Journal of Finsler Geometry and its Applications
Vol. 6, No. 2 (2025), pp 132-146
https://doi.org/10.22098/jfga.2025.17375.1159

Geometric aspects of η-Ricci soliton in Lorentzian β-Kenmotsu manifold

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Abstract. The object of the paper is to study η-Ricci solitons on Lorentzian β-Kenmotsu manifolds, subject to specific curvature conditions. We recall some basic knowledge on Lorentzian β-Kenmotsu manifolds. Then, we deal with η-Ricci solitons on Lorentzian β-Kenmotsu manifolds. Next, we study the η-Ricci solitons in ϕ-projectively semi symmetric Lorentzian β-Kenmotsu manifolds. Afterward, we investigate η-Ricci solitons in Lorentzian β-Kenmotsu manifolds admitting Codazzi type of Ricci tensor and cyclic parallel Ricci tensor. Additionally, we consider η-Ricci solitons on recurrent Lorentzian β-Kenmotsu manifolds. A concrete example has demonstrated the existence of η-Ricci solitons in a Lorentzian β-Kenmotsu manifold.

Keywords: η-Ricci solitons, Einstein manifold, Lorentzian β-Kenmotsu manifold, Projective curvature tensor.

1. Introduction

In the annals of mathematical history, the year 1982 heralded Hamilton's [19] introduction of the Ricci flow, accompanied by a rigorous proof of its

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AMS 2020 Mathematics Subject Classification: 53C05, 53C20,53C25, 53D15

existence. The Ricci flow was developed to support "Thurston's geometric conjecture" which denotes that every closed 3-dimensional manifold admits a geometric decomposition. He demonstrated all compact manifolds into 4-dimensions having a positive curvature.
The equation governing the Ricci flow is

$$\frac{\partial}{\partial t} g(t) = -2Ric,$$

where, 'Ric' is the Ricci tensor of the metric $g(t)$. A solution to this equation (or a Ricci flow) is a one-parameter family of metrics $g(t)$, parameterized by t in a non-degenerate interval I, on a smooth manifold $\mathcal{M}$ satisfying the Ricci flow equation. If I has an initial point t_0, then $(\mathcal{M}, g(t_0))$ is called the initial condition or the initial metric for the Ricci flow (or of the solution) [26].

On a compact Riemannian manifold $\mathcal{M}$ with metric g, a Ricci-soliton is a similar solution to the Ricci-flow, evolving only by a one parameter family of diffeomorphisms and scalings. The Ricci soliton is rigorously characterized by the following equation

$$L_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where L_V is the Lie derivative in the V direction, S is Ricci curvature tensor, g is a Riemannian metric, V is a vector field and λ is a scalar. Metrices satisfying (1.1) are interesting and useful in physics and are often referred to as quasi-Einstein metrics [8, 9]. Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t} g(t) = -2Ric$, projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise blow-up limits for the Ricci flow on compact manifolds. Theoretical physicists have also been investigating the equation of Ricci solitons, exploring its potential connections with string theory. The initial contribution in this direction is due to Friedman [16], who discusses some of its aspects.

The notion of η-Ricci soliton is more inclusive than the conventional Ricci-flow. This idea was put forward by J. T. Cho and Makoto Kimura [10], and they gave its equation by

$$L_\xi g + 2S = -2\lambda g - 2\mu \eta \otimes \eta, \tag{1.2}$$

where, λ and μ are constants.

A Ricci soliton is said to be trivial, if V is either zero or Killing on $\mathcal{M}$. Ricci soliton is considered as a generalization of Einstein metric and often arises as a fixed point of Hamiltons Ricci flow. In [32], Pigoli-Rigoli-Rimoldi-Setti generalized the notion of Ricci soliton to Ricci almost soliton by allowing the soliton constant λ to be a smooth function. In this case, we denote it by $(\mathcal{M}_n, g, V, \lambda)$.

The Ricci soliton is shrinking, steady, and expanding depending on $\lambda < 0$, $\lambda = 0$, $\lambda > 0$, respectively. Otherwise, it will be called indefinite. Moreover, if the potential vector field V is the gradient of some smooth function u on $\mathcal{M}_n$, i.e., $V = Du$, where, D is the gradient operator of g on $\mathcal{M}_n$, then the Ricci soliton is called a gradient Ricci soliton and the soliton becomes

$$Hess\ \ u + Ric = \lambda g,$$

where, $Hess\ \ u$ denotes the $Hessian$ of u. The function u is known as the potential function.

The works of Blaga [4, 5] and Prakasha et al. [33] related to η-Ricci solitons are acknowledged in this paper. In particular, if $\mu = 0$, then the notion of η-Ricci solitons (g, V, λ, μ) reduces to the notion of Ricci solitons (g, V, λ). If $\mu \neq 0$, then the η-Ricci solitons are called proper η-Ricci solitons. We refer to [3, 6, 28] and references therein for a survey and further references on the geometry of Ricci solitons on pseudo-Riemannian manifolds.

Ricci solitons and η-Ricci solitons are considered by many authors in different contexts for instant: on Kahler manifolds [11], on contact and Lorentzian manifolds [1, 2], on K-contact manifolds [36], on α-Sasakian manifolds [25], on Ricci soliton in Lorentzian Para-Sasakian manifolds [24], on 3-dimensional Kenmotsu manifolds [23], on Para-Sasakian manifolds [22], on η-Ricci soliton in Lorentzian Para-Sasakian manifolds [39], on η-Ricci solitons on Para-Kenmotsu manifolds [38], on Para-Sasakian manifolds satisfying pseudo-symmetry curvature conditions [37], on Conformal η-Ricci Soliton in Lorentzian Para-Kenmotsu manifold [31, 35], on some geometric properties of η-Ricci solitons on three-dimensional quasi-para-Sasakian manifolds [30], etc. We also refer to similar studies in [7] and [27]. In 2017, Yaning Wang [41] proved that if cosymplectic manifold $\mathcal{M}_3$ admits a Ricci soliton, then either $\mathcal{M}_3$ is locally flat or the potential vector field is an infinitesimal contact transformation. Also, in [29], authors have provided some insight on trans-Sasakian manifolds. Dey et al. [12] also have set up some new results on conformal η-Einstein soliton. Very recently, η-Ricci soliton and Yamabe soliton and their generalizations and related research have been studied by many authors [13, 14, 15, 18].

Therefore, with the above studies as motivation, we now turn our attention to the behavior of η-Ricci soliton in Lorentzian β-Kenmotsu manifold.

The following sections provide an outline of how this paper is organized: After a succinct introduction, in Section 2, we recall some basic knowledge on trans-Sasakian manifolds. Section 3 deals with η-Ricci soliton on Lorentzian β- Kenmotsu manifold. In the next section, we study the η-Ricci solitons in ϕ-projectively semi symmetric Lorentzian β-Kenmotsu manifolds. In Section 5, we have evolved η-Ricci solitons in Lorentzian β-Kenmotsu manifolds admitting Codazzi type of Ricci tensor and cyclic parallel Ricci tensor. Section 6 deals with η-Ricci solitons on recurrent Lorentzian β-Kenmotsu manifolds. In the last section, we show an example to illustrate the existence of η-Ricci soliton on 3-dimensional Lorentzian β-Kenmotsu manifold.

2. Some Preliminaries on Lorentzian β-Kenmotsu Manifold

An n-dimensional differentiable manifold $\mathcal{M}$ is said to be Lorentzian β-Kenmotsu manifold, if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and Lorentzian metric g which satisfy [34]:

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi\xi = 0, \tag{2.2}$$

$$\eta(\phi L) = 0, \tag{2.3}$$

$$\phi^2 L = L + \eta(L)\xi, \tag{2.4}$$

$$g(L, \xi) = \eta(L), \tag{2.5}$$

$$g(\phi(L), \phi(M)) = g(L, M) + \eta(L)\eta(M), \tag{2.6}$$

for all $L, M \in T(\mathcal{M})$.

Also, a Lorentzian β-Kenmotsu manifold $\mathcal{M}$ satisfies:

$$\nabla_L \xi = -\beta[L + \eta(L)\xi], \tag{2.7}$$

$$\nabla_L \eta(M) = \beta[g(L, M) - \eta(L)\eta(M)], \tag{2.8}$$

$$(\nabla_L \phi)M = \beta[g(\phi L, M) + \eta(M)\phi N], \tag{2.9}$$

where, '∇' denotes the operator of covariant differentiation with respect to the Lorentzian metric g. Further, on a Lorentzian β-Kenmotsu manifold $\mathcal{M}$, the following relations hold [34]:

$$\eta(R(L, M)N) = \beta^2[g(L, N)\eta(M) - g(M, N)\eta(L)], \tag{2.10}$$

$$R(\xi, L)M = \beta^2(\eta(M)L - g(L, M)\xi), \tag{2.11}$$

$$R(L, M)\xi = \beta^2(\eta(L)M - \eta(M)L), \tag{2.12}$$

$$S(L, \xi) = -(n-1)\beta^2\eta(L), \tag{2.13}$$

$$Q\xi = -(n-1)\beta^2\xi, \tag{2.14}$$

$$S(\xi, \xi) = (n-1)\beta^2, \tag{2.15}$$

$$g(\xi, \xi) = \eta(\xi) = -1, \tag{2.16}$$

where Q, R, S are the Ricci operator, Riemannian curvature and Ricci tensor, respectively, and β is a constant. Also, S and Q are related by

$$S(L, M) = g(QL, M)$$

for all $L, M \in \chi(\mathcal{M})$.

Definition 2.1. *An n-dimensional Lorentzian β-Kenmotsu manifold with constants a, b and c and vector fields L, M defined on $\mathcal{M}$, is called a Generalized η-Einstein manifold if it satisfies the condition*

$$S(L, M) = a\ g(L, M) + b\ \eta(L)\eta(M) + cg(\phi L, M).$$

Moreover, this Generalized η-Einstein manifold is called [21] η-Einstein, Einstein and a special type of generalized η-Einstein manifold according as $c = 0$, $b = c = 0$ and $b = 0$, respectively.

Definition 2.2. *An n-dimensional Lorentzian β-Kenmotsu manifold is called Projective curvature tensor P, if*

$$P(L, M)N = R(L, M)N - \frac{1}{(n-1)}[S(M, N)L - S(L, N)M], \tag{2.17}$$

where, R is the Riemannian curvature tensor and r is the scalar curvature of the manifold.

3. η-Ricci soliton on Lorentzian β-Kenmotsu Manifold

Assume that a Lorentzian β-Kenmotsu manifold admits an η-Ricci soliton (g, ξ, λ, μ) [21]. Then equation (1.2) holds, and therefore, we have

$$(\mathcal{L}_\xi g)(L, M) + 2S(L, M) + 2\lambda g(L, M) + 2\mu\eta(L)\eta(M) = 0. \tag{3.1}$$

As we know, in a Lorentzian β-Kenmotsu manifold following relation holds:

$$(\mathcal{L}_\xi g)(L, M) = g(\nabla_L \xi, M) + g(L, \nabla_M \xi) = -2\beta g(\phi L, \phi M). \tag{3.2}$$

On combining equation (3.1) and equation (3.2), we get

$$S(L, M) = -\lambda g(L, M) + \beta g(\phi L, \phi M) - \mu\eta(L)\eta(M). \tag{3.3}$$

It provides

$$QL = -\lambda L + \beta L + (\beta - \mu)\eta(L)\xi. \tag{3.4}$$

Putting $M = \xi$ in equation (3.3) and using equation (2.1), equation (2.2), equation (2.3) and equation (2.5), we have

$$S(L, \xi) = (\mu - \lambda)\eta(L). \tag{3.5}$$

From equation (2.13) and equation (3.5), it follows that

$$\mu - \lambda = -(n-1)\beta^2. \tag{3.6}$$

Consequently, in view of equation (3.3) and equation (3.6), we can state the following theorem:

Theorem 3.1. *If an n-dimensional Lorentzian β-Kenmotsu manifold with the structure (g, ξ, λ, μ) admits η-Ricci soliton, then the manifold becomes generalized η-Einstein manifold of the form:*

$$S(L, M) = -\lambda g(L, M) + \beta g(\phi L, \phi M) - \mu\eta(L)\eta(M).$$

and

$$\mu - \lambda = -(n-1)\beta^2.$$

In case, if we take μ=0 in equation (3.3) and equation (3.6), then we obtain

$$S(L,M) = -\lambda g(L,M) + \beta g(\phi L, \phi M), \tag{3.7}$$

and

$$\lambda = (n-1)\beta^2, \tag{3.8}$$

respectively. As a consequence, we can state the following corollary:

Corollary 3.2. *If an n-dimensional Lorentzian β-Kenmotsu manifold with the structure (g, ξ, λ) admits Ricci soliton, then the manifold becomes a special type of generalized η-Einstein manifold and its Ricci soliton is always expanding.*

Now, we assume that an Lorentzian β-Kenmotsu manifold with the structure (g, V, λ, μ) is Ricci soliton, such that V is pointwise collinear with ξ, [21] i.e., $V = b\xi$, where b is a function then with the help of equation (1.2). Then, we get

$$bg(\nabla_L \xi, M) + (Lb)\eta(M) + bg(L, \nabla_M \xi) + (Mb)\eta(L) + 2S(L,M) + 2\lambda g(L,M)$$
$$+2\mu\eta(L)\eta(M) = 0,$$

which in view of equation (2.7) converts in the form

$$-2b\beta g(\phi L, \phi M) + (Lb)\eta(M) + (Mb)\eta(L) + 2S(L,M) + 2\lambda g(L,M) \tag{3.9}$$
$$+2\mu\eta(L)\eta(M) = 0.$$

Taking $M = \xi$ in equation (3.9) and making use of equation (2.1), equation (2.3), equation (2.5) and equation (2.12), we infer

$$-(Lb) + [(b\xi) - 2(n-1)\beta^2 + 2\lambda - 2\mu]\eta(M) = 0. \tag{3.10}$$

Again, if we take $L = \xi$ in equation (3.10) and make use of equation (2.1), we find

$$(b\xi) - (n-1)\beta^2 + \lambda - \mu = 0. \tag{3.11}$$

Taking together the equation (3.10) and equation (3.11), it follows that

$$db = [-(n-1)\beta^2 + \lambda - \mu]\eta. \tag{3.12}$$

If we now apply the exterior derivative 'd' on equation (3.12), we arrive at

$$[-(n-1)\beta^2 + \lambda - \mu]d\eta = 0.$$

which yields

$$\mu - \lambda = -(n-1)\beta^2, d\eta \neq 0. \tag{3.13}$$

Consequently, from equation (3.12) and equation (3.13), we obtain $db = 0$, i.e., b is a constant. For this reason, equation (3.9) takes the form

$$S(L,M) = -\lambda g(L,M) + b\beta g(\phi L, \phi M) - \mu\eta(L)\eta(M). \tag{3.14}$$

Hence, in view of equation (3.13) and equation (3.14), we can state the following theorem:

Theorem 3.3. *If an n-dimensional Lorentzian β-Kenmotsu manifold with the structure (g, ξ, λ, μ) admits η-Ricci soliton, such that V is pointwise collinear with ξ, then V is a constant multiple of ξ and the manifold becomes a generalized η-Einstein manifold of the form $S(L,M) = -\lambda g(L,M) + b\beta g(\phi L, \phi M) - \mu\eta(L)\eta(M)$ and $\mu - \lambda =$ (n -1)β^2.*

4. ϕ-projectively semi symmetric Lorentzian β-Kenmotsu manifolds admitting η-Ricci solitons

In this section, we focus on the study of η-Ricci solitons in ϕ-projectively semi symmetric Lorentzian β-Kenmotsu manifolds. Firstly, we see the definition of ϕ-projectively semi symmetric. After that, we evaluate some results using the definition of ϕ-projectively semi symmetric.

Definition 4.1. *An n-dimensional Lorentzian β-Kenmotsu manifold is said to be ϕ-projectively semi symmetric, if* [20]

$$P(L,M)\phi = 0,$$

for all L, M on $\in \chi(\mathcal{M})$.

Let $\mathcal{M}$ be an n-dimensional ϕ-projectively semi symmetric Lorentzian β-Kenmotsu manifold admitting η-Ricci soliton. Therefore

$$P(L,M)\phi = 0.$$

which yields

$$(P(L,M)\phi)N = P(L,M)\phi N - \phi P(L,M)N = 0, \tag{4.1}$$

for any vector fields L, M, $N \in T(\mathcal{M})$. From equation (2.15), it follows that

$$P(L,M)\phi N = R(L,M)\phi N - \frac{1}{(n-1)}\Big[S(M,\phi N)L - S(L,\phi N)M\Big], \tag{4.2}$$

$$\phi P(L,M)N = \phi R(L,M)N - \frac{1}{n-1}\Big[S(M,N)\phi L - S(L,N)\phi M\Big]. \tag{4.3}$$

On combining the equation (4.1), equation (4.2) and equation (4.3), we obtain

$$R(L,M)\phi N - \phi R(L,M)N - \frac{1}{n-1}\Big[S(M,\phi N)L - S(L,\phi N)M\Big] \tag{4.4}$$
$$+\frac{1}{n-1}\Big[S(M,N)\phi L - S(L,N)\phi M\Big] = 0.$$

Putting $M = \xi$ and using equation (2.3), equation (2.9) and equation (2.12), we get

$$S(L, \phi N) = -(n-1)\beta^2(g(L, \phi N). \tag{4.5}$$

Taking into consideration equation (3.3), equation (4.5) converts into

$$[-\lambda + \beta + (n-1)\beta^2]g(L, \phi N) = 0. \tag{4.6}$$

Consequently, $\lambda = \beta + (n\text{-}1)\beta^2$ and hence from equation (3.6), we get $\mu = \beta$.

Now, we are ready to state the following theorem:

Theorem 4.2. *If an n-dimensional ϕ-projectively semi symmetric Lorentzian β-Kenmotsu manifold with the structure (g, ξ, λ, μ) admits η-Ricci soliton, then $\lambda = \beta + (n\text{-}1)\beta^2$ and $\mu = \beta$.*

Again, from the relations equation (3.3), equation (3.6) and equation (4.6), we obtain

$$S(L, M) = -(n-1)\beta^2 g(L, M). \tag{4.7}$$

This leads to the following corollary:

Corollary 4.3. *An n-dimensional ϕ-projectively semi symmetric Lorentzian β-Kenmotsu manifold with the structure (g, ξ, λ, μ) that admits η-Ricci soliton is an Einstein manifold.*

5. η-Ricci solitons in Lorentzian β-Kenmotsu manifolds admitting Codazzi type of Ricci tensor and cyclic parallel Ricci tensor

In this section, we study η-Ricci solitons in Lorentzian β-Kenmotsu manifolds admitting Codazzi type of Ricci tensor and cyclic parallel Ricci tensor [17].

Definition 5.1. *An n-dimensional Lorentzian β-Kenmotsu manifold is said to have Codazzi type of Ricci tensor, if its Ricci tensor S of type (0,2) is non-zero and satisfies the following condition:*

$$(\nabla_L S)(M, N) = (\nabla_M S)(L, N),$$

for all $L, M, N \in T(\mathcal{M})$.

Taking covariant derivative of equation (3.3) and making use of equation (2.7) and equation (2.14), we find

$$(\nabla_L S)(M, N) = \beta^2\Big[g(L, M)\eta(N) - g(L, N)\eta(M)\Big] - \beta\mu\Big[\eta(N)g(\phi L, \phi M) \tag{5.1}$$
$$+ g(\phi L, \phi N)\eta(M)\Big].$$

If, we take Ricci tensor S as Codazzi type then, we have

$$(\nabla_L S)(M, N) = (\nabla_M S)(L, N). \tag{5.2}$$

Taking into consideration equation (5.1), equation (5.2) converts into

$$\beta^2\Big[\eta(L)g(M,N) - g(L,N)\eta(M)\Big] - \beta\mu\Big[\eta(M)g(\phi L,\phi N) + \eta(L)g(\phi M,\phi N)\Big],$$

Putting L=ξ and using equation (2.1)-(2.3), equation (2.5) and equation (2.6), we get

$$g(\phi M,\phi N)[\beta\mu - \beta^2] = 0, \tag{5.3}$$

As we know, $g(\phi M,\phi N) \neq 0$,

$$\mu = \beta \tag{5.4}$$

and

$$\lambda = \beta + (n-1)\beta^2. \tag{5.5}$$

Now from above, we are able to state the following theorem:

Theorem 5.2. *Let an n-dimensional Lorentzian β-Kenmotsu manifold $\mathcal{M}$ with the structure ($\mathcal{M}$, g, ξ, λ, μ) admits η-Ricci soliton. If this manifold $\mathcal{M}$ has Ricci tensor of Codazzi type, then λ= β +(n-1)β^2 and μ=β.*

Definition 5.3. *A Lorentzian β-Kenmotsu manifold n-dimensional is said to have cyclic parallel Ricci tensor, if its Ricci tensor S of type (0,2) is non-zero and satisfies the following condition:*

$$(\nabla_L S)(M,N) + (\nabla_M S)(N,L) + (\nabla_N S)(L,M) = 0, \tag{5.6}$$

for all $L, M, N \in T(\mathcal{M})$.

If, we assume (g, ξ, λ, μ) as η-Ricci soliton in an n-dimensional Lorentzian β-Kenmotsu manifold having cyclic parallel Ricci tensor, then equation (5.6) holds.
Now, if we take covariant derivative of equation (3.3) and make use of equation (2.8) and equation (2.9), we obtain

$$(\nabla_L S)(M,N) = \beta^2\Big[g(L,M)\eta(N) - g(L,N)\eta(M)\Big] - \beta\mu\Big[\eta(N)g(\phi L,\phi M) + g(\phi L,\phi N)\eta(M)\Big]. \tag{5.7}$$

In the similar manner, we have

$$(\nabla_M S)(N,L) = \beta^2\Big[g(M,N)\eta(L) - g(M,L)\eta(N)\Big] - \beta\mu\Big[\eta(L)g(\phi M,\phi N) + g(\phi M,\phi L)\eta(N)\Big], \tag{5.8}$$

and

$$(\nabla_N S)(L,M) = \beta^2\Big[g(N,L)\eta(M) - g(M,L)\eta(N)\Big] - \beta\mu\Big[\eta(M)g(\phi N,\phi L) + g(\phi N,\phi M)\eta(L)\Big]. \tag{5.9}$$

By using equation (5.7) and equation (5.8) in equation (5.6), we get

$$\beta\mu\Big[\eta(N)g(\phi L,\phi M)+\eta(M)g(\phi L,\phi N)+\eta(L)g(\phi M,\phi N)\Big]=0,$$

Putting N=ξ, above equation reduces to

$$\beta\mu g(\phi L,\phi M)=0. \tag{5.10}$$

As we know, the manifold under consideration is non-cosymplectic and

$$g(\phi L,\phi M)\neq 0.$$

Therefore, equation (5.10) provides μ=0. Thus, the η-Ricci soliton becomes Ricci soliton.
As a consequence, we can state the following theorem:

Theorem 5.4. *An n-dimensional non-cosymplectic Lorentzian β- Kenmotsu manifold admitting η-Ricci soliton, whose Ricci tensor is of Codazzi type becomes a Ricci soliton.*

6. η-Ricci solitons on recurrent Lorentzian β-Kenmotsu manifolds

Again in this section, we evaluate some results using definition of η-Ricci solitons on recurrent Lorentzian β-Kenmotsu manifolds [40].

Definition 6.1. *An n-dimensional Lorentzian β-Kenmotsu manifold is said to be recurrent, if there exists a non-zero* 1*-form A such that* [21]

$$(\nabla_L R)(M,N)U=A(L)R(M,N)U, \tag{6.1}$$

for all vector fields L,M,N and U on $\mathcal{M}$. If the 1*-form A vanishes, then the manifold reduces to a symmetric manifold.*

Assume that, $\mathcal{M}$ is a recurrent Lorentzian β-Kenmotsu manifold. Therefore, the curvature tensor of the manifold satisfies equation (6.1). By a suitable contraction of equation (6.1), we get

$$(\nabla_L S)(N,U)=A(L)S(N,U). \tag{6.2}$$

The above equation implies that

$$(\nabla_L S)(N,U)-S(\nabla_L N,U)-S(N,\nabla_L U)=A(L)S(N,U), \tag{6.3}$$

Putting $U=\xi$ and using equation (2.6) and equation (2.12), we have

$$S(N,\phi^2 L)=(n-1)\beta^2 g(\phi L,\phi N)-\beta(n-1)A(L)\eta(N). \tag{6.4}$$

Taking into consideration equation (3.3), equation (6.4) takes the form

$$\beta g(\phi L,\phi N)=\Big[(\lambda+(n-1)\beta^2)\Big]g(\phi L,\phi N)-(n-1)\beta A(L)\eta. \tag{6.5}$$

Let us suppose that the associated 1-form A is equal to the associated 1-form η, then from equation (6.4), we get

$$\beta g(\phi L, \phi N) = [\lambda + (n-1)\beta^2] g(\phi L, \phi N) - (n-1)\beta\eta(L)\eta(N). \tag{6.6}$$

Replacing N by $\phi(N)$ in equation (6.6), we get

$$[\lambda + (n-1)\beta^2 - \beta] g(\phi L, N) = 0. \tag{6.7}$$

Now, since $g(\phi L, N) \neq 0$. We obtain
$\lambda = -(n-1)\beta^2 + \beta$ and hence from equation (3.6), we have $\mu = \beta$.
Hence, we can state the following theorem:

Theorem 6.2. *If an n-dimensional recurrent Lorentzian β-Kenmotsu manifold with the structure (g, ξ, λ, μ) is η-Ricci soliton, then*

$$\lambda = -(n-1)\beta^2 + \beta,$$

and

$$\mu = \beta. \tag{6.8}$$

Now, from the relations equation (3.3), equation (3.6) and equation (6.7), we obtain

$$S(L, M) = -(n-1)\beta^2 g(L, M). \tag{6.9}$$

This leads to the following corollary:

Corollary 6.3. *An n-dimensional recurrent Lorentzian β-Kenmotsu manifold with the structure (g, ξ, λ, μ) admitting η-Ricci soliton is an Einstein manifold.*

7. Examining a 3-Dimensional Lorentzian β-Kenmotsu manifold: An Illustrative Example

We take into consideration the three-dimensional manifold $\mathcal{M}=(l, m, n) \in R^3, n \neq 0$, where (l, m, n) are the standard coordinates of R^3. For each point on $\mathcal{M}$, the vector fields

$$e_1 = n^2 \frac{\partial}{\partial l}, e_2 = n^2 \frac{\partial}{\partial m}, e_3 = \frac{\partial}{\partial n},$$

are linearly independent.
Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0,$$
$$g(e_1, e_1) = g(e_2, e_2) = 1, g(e_3, e_3) = -1.$$

Let η be the 1-form defined by

$$\eta(N) := g(N, e_3), \quad \text{for any } N \in \chi(\mathcal{M}).$$

Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.$$

Then utilizing the linearity of g and ϕ, we get

$$\eta(e_3) = 1, \phi^2 N = -N + \eta(N)e_3,$$

$$g(\phi N, \phi U) = g(N,U) - \eta(N)\eta(U),$$

for any $N, U \in \chi(\mathcal{M})$.
Now, through direct computation, we have

$$[e_1, e_2] = 0, [e_2, e_3] = -\frac{2}{n}e_2, [e_1, e_3] = -\frac{2}{n}e_1.$$

The Koszul's formula below describes the Riemannian connection ∇ associated with the metric tensor g

$$\begin{aligned} 2g(\nabla_L M, N) &= Lg(M,N) + Mg(N,L) - Ng(L,M) \\ &\quad -g\Big(L, [M,N] - g(M,[L,N] + g(N,[L,M]\Big). \end{aligned} \tag{7.1}$$

Using (7.1), we have

$$2g\Big(\nabla_{e_1} e_3, e_1\Big) = 2g\Big(-\frac{2}{z}e_1, e_1\Big),$$

$$2g(\nabla_{e_1} e_3, e_2) = 0,$$

and

$$2g(\nabla_{e_1} e_3, e_3) = 0.$$

Therefore,

$$\nabla_{e_1} e_3 = -\frac{2}{n}e_1.$$

In a similar way, we get

$$\nabla_{e_2} e_3 = -\frac{2}{n}e_2, \quad \nabla_{e_3} e_3 = 0,$$

which further yields

$$\begin{aligned} &\nabla_{e_1} e_2 = 0, && \nabla_{e_1} e_1 = \frac{2}{n}e_3, && \nabla_{e_2} e_2 = \frac{2}{n}e_3, \\ &\nabla_{e_2} e_1 = 0, && \nabla_{e_3} e_2 = 0, && \nabla_{e_3} e_1 = 0. \end{aligned}$$

From the above, it follows that the manifold satisfies

$$\nabla_L \xi = \beta^2, \quad \text{for } \xi = e_3,$$

where $\beta^2 = 2/N$. Thus, we can assert that, $\mathcal{M}$ is β-Kenmotsu manifold. In this way, we know

$$R(L,M)N = \nabla_L \nabla_M N - \nabla_M \nabla_L N - \nabla_{[L,M]} N. \tag{7.2}$$

With the above mentioned formula and using (7.2), it's easy to verify that

$$R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = -\frac{6}{n^2}e_2, \quad R(e_1, e_3)e_3 = -\frac{6}{n^2}e_1,$$

$$R(e_1,e_2)e_3=-\frac{4}{n^2}e_1,\quad R(e_1,e_2)e_3=-\frac{6}{n^2}e_3,\quad R(e_1,e_2)e_3=0,$$
$$R(e_1,e_2)e_3=-\frac{4}{n^2}e_2,\quad R(e_1,e_2)e_3=0,\quad R(e_1,e_2)e_3=-\frac{6}{n^2}e_3.$$

Using the expression for curvature tensor provided above, we derive

$$S(e_1,e_1)=g\Big(R(e_1,e_2)e_2,e_1\Big)+g\Big(R(e_1,e_3)e_3,e_1\Big)=-\frac{10}{n^2}.$$

Likewise, we have

$$S(e_2,e_2)=-\frac{10}{n^2},$$

and

$$S(e_3,e_3)=-\frac{12}{n^2}.$$

Considering the given expressions for curvature tensors and Ricci tensor, we are now able to conclude that

$$P(e_1,e_2)e_3=P(e_1,e_3)e_3=P(e_2,e_3)e_3=0,$$

i.e., $\mathcal{M}$ is projectively flat.

Acknowledgment: The authors express their sincere gratitude to the anonymous reviewer for their helpful feedback and constructive recommendations that have improved the clarity and quality of the manuscript.

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Received: 22.05.2025
Accepted: 01.07.2025