(\overset{2}{\pi^*}\, T\,\overset{2}{M})$ *and* $\overset{1}{X}\in\chi(\overset{\circ}{T}\,\overset{1}{M})$ *then*

(i) $\nabla_{\overset{1}{X}}\overset{1}{\xi}=\overset{1}{\nabla}_{\overset{1}{X}}\overset{1}{\xi}$ *where* $\overset{1}{\nabla}$ *is the Chern connection associated with* $(\overset{1}{M},\overset{1}{F})$.

(ii) $\nabla_{\overset{1}{X}}\overset{2}{\xi}=\frac{1}{f}\overset{1}{X}(f)\,\overset{2}{\xi}$.

*Proof.* $(i)$ From the relation of $g$-almost compatibility of $\nabla$, we obtain

$$\begin{aligned}
2g(\nabla_{\overset{1}{X}}\overset{1}{\xi},\overset{2}{\xi}) &= \overset{1}{X}[g(\overset{1}{\xi},\overset{2}{\xi})]+\mathbf{h}(\overset{1}{\xi})[g(\overset{1}{\pi}_*\overset{1}{X},\overset{2}{\xi})]-\mathbf{h}(\overset{2}{\xi})[g(\overset{1}{\xi},\overset{1}{\pi}_*\overset{1}{X})]\\
&\quad -g(\overset{1}{\pi}_*\overset{1}{X},[\overset{1}{\xi},\overset{2}{\xi}])-g(\overset{1}{\xi},[\overset{1}{\pi}_*\overset{1}{X},\overset{2}{\xi}])+g(\overset{2}{\xi},[\overset{1}{\pi}_*\overset{1}{X},\overset{1}{\xi}])\\
&\quad +\mathcal{A}(\theta(\overset{1}{X}),\overset{1}{\xi},\overset{2}{\xi})+\mathcal{A}(\theta(\mathbf{h}(\overset{1}{\xi})),\overset{1}{\pi}_*\overset{1}{X},\overset{2}{\xi})-\mathcal{A}(\theta(\mathbf{h}(\overset{2}{\xi})),\overset{1}{\pi}_*\overset{1}{X},\overset{1}{\xi})\\
&= 0.
\end{aligned}$$

$(ii)$ For $\overset{2}{\xi},\overset{2}{\eta}\in\Gamma(\overset{2}{\pi^*}\,T\,\overset{2}{M})$,

$$\begin{aligned}
2g(\nabla_{\overset{1}{X}}\overset{2}{\xi},\overset{2}{\eta}) &= \overset{1}{X}[g(\overset{2}{\xi},\overset{2}{\eta})]\\
&\overset{(6.3)}{=} \overset{1}{X}[(f\circ\overset{1}{p})^2\,\overset{2}{g}(\overset{2}{\xi},\overset{2}{\eta})]
\end{aligned}$$

and the relation in $(ii)$ follows. □

As a direct consequence, we have

**Corollary 6.7.** *Let* $(\overset{1}{M},\overset{1}{F})$ *and* $(\overset{2}{M},\overset{2}{F})$ *be two Finslerian manifolds. On a warped product manifold* $M=\overset{1}{M}\times_f\overset{2}{M}$, *if* $\overset{1}{\xi},\overset{1}{\eta}\in\Gamma(\overset{1}{\pi^*}\,T\,\overset{1}{M})$, $\overset{1}{X},\overset{1}{Y}\in\chi(\overset{\circ}{T}\,\overset{1}{M})$ *and* $\overset{2}{X}\in\chi(\overset{\circ}{T}\,\overset{2}{M})$ *then*

(i) $\boldsymbol{R}(\overset{1}{\xi},\overset{1}{\eta},\overset{1}{X},\overset{1}{Y})=\overset{1}{\boldsymbol{R}}(\overset{1}{\xi},\overset{1}{\eta},\overset{1}{X},\overset{1}{Y})$.

(ii) $\boldsymbol{R}(\overset{1}{\xi},\overset{1}{\eta},\overset{2}{X},\overset{1}{Y})=0$.

*Proof.* The proof follows from the Proposition 6.6. □

6.3. **Proof of the Theorem 1.3.** We consider the special case where the conformal factor only depends on the base manifold $\overset{1}{M}$ of the product $\overset{1}{M}\times\overset{2}{M}$.

*Proof.* A Finslerian metric $F$ on a cylinder $\mathbb{R}\times\overset{2}{M}$ can be written as $F=\sqrt{t^2+\overset{2}{F}{}^2}$ where $\overset{2}{F}$ is a Finslerian metric on $\overset{2}{M}$. Further, if $F$ is locally conformal to the $R$-Einstein metric $e^{u(t)}F$, then by Proposition 1.1, we have

Case 1 : if $i = j = 1$, that is $t = y^i = y^j$, the equation (1.1) becomes

$$\begin{aligned} 0 &= \mathbf{E}_F(\partial_t, \hat{\partial}_t) - (n-2)\left(\nabla_t \nabla_t u - \nabla_t u \nabla_t u\right) \\ &\quad + \frac{(n-2)}{n}\left(\nabla^d \nabla_d u - \nabla^d u \nabla_d u\right) g_{tt} \\ &\quad + \frac{(n-1)}{2nF}\left(\nabla_r u \nabla^q u\right) \frac{\partial(F^2 g^{rs} - 2y^r y^s)}{\partial y^q} g^{kl} \mathcal{A}_{skl} g_{tt} \\ &= \mathbf{Ric}_F(\partial_t, \hat{\partial}_t) - \frac{1}{n}\mathbf{Scal}_F^H g_{tt} \\ &\quad -(n-2)\left(\nabla_t \nabla_t u - \nabla_t u \nabla_t u\right) \\ &\quad + \frac{(n-2)}{n} g^{td}\left(\nabla_t \nabla_d u - \nabla_t u \nabla_d u\right) g_{tt} \\ &\quad + \frac{(n-1)}{2nF}\left(\nabla_t u \nabla^t u\right) \frac{\partial(F^2 g^{tt} - 2t^2)}{\partial t} g^{kl} \mathcal{A}_{tkl} \end{aligned}$$

since $u = u(t)$ and $y^q = t$ is a coordinate on $\mathbb{R}$. It follows that

$$\begin{aligned} 0 &= -\frac{1}{n}\mathbf{Scal}_F^H \\ &\quad -(n-2)\left(u^{''} - u^{'2}\right) + \frac{(n-2)}{n}\left(u^{''} - u^{'2}\right) \\ &\quad + \frac{(n-1)}{2nF}\left(u^{'2}\right) \frac{\partial(F^2 - 2t^2)}{\partial t} g^{kl} \times 0 \\ &= \mathbf{Scal}_{\overset{2}{F}}^H + (n-1)(n-2)(u^{''} - u^{'2}). \end{aligned} \tag{6.9}$$

Case 2 : if $i = 1$ and $j \in \{2, 3, ..., n\}$ or $j = 1$ and $i \in \{2, 3, ..., n\}$ that is $t \neq y^i$ or $t \neq y^j$, by the Proposition 6.7 and by the fact that $u = u(t)$, each term in the left-hand side of the equation (1.1) vanishes.

Case 3 : if $i, j \in \{2, 3, ..., n\}$ that is $t \neq y^i$ and $t \neq y^j$, the equation (1.1) becomes

$$\begin{aligned} 0 &= \mathbf{E}_F(\partial_\alpha, \hat{\partial}_\beta) - (n-2)\left(\nabla_\beta \nabla_\alpha u - \nabla_\alpha u \nabla_\beta u\right) \\ &\quad + \frac{(n-2)}{n}\left(\nabla^d \nabla_d u - \nabla^d u \nabla_d u\right) g_{\alpha\beta} \\ &\quad + \frac{(n-1)}{2nF}\left(\nabla_r u \nabla^q u\right) \frac{\partial(F^2 g^{rs} - 2y^r y^s)}{\partial y^q} g^{kl} \mathcal{A}_{skl} g_{\alpha\beta} \\ &= \mathbf{Ric}_{\overset{2}{F}}(\partial_\alpha, \hat{\partial}_\beta) - \frac{1}{n}\mathbf{Scal}_{\overset{2}{F}}^H \overset{2}{g}_{\alpha\beta} \,. \end{aligned} \tag{6.10}$$

Therefore $\widetilde{F} = e^u F = e^u \sqrt{t^2 + \overset{2}{F}{}^2}$ is locally an Einstein metric if and only if

$$\begin{aligned} 0 &= \begin{cases} \mathbf{Scal}_{\overset{2}{F}}^H + (n-1)(n-2)(u^{''} - u^{'2}) \\ \mathbf{Ric}_{\overset{2}{F}}(\partial_\alpha, \hat{\partial}_\beta) - \frac{1}{n}\mathbf{Scal}_{\overset{2}{F}}^H \overset{2}{g}_{\alpha\beta} \end{cases} \\ &= \begin{cases} \mathbf{Scal}_{\overset{2}{F}}^H + (n-1)(n-2)(u^{''} - u^{'2}) \\ \mathbf{E}_{\overset{2}{F}}(\partial_\alpha, \hat{\partial}_\beta) \text{ for } \alpha, \beta \in \{2, ..., n\}. \end{cases} \end{aligned} \tag{6.11}$$

From the system (6.11), $\overset{2}{F}$ is $R$-Einstein. By the Lemma 3.10, $\mathbf{Scal}^H_{\overset{2}{F}} = constant$. Denote this constant by $s$. The system (6.11) becomes $u^{''} - u^{'2} + \frac{s}{(n-1)(n-2)} = 0$ or equivalently

$$u^{''} - u^{'2} + s^* = 0 \tag{6.12}$$

where $s^* := \frac{s}{(n-1)(n-2)}$. We set $e^u = \varphi^{-1} = \frac{1}{\varphi}$. Then

$$u^{'} = -\frac{\varphi^{'}}{\varphi},\ u^{'2} = \frac{\varphi^{'2}}{\varphi^2} \quad \text{and} \quad u^{''} = \frac{-\varphi^{''}\varphi + \varphi^{'2}}{\varphi^2}.$$

The equation (6.12) becomes

$$\varphi^{''} - \varphi s^* = 0. \tag{6.13}$$

We distinguish three cases :

(i) $s^* = 0$. The general solution $\varphi$ of the equation (6.13) is

$$\varphi(t) = c_1 t + c_2.$$

Thus, the conformal factor satisfies $e^{u(t)} = \varphi^{-1}(t) = \alpha t + \beta$. In particular, if $\alpha = 0$ then $\beta$ must be positive since $e^u$ is the conformal factor.

(ii) $s^* > 0$. The general solution $\varphi$ of the equation (6.13) is

$$\varphi(t) = cosh\left(\sqrt{s^*}t + \gamma\right).$$

(iii) $s^* < 0$. The general solution $\varphi$ of the equation (6.13) is $\varphi(t) = c_5 cos\left(\sqrt{-s^*}t\right) + c_6 sin\left(\sqrt{-s^*}t\right)$. Setting $c_5 = \mu cos\theta$ and $c_6 = -\mu sin\theta$ the last relation can be expressed as $\varphi(t) = \mu\left[cos\left(\sqrt{-s^*}t + \theta\right)\right]$.

Conversely, if one of the cases $(i)$, $(ii)$ and $(iii)$ is holds then $e^u F$ is $R$-Einstein. □

**Example 6.8.** *Let $\overset{2}{F}$ be a Finslerian metric on the sphere $\mathbb{S}^{n-1}$ with positive constant flag curvature $k = 1$. We can show $\overset{2}{F}$ is of horizantal scalar curvature $\boldsymbol{Scal}^H_{\overset{2}{F}} = (n-1)(n-2)$. Then the Finslerian metric $F = \sqrt{t^2 + \overset{2}{F}{}^2}$ is locally conformal to the R-Einstein metric $\widetilde{F} = cosh^{-1}tF$ for $t \in (1, \infty)$.*

## 7. Non-product metrics locally conformally $R$-Einstein

We give the following.

**Definition 7.1.** *Let $(M, F)$ be a Finslerian manifold of dimension $n \geq 3$.*

(1) *The horizontal Schouten tensor of $(M, F)$ is the $(0,0;1,1)$-tensor given by*

$$\boldsymbol{S}^H_F = \frac{1}{n-2}\left(\boldsymbol{Ric}^H_F - \frac{1}{2(n-1)}\boldsymbol{Scal}^H_F \underline{g}\right). \tag{7.1}$$

(2) *The horizontal Weyl tensor of* $(M, F)$ *is the* $(0, 0; 2, 2)$*-tensor defined by*

$$\boldsymbol{W}_F^H = \boldsymbol{R} - g \odot \boldsymbol{S}_F^H. \tag{7.2}$$

*Its components in a local coordinate are defined as follows,*

$$\begin{aligned} \boldsymbol{W}_F(\partial_l, \partial_i, \hat{\partial}_j, \hat{\partial}_k) \quad = \quad & \boldsymbol{R}(\partial_l, \partial_i, \hat{\partial}_j, \hat{\partial}_k) \\ & - g_{lj} \boldsymbol{S}_F(\partial_i, \hat{\partial}_k) - g_{ik} \boldsymbol{S}_F(\partial_l, \hat{\partial}_j) \\ & + g_{lk} \boldsymbol{S}_F(\partial_i, \hat{\partial}_j) + g_{ij} \boldsymbol{S}_F(\partial_l, \hat{\partial}_k). \end{aligned} \tag{7.3}$$

(3) *The horizontal tensor of type Cotton-York of* $(M, F)$ *is the* $(0, 0; 1, 2)$*-tensor* $\boldsymbol{C}_F^H$ *defined by*

$$\boldsymbol{C}_F^H(\xi, X, Y) = \left(\nabla_X \boldsymbol{S}_F^H\right)(\xi, Y) - \left(\nabla_Y \boldsymbol{S}_F^H\right)(\xi, X) \tag{7.4}$$

*for every* $\xi \in \Gamma(\pi^* TM)$ *and* $X, Y \in \chi(\mathring{T}M)$*. In a local chart,*

$$\boldsymbol{C}_F(\partial_i, \hat{\partial}_j, \hat{\partial}_k) \quad = \quad (\nabla_j \boldsymbol{S}_F)(\partial_i, \hat{\partial}_k) - (\nabla_k \boldsymbol{S}_F)(\partial_i, \hat{\partial}_j). \tag{7.5}$$

In dimension greater than 3, we introduce in Finselrian Geometry the following tensor.

**Definition 7.2.** *The horizontal tensor of type Bach of a Finslerian manifold* $(M, F)$ *is the* $(1, 1; 0)$*-tensor* $\boldsymbol{B}_F^H$ *defined by*

$$\boldsymbol{B}_F(\partial_i, \hat{\partial}_j) = \nabla^k \boldsymbol{C}_F(\partial_i, \hat{\partial}_j, \hat{\partial}_k) + \boldsymbol{S}_F^{lk} \boldsymbol{W}_F(\partial_l, \partial_i, \hat{\partial}_k, \hat{\partial}_j). \tag{7.6}$$

We have the following properties.

**Lemma 7.3.** *Let* $(M, F)$ *be a Finslerian manifold of dimension* $n \geq 3$*. Then,*

(1) *the horizontal tensors of Weyl and of Cotton-York tensors are related as follows :*

$$\nabla^l \boldsymbol{W}_F(\partial_l, \partial_i, \hat{\partial}_j, \hat{\partial}_k) = (n-3) \boldsymbol{C}_F(\partial_i, \hat{\partial}_j, \hat{\partial}_k)$$

(2) *if* $F$ *is horizontally an Einstein metric then its horizontal Cotton-York tensor vanishes*

$$\boldsymbol{C}_F(\partial_i, \hat{\partial}_j, \hat{\partial}_k) = 0..$$

*Proof.* (1) Contracting the Finslerin second Bianchi identity given in Lemma 3.9 we get $g^{sl}\Big[\nabla_j \mathbf{R}_{liks} + \nabla_k \mathbf{R}_{lisj} + \nabla_s \mathbf{R}_{lijk}\Big] = 0$. Equivalent

$$-\nabla_j \mathbf{Ric}_F(\partial_i, \hat{\partial}_k) + \nabla_k \mathbf{Ric}_F(\partial_i, \hat{\partial}_j) + \nabla^l \mathbf{R}_{lijk} = 0.$$

Using this relation we have

$$\begin{aligned}
\nabla^l \mathbf{W}_F(\partial_l, \partial_i, \hat{\partial}_j, \hat{\partial}_k) &= g^{ls}\nabla_s \mathbf{W}_F(\partial_l, \partial_i, \hat{\partial}_j, \hat{\partial}_k) \\
&\overset{(7.3)}{=} g^{ls}\nabla_s\Big\{\mathbf{R}(\partial_l, \partial_i, \hat{\partial}_j, \hat{\partial}_k) \\
&\quad -\frac{1}{n-2}\Big[\mathbf{Ric}_F(\partial_l, \hat{\partial}_j)g_{ik} - \mathbf{Ric}_F(\partial_i, \hat{\partial}_j)g_{lk} \\
&\quad +\mathbf{Ric}_F(\partial_i, \hat{\partial}_k)g_{lj} - \mathbf{Ric}_F(\partial_l, \hat{\partial}_k)g_{ij}\Big] \\
&\quad +\frac{\mathbf{Scal}^H_F}{(n-1)(n-2)}\left[g_{ij}g_{lk} - g_{ik}g_{lj}\right]\Big\} \\
&= \frac{n-3}{n-2}\Big(\nabla_j \mathbf{Ric}_F(\partial_i, \hat{\partial}_k) - \nabla_k \mathbf{Ric}_F(\partial_i, \hat{\partial}_j)\Big) \\
&\quad -\frac{n-3}{2(n-1)(n-2)}\Big(\nabla_j \mathbf{Scal}^H_F g_{ik} - \nabla_k \mathbf{Scal}^H_F g_{ij}\Big) \\
&\overset{(7.5)}{=} (n-3)\mathbf{C}_F(\partial_i, \hat{\partial}_j, \hat{\partial}_k).
\end{aligned}$$

(2) If $F$ is $R$-Einstein then, by Lemma 3.10, $\mathbf{Scal}^H_F$ is constant. Hence,

$$\begin{aligned}
\mathbf{C}_F(\partial_i, \hat{\partial}_j, \hat{\partial}_k) &= \left(\nabla_j \mathbf{S}_F\right)(\partial_i, \hat{\partial}_k) - \left(\nabla_k \mathbf{S}_F\right)(\partial_i, \hat{\partial}_j) \\
&= \nabla_j\Big[\frac{1}{n-2}\Big(\mathbf{Ric}^H_F - \frac{1}{2(n-1)}\mathbf{Scal}^H_F \underline{g}\Big)\Big](\partial_i, \hat{\partial}_k) \\
&\quad -\nabla_k\Big[\frac{1}{n-2}\Big(\mathbf{Ric}^H_F - \frac{1}{2(n-1)}\mathbf{Scal}^H_F \underline{g}\Big)\Big](\partial_i, \hat{\partial}_j).
\end{aligned}$$

Hence, formula (3.5) implies relation (7.7).

□

**Lemma 7.4.** *Let $F$ be a Finslerian metric on a manifold of dimension $n \geq 3$. If $\widetilde{F}$ is a conformal deformation of $F$, with $\widetilde{F} = e^u F$, then*

(1) *the horizontal Schouten tensor behaves as follows :*

$$\widetilde{\boldsymbol{S}}^H_{\widetilde{F}}(\partial_i, \hat{\partial}_j) = \boldsymbol{S}^H_F(\partial_i, \hat{\partial}_j) - \nabla_j\nabla_i u + \nabla_i u \nabla_j u + h g_{ij} \qquad (7.7)$$

*where*

$$\begin{aligned}
h: &= -\frac{1}{2}\nabla^k u \nabla_k u + \frac{\nabla^s u g^{kl}}{n(n-1)}\Big[(n+8)\mathcal{B}^{s_1}_l \mathcal{A}_{sks_1} - 2\mathcal{B}^{s_1}_s \mathcal{A}_{lks_1}\Big] \\
&\quad +\frac{1}{2n(n-1)}g^{kl}g^{rs}\Big\{\Big[g(\Theta(\hat{\partial}_s, \boldsymbol{h}(\Theta_{lr})), \partial_k) - g(\Theta(\hat{\partial}_l, \boldsymbol{h}(\Theta_{rs})), \partial_k)\Big] \\
&\quad +\Big[g((\nabla_s\Theta)_{lr}, \partial_k) - g((\nabla_l\Theta)_{rs}, \partial_k)\Big]\Big\}.
\end{aligned}$$

(2) *if a horizontal* $(1,1,0)$*-tensor* $\boldsymbol{T}_F^H$ *satisfies* $\boldsymbol{T}_F(\partial_i,\hat{\partial}_j)=\boldsymbol{T}_F(\pi_*\hat{\partial}_j,\boldsymbol{h}(\partial_i))$, *for any* $i,j=1,...,n$, *then*

$$\begin{aligned}\left(\widetilde{\nabla}_j\boldsymbol{T}_F\right)(\partial_i,\hat{\partial}_k) &= \left(\nabla_j\boldsymbol{T}_F\right)(\partial_i,\hat{\partial}_k)\\ &\quad -2\nabla_j u\,\boldsymbol{T}_{ik}-\nabla_i u\,\boldsymbol{T}_{jk}-\nabla_k u\,\boldsymbol{T}_{ij}\\ &\quad +g_{ij}\,\boldsymbol{T}_F(\nabla u,\partial_k)+g_{jk}\,\boldsymbol{T}_F(\partial_i,\boldsymbol{h}(\nabla u))\\ &\quad -\boldsymbol{T}_F(\partial_i,\boldsymbol{h}(\Theta)_{jk})-\boldsymbol{T}_F(\Theta_{ij},\hat{\partial}_k). \end{aligned}\tag{7.8}$$

(3) *the horizontal Cotton-York tensor behaves as follows :*

$$\widetilde{\boldsymbol{C}}_{\widetilde{F}}(\partial_i,\hat{\partial}_j,\hat{\partial}_k)=\boldsymbol{C}_F(\partial_i,\hat{\partial}_j,\hat{\partial}_k)+\boldsymbol{W}_F(\nabla u,\partial_i,\hat{\partial}_j,\hat{\partial}_k)+\Psi_u^{\boldsymbol{C}_F^H}(\partial_i,\hat{\partial}_j,\hat{\partial}_k)$$

*where*

$$\begin{aligned}\Psi_u^{\boldsymbol{C}_F^H}(\partial_i,\hat{\partial}_j,\hat{\partial}_k) &= \nabla_j\Big(\nabla_i u\nabla_k u+hg_{ik}\Big)-\nabla_k\Big(\nabla_i u\nabla_j u+hg_{ij}\Big)\\ &\quad +\Gamma^l_{ik}\nabla_l u\nabla_j u-\Gamma^l_{ij}\nabla_l u\nabla_k u\\ &\quad +g_{jl}\nabla^l u\Big(\nabla_k\nabla_i u-hg_{ik}\Big)-g_{kl}\nabla^l u\Big(\nabla_j\nabla_i u-hg_{ij}\Big)\\ &\quad +g_{ij}\nabla_k\nabla_{\boldsymbol{h}(\nabla u)}u+g_{ik}\nabla_j\nabla_{\boldsymbol{h}(\nabla u)}u\\ &\quad -\nabla_j\nabla_{\Theta_{ik}}u+\nabla_{\Theta_{ik}}u\nabla_j u+\nabla_k\nabla_{\Theta_{ij}}u-\nabla_{\Theta_{ij}}u\nabla_k u. \end{aligned}$$

*Proof.* The assertion (1) in Lemma 7.4 is obtained by using the relation (7.1) and the lemmas 5 and 6 in [11].

To obtain the relation (7.8) in Lemma 7.4, we consider a $(1,1,0)$-tensor $\mathbf{T}_F^H$ on $(M,F)$. Then for every vector fields $X,Y$ on $\mathring{T}M$ and for any section $\xi$ of the vector bundle $\pi^*TM$, we obtain

$$\left(\widetilde{\nabla}_X\mathbf{T}_F^H\right)(\xi,Y)=\widetilde{\nabla}_X(\mathbf{T}_F^H(\xi,Y))-\mathbf{T}_F^H(\widetilde{\nabla}_X\xi,Y)-\mathbf{T}_F^H(\xi,\mathbf{h}(\widetilde{\nabla}_X\pi_*Y))\tag{7.9}$$

where $\widetilde{\nabla}_X$ is the covariant derivative with respect to $\widetilde{F}$ in a given direction $X$. We have

$$\begin{aligned}\left(\widetilde{\nabla}_X\mathbf{T}_F^H\right)(\xi,Y) &= \nabla_X(\mathbf{T}_F^H(\xi,Y))-\mathbf{T}_F^H\Big(\nabla_X\xi+du(\pi_*X)\xi+du(\xi)\pi_*X\\ &\quad -g(\pi_*X,\xi)\nabla u+\Theta(X,\mathbf{h}(\xi)),Y\Big)\\ &\quad -\mathbf{T}_F^H(\xi,\mathbf{h}(\nabla_X\pi_*Y+du(\pi_*X)\pi_*Y\\ &\quad +du(\pi_*Y)\pi_*X-g(\pi_*X,\pi_*Y)\nabla u+\Theta(X,Y)))\\ &= \nabla_X(\mathbf{T}_F(\xi,\hat{Y}))-2(\nabla_X u)\mathbf{T}_F(\xi,\hat{Y})\\ &\quad -(\nabla_{\mathbf{h}(\xi)}u)\mathbf{T}_F(\pi_*X,\hat{Y})-(\nabla_Y u)\mathbf{T}_F(\xi,\hat{X})\\ &\quad +g(\xi,\pi_*X)\mathbf{T}_F(\nabla u,\hat{Y})+g(\pi_*X,\pi_*Y)\mathbf{T}_F(\xi,\hat{\nabla u})\\ &\quad -\mathbf{T}_F(\xi,\mathbf{h}(\Theta(X,Y))-\mathbf{T}_F(\Theta(\mathbf{h}(\xi),X),Y). \end{aligned}$$

Setting $\xi=\partial_i$, $X=\hat{\partial}_j$ and $Y=\hat{\partial}_k$, we obtain the relation.

From the these two properties, we obtain the assertion (3) in the Lemma 7.4. □

7.1. **Proof of the Theorem 1.4 in dimension** $n = 3$**.** Let $F$ and $\widetilde{F}$ be two conformal Finslerian metric, with $\widetilde{F} = e^u F$, on a manifold of dimension $n \geq 3$. Then

$$\begin{aligned}
\widetilde{\mathbf{C}}_{\widetilde{F}}(\partial_i, \hat{\partial}_j, \hat{\partial}_k) &\overset{(7.5)}{=} \left(\widetilde{\nabla}_j \widetilde{\mathbf{S}}_{\widetilde{F}}\right)(\partial_i, \hat{\partial}_k) - \left(\widetilde{\nabla}_k \widetilde{\mathbf{S}}_{\widetilde{F}}\right)(\partial_i, \hat{\partial}_j) \\
&\overset{(7.8)}{=} \left(\nabla_j \widetilde{\mathbf{S}}_{\widetilde{F}}\right)(\partial_i, \hat{\partial}_k) - \left(\nabla_k \widetilde{\mathbf{S}}_{\widetilde{F}}\right)(\partial_i, \hat{\partial}_j) \\
&\quad -\nabla_j u \widetilde{\mathbf{S}}_{\widetilde{F}}(\partial_i, \hat{\partial}_k) + \nabla_k u \widetilde{\mathbf{S}}_{\widetilde{F}}(\partial_i, \hat{\partial}_j) \\
&\quad +g_{ij} \widetilde{\mathbf{S}}_{\widetilde{F}}(\nabla u, \partial_k) - g_{ik} \widetilde{\mathbf{S}}_{\widetilde{F}}(\nabla u, \partial_j) \\
&\quad +\widetilde{\mathbf{S}}_{\widetilde{F}}(\Theta_{ik}, \hat{\partial}_j) - \widetilde{\mathbf{S}}_{\widetilde{F}}(\Theta_{ij}, \hat{\partial}_k). \\
&= \nabla_j \left[\widetilde{\mathbf{S}}_{\widetilde{F}}(\partial_i, \hat{\partial}_k)\right] - \widetilde{\mathbf{S}}_{\widetilde{F}}\big(\nabla_j \partial_i, \hat{\partial}_k\big) \\
&\quad -\widetilde{\mathbf{S}}_{\widetilde{F}}\big(\partial_i, \mathbf{h}(\nabla_j \pi_* \hat{\partial}_k)\big) - \Big\{\nabla_k \left[\widetilde{\mathbf{S}}_{\widetilde{F}}(\partial_i, \hat{\partial}_j)\right] \\
&\quad -\widetilde{\mathbf{S}}_{\widetilde{F}}\big(\nabla_k \partial_i, \hat{\partial}_j\big) - \widetilde{\mathbf{S}}_{\widetilde{F}}\big(\partial_i, \mathbf{h}(\nabla_k \pi_* \hat{\partial}_j)\big)\Big\} \\
&\quad -\nabla_j u \widetilde{\mathbf{S}}_{\widetilde{F}}(\partial_i, \hat{\partial}_k) + \nabla_k u \widetilde{\mathbf{S}}_{\widetilde{F}}(\partial_i, \hat{\partial}_j) \\
&\quad +g_{ij} \widetilde{\mathbf{S}}_{\widetilde{F}}(\nabla u, \partial_k) - g_{ik} \widetilde{\mathbf{S}}_{\widetilde{F}}(\nabla u, \partial_j) \\
&\quad +\widetilde{\mathbf{S}}_{\widetilde{F}}(\Theta_{ik}, \hat{\partial}_j) - \widetilde{\mathbf{S}}_{\widetilde{F}}(\Theta_{ij}, \hat{\partial}_k) \\
&= \mathbf{C}_{\widetilde{F}}(\partial_i, \hat{\partial}_j, \hat{\partial}_k) \\
&\quad +\nabla_j \Big[-\nabla_k g_{il} \nabla^l u + \nabla_i u \nabla_k u + h g_{ik}\Big] \\
&\quad -\nabla_k \Big[-\nabla_j g_{il} \nabla^l u + \nabla_i u \nabla_j u + h g_{ij}\Big] \\
&\quad +\Big[-\nabla_j \nabla_{\nabla_k \partial_i} u + \nabla_{\nabla_k \partial_i} u \nabla_j u + h g(\nabla_k \partial_i, \pi_* \hat{\partial}_j)\Big] \\
&\quad -\Big[-\nabla_k \nabla_{\nabla_j \partial_i} u + \nabla_{\nabla_j \partial_i} u \nabla_k u + h g(\nabla_j \partial_i, \pi_* \hat{\partial}_k)\Big] \\
&\quad -g_{jl} \nabla^l u \Big[\mathbf{S}^H_F(\partial_i, \hat{\partial}_k) - \nabla_k \nabla_i u + \nabla_i u \nabla_k u + h g_{ik}\Big] \\
&\quad +g_{kl} \nabla^l u \Big[\mathbf{S}^H_F(\partial_i, \hat{\partial}_j) - \nabla_j \nabla^l u \partial_l + \nabla_i u \nabla_k u + h g_{ij}\Big] \\
&\quad +g_{ij} \Big[\nabla^l u \mathbf{S}^H_F(\partial_l, \hat{\partial}_k) - \nabla_k \nabla_{\mathbf{h}(\nabla u)} u\Big] \\
&\quad -g_{ik} \Big[\nabla^l u \mathbf{S}^H_F(\partial_l, \hat{\partial}_j) - \nabla_j \nabla_{\mathbf{h}(\nabla u)} u\Big] \\
&\quad +\Big[\mathbf{S}^H_F(\Theta_{ik}, \hat{\partial}_j) - \nabla_j \nabla_{\Theta_{ik}} u + \nabla_{\Theta_{ik}} u \nabla_j u\Big] \\
&\quad -\Big[\mathbf{S}^H_F(\Theta_{ij}, \hat{\partial}_k) - \nabla_k \nabla_{\Theta_{ij}} u + \nabla_{\Theta_{ij}} u \nabla_k u\Big].
\end{aligned}$$

Therefore

$$\widetilde{\mathbf{C}}_{\widetilde{F}}(\partial_i, \hat{\partial}_j, \hat{\partial}_k) = \mathbf{C}_{\widetilde{F}}(\partial_i, \hat{\partial}_j, \hat{\partial}_k) + \nabla^l u \mathbf{W}_F(\partial_l, \partial_i, \hat{\partial}_j, \hat{\partial}_k) + \Psi_u^{\mathbf{C}^H_F}(\partial_i, \hat{\partial}_j, \hat{\partial}_k),$$

where

$$\begin{aligned}\Psi_u^{\mathbf{C}_F^H}(\partial_i,\hat{\partial}_j,\hat{\partial}_k) &= \nabla_j\Big(\nabla_i u\nabla_k u + hg_{ik}\Big) - \nabla_k\Big(\nabla_i u\nabla_j u + hg_{ij}\Big)\\ &\quad +\Gamma_{ik}^l\nabla_l u\nabla_j u - \Gamma_{ij}^l\nabla_l u\nabla_k u + g_{jl}\nabla^l u\Big(\nabla_k\nabla_i u - hg_{ik}\Big)\\ &\quad -g_{kl}\nabla^l u\Big(\nabla_j\nabla_i u - hg_{ij}\Big) + g_{ij}\nabla_k\nabla_{\mathbf{h}(\nabla u)}u + g_{ik}\nabla_j\nabla_{\mathbf{h}(\nabla u)}u\\ &\quad -\nabla_j\nabla_{\Theta_{ik}}u + \nabla_{\Theta_{ik}}u\nabla_j u + \nabla_k\nabla_{\Theta_{ij}}u - \nabla_{\Theta_{ij}}u\nabla_k u.\end{aligned}$$

If $\widetilde{F}$ is $R$-Einstein then by the Lemma 7.3

$$\mathbf{C}_F(\partial_i,\hat{\partial}_j,\hat{\partial}_k) + \mathbf{W}_F(\nabla u,\partial_i,\hat{\partial}_j,\hat{\partial}_k) + \Psi_u^{\mathbf{C}_F^H}(\partial_i,\hat{\partial}_j,\hat{\partial}_k) = 0. \tag{7.10}$$

When $n=3$, the tensor $\mathbf{W}_F^H$ vanishes and hence the equation (7.10) reduces to

$$\mathbf{C}_F(\partial_i,\hat{\partial}_j,\hat{\partial}_k) + \Psi_u^{\mathbf{C}_F^H}(\partial_i,\hat{\partial}_j,\hat{\partial}_k) = 0. \tag{7.11}$$

If $u = constant$ then $\mathbf{C}_F^H \equiv 0$.

7.2. **Proof of the Theorem 1.4 in dimension** $n=4$**.** From the Lemma 7.3, if $\widetilde{F}$ is $R$-Einstein metric then $\widetilde{\mathbf{C}}_{\widetilde{F}}$ vanishes. Then the equation (7.10) holds. Applying $\nabla^k$ to this equation, using the Definition 7.1 and the equation (7.10) again, we get

$$\begin{aligned}0 &= \mathbf{B}_F(\partial_i,\hat{\partial}_j) - \mathbf{S}_F^{lk}\mathbf{W}_F(\partial_l,\partial_i,\hat{\partial}_k,\hat{\partial}_j)\\ &\quad - \left[\nabla^k\nabla^l u - (n-3)\nabla^k u\nabla^l u\right]\mathbf{W}_F(\partial_l,\partial_i,\hat{\partial}_k,\hat{\partial}_j)\\ &\quad -\nabla^k\Psi_u^{\mathbf{C}_F^H}(\partial_i,\hat{\partial}_j,\hat{\partial}_k) + \Psi_u^{\mathbf{C}_F^H}(\partial_i,\hat{\partial}_j,\hat{\partial}_k).\end{aligned} \tag{7.12}$$

Since $\widetilde{F}$ is locally an $R$-Einstein metric, the equation (1.1) is equivalent to

$$\begin{aligned}0 &= \mathbf{S}_F(\partial_i,\hat{\partial}_j) - \frac{1}{n}\mathbf{J}_F^H g_{ij} - \nabla_j\nabla_i u + \nabla_i u\nabla_j u\\ &\quad +\frac{1}{n}\left(\nabla^d\nabla_d u - \nabla^d u\nabla_d u\right)g_{ij}\\ &\quad +\frac{(n-1)}{2n(n-2)F}\left(\nabla_r u\nabla^q u\right)\frac{\partial(F^2g^{rs}-2y^ry^s)}{\partial y^q}g^{kl}\mathcal{A}_{skl}g_{ij}\end{aligned}$$

where $\mathbf{J}_F^H$ is the trace of $\mathbf{Scal}_F^H$. Raising both indices and applying $\mathbf{W}_F(\partial_l,\partial_i,\hat{\partial}_j,\hat{\partial}_k)$ to this equation, using the relation (7.7) in Lemma 7.4 and the equation (7.12) we obtain

$$\begin{aligned}0 &= \mathbf{B}_F(\partial_i,\hat{\partial}_j) + (n-4)\mathbf{W}_F(\nabla u,\partial_i,\hat{\partial}_j,\nabla u)\\ &\quad + \left[(n-3)\nabla^k u - \nabla^k\right]\Psi_u^{\mathbf{C}_F^H}(\partial_i,\hat{\partial}_j,\hat{\partial}_k).\end{aligned}$$

Therefore, in dimension $n=4$, we have $\mathbf{B}_F(\partial_i,\hat{\partial}_j)+\Big(\nabla^k u-\nabla^k\Big)\Psi_u^{\mathbf{C}_F^H}(\partial_i,\hat{\partial}_j,\hat{\partial}_k) = 0$.

If $u = constant$ then $\mathbf{B}_F^H \equiv 0$.

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