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Geometry of warped tangent bundles with Ricci-Flatness and shrinking solitions

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Abstract. In this paper, we investigate the geometric structure of the tangent bundle of a warped product of two pseudo-Riemannian manifolds. Let (M, g) and $(\overline{M}, \overline{g})$ be smooth pseudo-Riemannian manifolds, and consider the warped product manifold $(M \times_f \overline{M}, g + e^{2f}\overline{g})$, where f is a smooth warping function. We construct a Sasaki-Matsumoto type lift of the warped metric to define a pseudo-Riemannian metric on the tangent bundle, which depends on a pair of smooth scalar functions and related to the total kinetic energy. We derive necessary and sufficient conditions under which the lifted metric on is Ricci-flat, expressed in terms of the curvature properties of the base manifold and the structure functions. Furthermore, we prove that, equipped with the metric, admits a one-parameter family of shrinking Ricci solitons.

Keywords: Ricci flat, warped pseudo Riemannian manifold, Ricci soliton.

1. Introduction

Ricci solitons, originally introduced in the context of Ricci flow by Hamilton, serve as self-similar solutions and provide a natural generalization of Einstein

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metrics. A Ricci soliton on a pseudo-Riemannian manifold (M^n, g) is a triple (X, ε), where X is a vector field and ε a real constant, satisfying the equation:

$$\mathcal{L}_X g + 2\,\mathrm{Ric}(g) + 2\varepsilon g = 0. \tag{1.1}$$

Such solitons are classified as shrinking, steady, or expanding depending on the sign of ε. In particular, if X is a Killing vector field and $\mathrm{Ric}(g) = 0$, the soliton reduces to a homothetic Einstein manifold (see [2]). Warped product manifolds are essential tools in differential geometry and mathematical physics, particularly in general relativity. Given two pseudo-Riemannian manifolds (M, g) and $(\overline{M}, \overline{g})$ and a smooth warping function $f : M \to \mathbb{R}$, the warped product $\widetilde{M} = M \times_f \overline{M}$ carries the metric $\widetilde{g} = g + e^{2f}\overline{g}$. The curvature and geometric behavior of $\widetilde{M}$ are closely tied to the properties of f, g, and g'. In this paper, we consider the complete lift of a warped product metric to the tangent bundle $T\widetilde{M}$ and analyze the resulting warped pseudo-Riemannian metric G on $T\widetilde{M}$. Using a Sasaki-Matsumoto type lift and a suitable energy-based deformation, we define a new metric G depending on functions $\phi(t)$ and $\psi(t)$ of the kinetic energy t. We then derive necessary and sufficient conditions under which the tangent bundle manifold $(T\widetilde{M}, G)$ is Ricci-flat. Furthermore, we show that $T\widetilde{M}$ admits a one-parameter family of shrinking Ricci solitons under certain geometric constraints. Our approach also relates the warped pseudo-Riemannian metric G to a Lagrangian formulation, enriching the geometric framework with variational principles. This study contributes to the interplay between warped products, tangent bundle geometry, and Ricci soliton theory, and may be relevant to mathematical models in geometry and physics.

2. **Preliminaries**

Let (M, g) and $(\overline{M}, \overline{g})$ be two smooth manifolds with $\dim M = m$ and $\dim \overline{M} = n$, respectively, and let $\widetilde{M} = M \times \overline{M}$ be the product of manifolds. Then a local coordinates system in $\widetilde{M}$ is denoted by $\mathbf{x}^a = (x^i, u^\alpha)$, where (x^i) and (u^α) are local coordinates system in M and $\overline{M}$, respectively. In this paper, the indexes $\{i, j, \cdots\}$, $\{\alpha, \beta, \cdots\}$ and $\{a, b, \cdots\}$ run over the ranges $\{1, 2, \cdots, m\}$, $\{1, 2, \cdots, n\}$ and $\{1, 2, \cdots, m, m+1, \cdots, m+n\}$, respectively.

Suppose that $(\widetilde{M}, \widetilde{g} = g + e^{2f}\overline{g})$ is warped product Riemannian manifolds with the warped function $f : M \to \mathbb{R}$. If we denote the Christoffel symbols of $\widetilde{M}$, M and $\overline{M}$ by $\widetilde{\Gamma}^a_{b\ c}$, $\Gamma^i_{j\ k}$ and $\overline{\Gamma}^\alpha_{\beta\ \gamma}$, respectively, then we have

$$\widetilde{\Gamma}^a_{b\ c} = \left(\widetilde{\Gamma}^i_{j\ k}, \widetilde{\Gamma}^i_{j\ \gamma}, \widetilde{\Gamma}^i_{\beta\ \gamma}, \widetilde{\Gamma}^\alpha_{j\ k}, \widetilde{\Gamma}^\alpha_{j\ \gamma}, \widetilde{\Gamma}^\alpha_{\beta\ \gamma}\right),$$

where

$$\begin{cases} \widetilde{\Gamma}^i_{j\ k} = \Gamma^i_{j\ k}, \ \ \widetilde{\Gamma}^i_{j\ \gamma} = 0, \ \ \widetilde{\Gamma}^i_{\beta\ \gamma} = -(e^{2f})^i \overline{g}_{\beta\gamma} \\ \widetilde{\Gamma}^\alpha_{j\ k} = 0, \ \ \widetilde{\Gamma}^\alpha_{j\ \gamma} = f_j \delta^\alpha_\gamma, \ \ \widetilde{\Gamma}^\alpha_{\beta\ \gamma} = \overline{\Gamma}^\alpha_{\beta\ \gamma}. \end{cases} \tag{2.1}$$

Here, $f_i := \frac{\partial f}{\partial x^i}$ and $(e^{2f})^i := g^{ij} \frac{\partial e^{2f}}{\partial x^j}$.

Suppose that, $\mathbf{x} \in \widetilde{M}$ and $\mathbf{y} \in T_{\mathbf{x}}\widetilde{M}$, where $\mathbf{x} = (x, u)$ and $\mathbf{y} = (y, v)$ and $T_{\mathbf{x}}\widetilde{M} = T_xM \oplus T_u\overline{M}$. Then, the warped Levi-Civita connection $(\widetilde{\Gamma}^a_{b\ c})$ of $\widetilde{g}$ defines a splitting

$$T(T\widetilde{M}) = V(T\widetilde{M}) \oplus H(T\widetilde{M}), \tag{2.2}$$

into *vertical* and *horizontal* sub bundles respectively. Locally, the integrable vertical distribution $V(T\widetilde{M})$ is spanned by $\{\frac{\partial}{\partial y^i}, \frac{\partial}{\partial v^\alpha}\}$, while the horizontal distribution $H(T\widetilde{M})$ is spanned by $\{\frac{\delta^*}{\delta x^i}, \frac{\delta^*}{\delta u^\alpha}\}$, where (see [1]),

$$\frac{\delta^*}{\delta x^i} := \frac{\partial}{\partial x^i} - \widetilde{\Gamma}^j_{i\ 0}\frac{\partial}{\partial y^j} - \widetilde{\Gamma}^\beta_{i\ 0}\frac{\partial}{\partial v^\beta}$$

$$\frac{\delta^*}{\delta u^\alpha} := \frac{\partial}{\partial u^\alpha} - \widetilde{\Gamma}^j_{\alpha\ 0}\frac{\partial}{\partial y^j} - \widetilde{\Gamma}^\beta_{\alpha\ 0}\frac{\partial}{\partial v^\beta},$$

and ([3, 5])

$$\widetilde{\Gamma}^j_{i\ 0} = \Gamma^j_{i\ 0},\ \ \widetilde{\Gamma}^\beta_{i\ 0} = f_i\delta^\beta_\gamma v^\gamma,$$

$$\widetilde{\Gamma}^j_{\alpha\ 0} = -(e^{2f})^j\overline{g}_{\alpha 0},\ \ \widetilde{\Gamma}^\beta_{\alpha\ 0} = \overline{\Gamma}^\beta_{\alpha\ 0} + f_jy^j\delta^\beta_\alpha.$$

Here, $\Gamma^j_{i\ 0} = \Gamma^j_{i\ k}y^k$, $\overline{\Gamma}^\beta_{\alpha\ 0} = \overline{\Gamma}^\beta_{\alpha\ \gamma}v^\gamma$, $\overline{g}_{\alpha 0} = \overline{g}_{\alpha\beta}v^\beta$ and $g_{i0} = g_{ij}y^j$.

3. Sasaki-Matsumoto type Lift on Tangent Bundle

Let $(\widetilde{M} = M \times_f \overline{M}, \widetilde{g})$ be the warped product Riemannian manifolds and $\widetilde{g}_{ab} = g_{ij} + e^{2f}\overline{g}_{\alpha\beta}$ be the warped metric. We define the *warped kinetic energy* (or warped energy density) by

$$t(x, u, y, v) := \frac{1}{2}\|\mathbf{y}\|^2_{\widetilde{g}} = \frac{1}{2}\Big(g_{ij}(x)y^iy^j + e^{2f}\overline{g}_{\alpha\beta}(u)v^\alpha v^\beta\Big), \tag{3.1}$$

where $\mathbf{y} \in T_{(x,u)}\widetilde{M} \cong T_xM \oplus T_u\overline{M}$. Certainly, $t = \widetilde{t} + e^{2f}\overline{t}$ where $\widetilde{t}(x, y) := \frac{1}{2}g_{ij}(x)y^iy^j$ and $\overline{t}(u, v) := \frac{1}{2}\overline{g}_{\alpha\beta}(u)v^\alpha v^\beta$. Also, we have

$$\frac{\partial t}{\partial y^k} = g_{k0}\ \ \&\ \ \frac{\partial t}{\partial v^\gamma} = e^{2f}\overline{g}_{\gamma 0} \tag{3.2}$$

Suppose, the functions $\varphi, \psi : [0, \infty) \to \mathbb{R}$ are smooth such that, $\varphi(t) > 0$ and $\varphi(t)+2t\psi(t) > 0$ for every t (see [7]). Then, the *warped Sasaki-Matsumoto type lift* of the $\widetilde{g}_{ab}$ (the symmetric $\widetilde{M}$-tensor on $T\widetilde{M}$) can be introduced as follows

$$G_{ab} := \varphi(t)(g_{ij} + e^{2f}\overline{g}_{\alpha\beta}) + \psi(t)(g_{i0} + e^{2f}\overline{g}_{\alpha 0})(g_{j0} + e^{2f}\overline{g}_{\beta 0}).$$

Now, the following warped pseudo Riemannian metric will be considered on $T\widetilde{M}$:

$$\begin{aligned} G = {} & 2(\varphi g_{ij} + \psi g_{i0}g_{j0})\delta^*y^i \otimes dx^j + 2(\varphi e^{2f}\overline{g}_{\alpha\beta} + e^{4f}\overline{g}_{\alpha 0}\overline{g}_{\beta 0})\delta^*v^\alpha \otimes du^\beta \\ & +2\psi e^{2f}g_{i0}\overline{g}_{\beta 0}\delta^*y^i \otimes du^\beta + 2\psi e^{2f}g_{j0}\overline{g}_{\alpha 0}\delta^*v^\alpha \otimes dx^j \end{aligned} \tag{3.3}$$

where

$$\delta^*y^i := dy^i + \widetilde{\Gamma}^i_{j\ 0}dx^j + \widetilde{\Gamma}^i_{\beta\ 0}du^\beta, \delta^*v^\alpha := dv^\alpha + \widetilde{\Gamma}^\alpha_{j\ 0}dx^j + \widetilde{\Gamma}^\alpha_{\beta\ 0}du^\beta$$

and $\{dx^i, du^\alpha, \delta^* y^i, \delta v^\alpha\}$ is the dual basis of adapted basis $\{\frac{\delta^*}{\delta x^i}, \frac{\delta^*}{\delta u^\alpha}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial v^\alpha}\}$. Therefor

$$G\Big(\frac{\delta^*}{\delta x^i}, \frac{\partial}{\partial y^j}\Big) = \varphi g_{ij} + \psi g_{i0} g_{j0},$$
$$G\Big(\frac{\delta^*}{\delta u^\alpha}, \frac{\partial}{\partial v^\beta}\Big) = \varphi e^{2f} \overline{g}_{\alpha\beta} + \psi e^{4f} \overline{g}_{\alpha 0} \overline{g}_{\beta 0},$$
$$G\Big(\frac{\delta^*}{\delta x^i}, \frac{\partial}{\partial v^\beta}\Big) = \psi e^{2f} g_{i0} \overline{g}_{\beta 0},$$
$$G\Big(\frac{\delta^*}{\delta u^\alpha}, \frac{\partial}{\partial y^j}\Big) = \psi e^{2f} g_{j0} \overline{g}_{\alpha 0},$$

and

$$G\Big(\frac{\delta^*}{\delta x^i}, \frac{\delta^*}{\delta x^j}\Big) = G\Big(\frac{\delta^*}{\delta u^\alpha}, \frac{\delta^*}{\delta u^\beta}\Big) = G\Big(\frac{\delta^*}{\delta x^i}, \frac{\delta^*}{\delta u^\beta}\Big) = G\Big(\frac{\delta^*}{\delta u^\alpha}, \frac{\delta^*}{\delta x^j}\Big) = 0,$$
$$G\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big) = G\Big(\frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta}\Big) = G\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial v^\beta}\Big) = G\Big(\frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial y^j}\Big) = 0.$$

Hence we have

$$(G) = \begin{pmatrix} 0 & 0 & G_{ij} & G_{i\beta} \\ 0 & 0 & G_{\alpha j} & G_{\alpha\beta} \\ G_{ij} & G_{i\beta} & 0 & 0 \\ G_{\alpha j} & G_{\alpha\beta} & 0 & 0 \end{pmatrix} \tag{3.4}$$

where

$$G_{ij} := \varphi g_{ij} + \psi g_{i0} g_{j0},$$
$$G_{i\beta} := \psi e^{2f} g_{i0} \overline{g}_{\beta 0} =: G_{\beta i},$$
$$G_{\alpha\beta} := \varphi e^{2f} \overline{g}_{\alpha\beta} + \psi e^{4f} \overline{g}_{\alpha 0} \overline{g}_{\beta 0}.$$

The following Proposition is a direct result of the above relations.

Proposition 3.1. *The matrix (G) is the positive definite symmetric, and has an inverse with the entries*

$$(\widetilde{H}) = \begin{pmatrix} 0 & 0 & \widetilde{H}^{ij} & \widetilde{H}^{i\beta} \\ 0 & 0 & \widetilde{H}^{\alpha j} & \widetilde{H}^{\alpha\beta} \\ \widetilde{H}^{ij} & \widetilde{H}^{i\beta} & 0 & 0 \\ \widetilde{H}^{\alpha j} & \widetilde{H}^{\alpha\beta} & 0 & 0 \end{pmatrix} \tag{3.5}$$

where,

$$\widetilde{H}^{ij} := \frac{1}{\varphi} g^{ij} - \frac{\psi}{s(t)} z(t) y^i y^j$$
$$\widetilde{H}^{i\beta} := \frac{\psi}{s(t)}\Big(\frac{-1}{\varphi} + \frac{2e^{2f}\varphi\psi}{r(t)} \overline{t} p(t)\Big) y^i v^\beta =: \widetilde{H}^{\beta i},$$
$$\widetilde{H}^{\alpha\beta} := \frac{e^{-2f}}{\varphi} \overline{g}^{\alpha\beta} - \frac{\psi}{r(t)} p(t) v^\alpha v^\beta.$$

Here,

$$s(t) := \varphi(t)\Big(\varphi(t) + 2\widetilde{t}\psi(t)\Big),$$
$$r(t) := \varphi(t)\Big(\varphi(t) + 2e^{2f}\overline{t}\psi(t)\Big),$$
$$p(t) := 1 + \frac{2\widetilde{t}\psi(t)}{4e^{2f}\psi^2\widetilde{\overline{t}t} - s(t)r(t)},$$
$$z(t) := 1 + \frac{2\overline{t}e^{2f}\psi(t)}{4e^{2f}\psi^2\widetilde{\overline{t}t} - s(t)r(t)}.$$

Therefore, $\widetilde{H}^{ab}(x,u,y,v)$ is a symmetric $\widetilde{M}$-tensor on $T\widetilde{M}$.

Remark 3.2. [3, 5] *Let $\widetilde{\nabla}$, ∇ and $\overline{\nabla}$ be the Levi-Civita connections of $\widetilde{M}$, M and $\overline{M}$ with respect to the metrics $\widetilde{g} = g + e^{2f}\overline{g}$, g and $\overline{g}$ respectively. Also, the components of curvature tensors of $\widetilde{M}$, M and $\overline{M}$ will be denoted by $\widetilde{R}^a_{bcd}$, R^i_{jkl} and $\overline{R}^{\alpha}_{\beta\gamma\lambda}$ respectively. Then, the warped Riemannian tensors are*

$$\begin{cases}
\widetilde{R}^h_{ijk} = R^h_{ijk} \\
\widetilde{R}^h_{i\beta\gamma} = g^{hl}(\nabla_i(f_l) + f_i f_l)e^{2f}\overline{g}_{\beta\gamma} = -\widetilde{R}^h_{\beta i\gamma} \\
\widetilde{R}^{\lambda}_{ijk} = \widetilde{R}^h_{\alpha jk} = \widetilde{R}^h_{i\beta k} = \widetilde{R}^h_{ij\gamma} = 0 \\
\widetilde{R}^{\lambda}_{ij\gamma} = \widetilde{R}^h_{\alpha\beta k} = 0 \\
\widetilde{R}^h_{\alpha\beta\gamma} = \widetilde{R}^{\lambda}_{\alpha\beta k} = \widetilde{R}^{\lambda}_{i\beta\gamma} = \widetilde{R}^{\lambda}_{\alpha j\gamma} = 0 \\
\widetilde{R}^{\lambda}_{i\beta k} = (\nabla_i(f_k) + f_i f_k)\delta^{\gamma}_{\beta} = -\widetilde{R}^{\lambda}_{\beta ik} \\
\widetilde{R}^{\lambda}_{\alpha\beta\gamma} = \overline{R}^{\lambda}_{\alpha\beta\gamma} + e^{2f}g^{ij}f_i f_j(\overline{g}_{\beta\gamma}\delta^{\lambda}_{\alpha} - \overline{g}_{\alpha\gamma}\delta^{\lambda}_{\beta})
\end{cases}$$

In the following we determine the warped Levi-Civita connection ∇^G of the warped pseudo Riemannian metric G defined by (3.3). First, the Lie brackets of the above vector fields are stated as follows.

Lemma 3.3.

$$\Big[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big] = \Big[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial v^\beta}\Big] = \Big[\frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial y^j}\Big] = \Big[\frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta}\Big] = 0, \tag{3.6}$$

$$\begin{cases}
\Big[\frac{\partial}{\partial y^i}, \frac{\delta^*}{\delta x^j}\Big] = -\Gamma^k_{i\ j}\frac{\partial}{\partial y^k} \\
\Big[\frac{\partial}{\partial y^j}, \frac{\delta^*}{\delta u^\beta}\Big] = \Big[\frac{\partial}{\partial v^\alpha}, \frac{\delta^*}{\delta x^j}\Big] = -\widetilde{\Gamma}^{\gamma}_{j\ \alpha}\frac{\partial}{\partial v^\gamma} \\
\Big[\frac{\partial}{\partial v^\alpha}, \frac{\delta^*}{\delta u^\beta}\Big] = -\widetilde{\Gamma}^k_{\alpha\ \beta}\frac{\partial}{\partial y^k} - \overline{\Gamma}^{\gamma}_{\alpha\ \beta}\frac{\partial}{\partial v^\gamma},
\end{cases} \tag{3.7}$$

$$\begin{cases} \left[\frac{\delta^*}{\delta x^i}, \frac{\delta^*}{\delta x^j}\right] = -R^h_{ij0}\frac{\partial}{\partial y^h} \\ \left[\frac{\delta^*}{\delta x^i}, \frac{\delta^*}{\delta u^\beta}\right] = -\widetilde{R}^h_{i\beta\lambda}v^\lambda\frac{\partial}{\partial y^h} - \widetilde{R}^\lambda_{i\beta k}y^k\frac{\partial}{\partial v^\lambda} \\ \left[\frac{\delta^*}{\delta u^\alpha}, \frac{\delta^*}{\delta x^j}\right] = -\widetilde{R}^h_{\alpha j\lambda}v^\lambda\frac{\partial}{\partial y^h} - \widetilde{R}^\lambda_{\alpha jk}y^k\frac{\partial}{\partial v^\lambda} \\ \left[\frac{\delta^*}{\delta u^\alpha}, \frac{\delta^*}{\delta u^\beta}\right] = -\overline{R}^\gamma_{\alpha\beta 0}\frac{\partial}{\partial v^\gamma}, \end{cases} \tag{3.8}$$

where, $R^h_{ij0} = R^h_{ijk}y^k$ *and* $\widetilde{R}^\gamma_{\alpha\beta 0} = \widetilde{R}^\gamma_{\alpha\beta\lambda}v^\lambda$.

Proposition 3.4. *Let* $(\widetilde{M} = M \times_f \overline{M}, g + e^{2f}\overline{g})$ *be a warped Riemannian manifold. Then, the Levi-Civita connection* ∇^G *of the warped pseudo Riemannian metric* G *defined by (3.3) on* $T\widetilde{M}$ *has the following expression*

$$\begin{cases} \nabla^G_{\partial/\partial y^i}\frac{\partial}{\partial y^j} = A^k_{ij}\frac{\partial}{\partial y^k}, \\ \nabla^G_{\partial/\partial y^i}\frac{\partial}{\partial v^\beta} = \hat{A}^k_{i\beta}\frac{\partial}{\partial y^k} \\ \nabla^G_{\partial/\partial v^\alpha}\frac{\partial}{\partial y^j} = \hat{B}^\gamma_{\alpha j}\frac{\partial}{\partial v^\gamma}, \\ \nabla^G_{\partial/\partial v^\alpha}\frac{\partial}{\partial v^\beta} = B^\gamma_{\alpha\beta}\frac{\partial}{\partial v^\gamma} \end{cases} \tag{3.9}$$

$$\begin{cases} \nabla^G_{\partial/\partial y^i}\frac{\delta^*}{\delta x^j} = C^k_{ij}\frac{\delta^*}{\delta x^k}, \\ \nabla^G_{\partial/\partial y^i}\frac{\delta^*}{\delta u^\beta} = \hat{C}^k_{i\beta}\frac{\delta^*}{\delta x^k} \\ \nabla^G_{\partial/\partial v^\alpha}\frac{\partial}{\partial y^j} = \hat{D}^\gamma_{\alpha j}\frac{\delta^*}{\delta u^\gamma}, \\ \nabla^G_{\partial/\partial v^\alpha}\frac{\partial}{\partial v^\beta} = D^\gamma_{\alpha\beta}\frac{\delta^*}{\delta u^\gamma} \end{cases} \tag{3.10}$$

$$\begin{cases} \nabla^G_{\frac{\delta^*}{\delta x^i}}\partial/\partial y^j = \Gamma^k_{i\ j}\frac{\partial}{\partial y^k} + C^k_{ij}\frac{\delta^*}{\delta x^k}, \\ \nabla^G_{\frac{\delta^*}{\delta x^i}}\partial/\partial v^\beta = \widetilde{\Gamma}^\gamma_{i\ \beta}\frac{\partial}{\partial v^\gamma} + \hat{C}^k_{i\beta}\frac{\delta^*}{\delta x^k} \\ \nabla^G_{\frac{\delta^*}{\delta u^\alpha}}\frac{\partial}{\partial y^j} = \widetilde{\Gamma}^\gamma_{\alpha\ j}\frac{\partial}{\partial v^\gamma} + \hat{D}^\gamma_{\alpha j}\frac{\delta^*}{\delta u^\gamma}, \\ \nabla^G_{\frac{\delta^*}{\delta u^\alpha}}\frac{\partial}{\partial v^\beta} = \widetilde{\Gamma}^k_{\alpha\ \beta}\frac{\partial}{\partial y^k} + \overline{\Gamma}^\gamma_{\alpha\ \beta}\frac{\partial}{\partial v^\gamma} + D^\gamma_{\alpha\beta}\frac{\delta^*}{\delta u^\gamma} \end{cases} \tag{3.11}$$

$$\begin{cases} \nabla^G_{\frac{\delta^*}{\delta x^i}}\frac{\delta^*}{\delta x^j} = E^k_{ij}\frac{\partial}{\partial y^k} + \Gamma^k_{i\ j}\frac{\delta^*}{\delta x^k}, \\ \nabla^G_{\frac{\delta^*}{\delta x^i}}\frac{\delta^*}{\delta u^\beta} = F^k_{i\beta}\frac{\partial}{\partial y^k} + \widetilde{\Gamma}^\gamma_{i\ \beta}\frac{\delta^*}{\delta u^\gamma}, \\ \nabla^G_{\frac{\delta^*}{\delta u^\alpha}}\frac{\delta^*}{\delta x^j} = P^\gamma_{\alpha j}\frac{\partial}{\partial v^\gamma} + \widetilde{\Gamma}^\gamma_{\alpha\ j}\frac{\delta^*}{\delta u^\gamma}, \\ \nabla^G_{\frac{\delta^*}{\delta u^\alpha}}\frac{\delta^*}{\delta u^\beta} = Q^\gamma_{\alpha\beta}\frac{\partial}{\partial v^\gamma} + \widetilde{\Gamma}^k_{\alpha\ \beta}\frac{\delta^*}{\delta x^k} + \overline{\Gamma}^\gamma_{\alpha\ \beta}\frac{\delta^*}{\delta u^\gamma}, \end{cases} \tag{3.12}$$

where the components A_{ij}^k, $\hat{A}_{i\beta}^k$, $\cdots$, $\hat{D}_{\alpha j}^\gamma$, $D_{\alpha\beta}^\gamma$, E_{ij}^k, $F_{i\beta}^k$, $P_{\alpha j}^\gamma$ and $Q_{\alpha\beta}^\gamma$ define $\widetilde{M}$-tensor fields on $T\widetilde{M}$ and are given by

$$\begin{cases} A_{ij}^k = \frac{\varphi'+\psi}{2\varphi}\Big(g_{i0}\delta_j^k + g_{j0}\delta_i^k\Big) + \Big[\frac{\psi'}{\varphi} - \frac{\psi z(t)}{s(t)}(\varphi' + \psi + 2\psi'\widetilde{t})\Big]g_{i0}g_{j0}y^k \\ +\Big(\frac{\psi}{\varphi} - \frac{2\psi\psi'\widetilde{t}}{s(t)}z(t)\Big)g_{ij}y^k, \\ \hat{A}_{i\beta}^k = e^{2f}\overline{g}_{\beta 0}\Big[\frac{\varphi'+\psi}{2\varphi}\delta_i^k + \Big(\frac{\psi'}{\varphi} - \psi\frac{4\psi' e^{2f}\overline{t}+\varphi'+\psi}{2r(t)}p(t)\Big)g_{i0}y^k\Big], \\ \hat{B}_{\alpha j}^\gamma = e^{2f}g_{j0}\Big[\frac{\varphi'+\psi}{2e^{2f}\varphi}\delta_\alpha^\gamma + \Big(\frac{\psi'}{\varphi} - \psi\frac{4\psi'\widetilde{t}+\varphi'+\psi}{2s(t)}z(t)\Big)\overline{g}_{\alpha 0}v^\gamma\Big], \\ B_{\alpha\beta}^\gamma = e^{2f}\frac{\varphi'+\psi}{2\varphi}\Big(\overline{g}_{\alpha 0}\delta_\beta^\gamma + \overline{g}_{\beta 0}\delta_\alpha^\gamma\Big) + e^{4f}\Big(\frac{\psi}{\varphi} - \frac{2e^{2f}\psi^2\overline{t}}{r(t)}p(t)\Big)\overline{g}_{\alpha\beta}v^\gamma \\ +e^{4f}\Big(\frac{\psi'}{\varphi} - \frac{\psi(\varphi'+\psi)-2e^{2f}\psi'\psi\overline{t}}{r(t)}p(t)\Big)\overline{g}_{\alpha 0}\overline{g}_{\beta 0}v^\gamma + \frac{\psi p(t)}{r(t)}\Big(\varphi' + \psi - 2e^{2f}\psi\overline{t}\Big), \\ C_{ij}^k = \frac{\varphi'-\psi}{2\varphi}\Big(g_{i0}\delta_j^k - (1 + \frac{2\varphi\psi\widetilde{t}}{s(t)}z(t))g_{ij}y^k - \frac{\varphi\psi}{s(t)}z(t)g_{i0}g_{j0}y^k\Big), \\ \hat{C}_{i\beta}^k = e^{2f}\frac{(\varphi'-\psi)\psi}{2s(t)}\Big(\frac{-1}{\varphi} + \frac{2\varphi\psi e^{2f}\overline{t}}{r(t)}p(t)g_{i0}\overline{g}_{\beta 0}y^k\Big), \\ \hat{D}_{\alpha j}^\gamma = e^{2f}\frac{(\varphi'-\psi)\psi}{2s(t)}\Big(\frac{-1}{\varphi} + \frac{2\varphi\psi e^{2f}\overline{t}}{r(t)}p(t)g_{j0}\overline{g}_{\alpha 0}v^\gamma\Big), \\ D_{\alpha\beta}^\gamma = e^{2f}\frac{\varphi'-\psi}{2\varphi}\Big(\overline{g}_{\alpha 0}\delta_\beta^\gamma - (1 + \frac{2e^{2f}\varphi\psi\overline{t}}{r(t)}p(t))\overline{g}_{\alpha\beta}v^\gamma - e^{2f}\frac{\varphi\psi}{r(t)}p(t)\overline{g}_{\alpha 0}\overline{g}_{\beta 0}v^\gamma\Big), \\ E_{ij}^k = -R_{hij0}g^{kh} + \frac{\varphi\psi}{s(t)}z(t)R_{0ij0}y^k \\ +\frac{1}{2}\Big(\frac{\psi}{\varphi}g^{kh} - \frac{\psi^2}{s(t)}z(t)y^k y^h\Big)(R_{0ih0}g_{j0} + R_{0jh0}g_{i0} - R_{0ij0}g_{h0}), \\ F_{i\beta}^k = \frac{e^{2f}}{s(t)}\Big(\frac{-1}{\varphi} + \frac{2\varphi\psi e^{2f}\overline{t}}{r(t)}p(t)\Big)\overline{R}_{\beta 00}^\gamma g_{i0}\overline{g}_{\gamma 0}y^k, \\ P_{\alpha j}^\gamma = \frac{\psi^2 e^{2f}}{s(t)}\Big(\frac{-1}{\varphi} + \frac{2\varphi\psi e^{2f}\overline{t}}{r(t)}p(t)\Big)R_{j00}^h g_{h0}\overline{g}_{\alpha 0}v^\gamma, \\ Q_{\alpha\beta}^\gamma = -\overline{R}_{\lambda\alpha\beta 0}\overline{g}^{\gamma\lambda} + \frac{\psi e^{2f}}{2}\Big(-\frac{1}{\varphi} + \frac{\varphi+2\psi e^{2f}\overline{t}}{r(t)}p(t)\Big)\overline{R}_{0\alpha\beta 0}v^\gamma \\ +\frac{\psi e^{2f}}{2}\Big(\frac{1}{\varphi}\overline{g}^{\gamma\lambda} - \frac{\psi e^{2f}}{r(t)}p(t)v^\gamma v^\lambda\Big)(\overline{R}_{0\alpha\lambda 0}\overline{g}_{\beta 0} + \overline{R}_{0\beta\lambda 0}\overline{g}_{\alpha 0}). \end{cases} \tag{3.13}$$

Proof. The proof is completed by direct calculations and the application of Remark 3.2 and Lemma 3.3. □

Let $\mathcal{R}$ be the curvature tensor field of the warped Levi-Civita connection ∇^G on $T\widetilde{M}$, then we have

$$\mathcal{R}(X,Y)Z = \nabla_X^G \nabla_Y^G Z - \nabla_Y^G \nabla_X^G Z - \nabla_{[X,Y]}^G Z$$

for any $X, Y, Z \in \Gamma(T\widetilde{M})$. By straightforward computations we obtain

$$\begin{cases} (1)\mathcal{R}(\delta^*_i,\delta^*_j)\delta^*_k = \Big(\nabla^{(1)}_{\delta^*_i} E^l_{jk} - \nabla^{(1)}_{\delta^*_j} E^l_{ik}\Big)\partial_l \\ +\Big(\nabla^{(1)}_{\delta^*_i}\Gamma^l_{j\ k} - \nabla^{(1)}_{\delta^*_j}\Gamma^l_{i\ k} + E^h_{jk}C^l_{hi} - E^h_{ik}C^l_{hj} + R^h_{ij0}C^l_{hk}\Big)\delta^*_l, \\ (2)\mathcal{R}(\delta^*_i,\delta^*_j)\partial_k = \Big(\nabla^{(1)}_{\delta^*_i}\Gamma^l_{j\ k} - \nabla^{(1)}_{\delta^*_j}\Gamma^l_{i\ k} + C^h_{jk}E^l_{hi} - C^h_{ik}E^l_{hj} \\ +R^h_{ij0}A^l_{hk}\Big)\partial_l + \Big(\nabla^{(1)}_{\delta^*_i} C^l_{jk} - \nabla^{(1)}_{\delta^*_j} C^l_{ik}\Big)\delta^*_l, \\ (3)\mathcal{R}(\delta^*_i,\partial_j)\delta^*_k = \Big(C^h_{jk}E^l_{ih} - E^h_{ik}A^l_{jh} - \frac{\partial E^l_{ik}}{\partial y^j}\Big)\partial_l + \Big(\nabla^{(1)}_{\delta^*_i} C^l_{jk}\Big)\delta^*_l, \\ (4)\mathcal{R}(\delta^*_i,\partial_j)\partial_k = \Big(\nabla^{(1)}_{\delta^*_i} A^l_{jk}\Big)\partial_l + \Big(A^h_{jk}C^l_{ih} - \frac{\partial C^l_{jk}}{\partial y^i} - C^h_{ik}C^l_{hj}\Big)\delta^*_l, \\ (5)\mathcal{R}(\partial_i,\partial_j)\delta^*_k = \Big(\frac{\partial C^l_{jk}}{\partial y^i} - \frac{\partial C^l_{ik}}{\partial y^j} + C^h_{jk}C^l_{ih} - C^h_{ik}C^l_{jh}\Big)\delta^*_l, \\ (6)\mathcal{R}(\partial_i,\partial_j)\partial_k = \Big(\frac{\partial A^l_{jk}}{\partial y^i} - \frac{\partial A^l_{ik}}{\partial y^j} + A^h_{jk}A^l_{ih} - A^h_{ik}A^l_{jh}\Big)\partial_l \end{cases} \tag{3.14}$$

$$\begin{cases} (7)\mathcal{R}(\delta^*_\alpha,\delta^*_\beta)\gamma^*_k = 2(e^{2f})^k\overline{R}_{\alpha\beta\gamma0}\partial_k + \Big(\nabla^{(2)}_{\delta^*_\alpha} Q^\mu_{\beta\gamma} - \nabla^{(2)}_{\delta^*_\beta} Q^\mu_{\alpha\gamma} + \widetilde{\Gamma}^k_{\beta\ \gamma}P^\mu_{\alpha k} \\ -\widetilde{\Gamma}^k_{\alpha\ \gamma}P^\mu_{\beta k}\Big)\partial_\mu + \Big(\frac{\delta^*\overline{\Gamma}^\mu_{\beta\ \gamma}}{\delta u^\alpha} - \frac{\delta^*\overline{\Gamma}^\mu_{\alpha\ \gamma}}{\delta u^\beta} + \widetilde{\Gamma}^k_{\beta\ \gamma}\widetilde{\Gamma}^\mu_{\alpha\ k} - \widetilde{\Gamma}^k_{\alpha\ \gamma}\widetilde{\Gamma}^\mu_{\beta\ k} + \overline{\Gamma}^\lambda_{\beta\ \gamma}\overline{\Gamma}^\mu_{\alpha\ \lambda} \\ -\overline{\Gamma}^\lambda_{\alpha\ \gamma}\overline{\Gamma}^\mu_{\beta\ \lambda} + Q^\lambda_{\beta\gamma}D^\mu_{\alpha\lambda} - Q^\lambda_{\alpha\gamma}D^m_{\beta\lambda}u\Big)\delta^*_\mu, \\ (8)\mathcal{R}(\delta^*_\alpha,\delta^*\beta)\partial_\gamma = \Big(D^\lambda_{\beta\gamma}\widetilde{\Gamma}^k_{\alpha\lambda} - D^\lambda_{\alpha\gamma}\widetilde{\Gamma}^k_{\beta\lambda}\Big)\delta^*_k + \Big(\frac{\delta^*\overline{\Gamma}^\mu_{\beta\ \gamma}}{\delta u^\alpha} - \frac{\delta^*\overline{\Gamma}^\mu_{\alpha\ \gamma}}{\delta u^\beta} + \widetilde{\Gamma}^k_{\beta\ \gamma}\widetilde{\Gamma}^\mu_{\alpha\ k} \\ -\widetilde{\Gamma}^k_{\alpha\ \gamma}\widetilde{\Gamma}^\mu_{\beta\ k} + \overline{\Gamma}^\lambda_{\beta\ \gamma}\overline{\Gamma}^\mu_{\alpha\ \lambda} - \overline{\Gamma}^\lambda_{\alpha\ \gamma}\overline{\Gamma}^\mu_{\beta\ \lambda} + D^\lambda_{\beta\gamma}Q^\mu_{\lambda\alpha} - D^\lambda_{\alpha\gamma}Q^\mu_{\lambda\beta} + \overline{R}^\lambda_{\alpha\beta0}B^\mu_{\lambda\gamma}\Big)\partial_\mu \\ +\Big(\nabla^{(2)}_{\delta^*_\alpha} D^\mu_{\beta\gamma} - \nabla^{(2)}_{\delta^*_\beta} D^\mu_{\alpha\gamma} - \widetilde{\Gamma}^k_{\beta\ \gamma}\hat{D}^\mu_{\alpha k} - \widetilde{\Gamma}^k_{\alpha\ \gamma}\hat{D}^\mu_{\beta k}\Big)\delta^*_\mu, \\ (9)\mathcal{R}(\delta^*_\alpha,\partial_\beta)\delta^*_\gamma = \Big(D^\lambda_{\beta\gamma}\widetilde{\Gamma}^k_{\lambda\alpha} - \widetilde{\Gamma}^l_{\alpha\beta}\hat{C}^k_{l\gamma}\Big)\delta^*_k + \Big(D^\lambda_{\beta\gamma}Q^\mu_{\alpha\lambda} - Q^\lambda_{\alpha\gamma}B^\mu_{\beta\lambda} \\ -\frac{\partial Q^\mu_{\alpha\gamma}}{\partial v^\beta}\Big)\partial_\mu + \Big(\nabla^{(2)}_{\delta^*_\alpha} D^\mu_{\beta\gamma} - \widetilde{\Gamma}^k_{\alpha\gamma}\hat{D}^\mu_{\beta k}\Big)\delta^*_\mu, \\ (10)\mathcal{R}(\delta^*_\alpha,\partial_\beta)\partial_\gamma = \Big(B^\lambda_{\beta\gamma}\widetilde{\Gamma}^k_{\alpha\ \lambda} - \frac{\partial\widetilde{\Gamma}^k_{\alpha\ \gamma}}{\partial v^\beta} - \widetilde{\Gamma}^l_{\alpha\ \beta}\hat{A}^k_{l\gamma}\Big)\partial_k \\ +\Big(\nabla^{(2)}_{\delta^*_\alpha} B^\mu_{\beta\gamma} - \widetilde{\Gamma}^l_{\alpha\ \gamma}\hat{B}^\mu_{l\beta}\Big)\partial_\mu + \Big(D^\lambda_{\alpha\gamma}D^\mu_{\beta\lambda} - D^\lambda_{\beta\gamma}D^\mu_{\alpha\lambda} - \frac{\partial D^\mu_{\alpha\gamma}}{\partial v^\beta}\Big)\delta^*_\mu, \\ (11)\mathcal{R}(\partial_\alpha,\partial_\beta)\delta^*_\gamma = \Big(\frac{\partial D^\mu_{\beta\gamma}}{\partial v^\alpha} - \frac{\partial D^\mu_{\alpha\gamma}}{\partial v^\beta} + D^\lambda_{\beta\gamma}D^\mu_{\alpha\lambda} - D^\lambda_{\alpha\gamma}D^\mu_{\beta\lambda}\Big)\delta^*_\mu, \\ (12)\mathcal{R}(\partial_\alpha,\partial_\beta)\partial_\gamma = \Big(\frac{\partial B^\mu_{\beta\gamma}}{\partial v^\alpha} - \frac{\partial B^\mu_{\alpha\gamma}}{\partial v^\beta} + B^\lambda_{\beta\gamma}B^\mu_{\alpha\lambda} - B^\lambda_{\alpha\gamma}B^\mu_{\beta\lambda}\Big)\partial_\mu \end{cases} \tag{3.15}$$

$$\begin{cases} (13)\mathcal{R}(\delta^*_i,\delta^*_j)\delta^*_\gamma = \Big(\frac{\delta^* F^k_{j\gamma}}{\delta x^i} + \frac{\delta^* F^k_{i\gamma}}{\delta x^j} + F^l_{j\gamma}\Gamma^k_{i\ l} - F^l_{i\gamma}\Gamma^k_{j\ l} + \widetilde{\Gamma}^\lambda_{j\ \gamma}F^k_{i\lambda} \\ -\widetilde{\Gamma}^\lambda_{i\ \gamma}F^k_{j\lambda}\Big)\partial_k + \Big(F^l_{j\gamma}C^k_{il} - F^l_{i\gamma}C^k_{jl} + R^l_{ij0}\hat{C}^k_{l\gamma}\Big)\delta^*_k, \\ (14)\mathcal{R}(\delta^*_i,\delta^*_j)\partial_\gamma = \Big(\hat{C}^l_{j\gamma}E^k_{il} - \hat{C}^l_{i\gamma}E^k_{jl} + R^l_{ij0}\hat{A}^k_{l\gamma}\Big)\partial_k \\ +\Big(\frac{\delta^*\hat{C}^k_{j\gamma}}{\delta x^i} - \frac{\delta^*\hat{C}^k_{i\gamma}}{\delta x^j} + \widetilde{\Gamma}^\lambda_{j\ \gamma}\hat{C}^k_{i\lambda} - \widetilde{\Gamma}^\lambda_{i\ \gamma}\hat{C}^k_{j\lambda} + \hat{C}^l_{j\gamma}\Gamma^k_{l\ i} - \hat{C}^l_{i\gamma}\Gamma^k_{l\ j}\Big)\delta^*_k, \\ (15)\mathcal{R}(\delta^*_i,\partial_j)\delta^*_\gamma = \Big(\hat{C}^l_{j\gamma}E^k_{il} - \frac{\partial F^k_{i\gamma}}{\partial y^j} - F^l_{i\gamma}A^k_{lj}\Big)\partial_k + \Big(\frac{\delta^*\hat{C}^k_{j\gamma}}{\delta x^i} + \hat{C}^l_{j\gamma}\Gamma^k_{l\ i} \\ -\widetilde{\Gamma}^\lambda_{i\ \gamma}\hat{C}^k_{j\lambda} - \Gamma^l_{i\ j}\hat{C}^k_{l\gamma}\Big)\delta^*_k, \\ (16)\mathcal{R}(\delta^*_i,\partial_j)\partial_\gamma = \Big(\frac{\delta^*\hat{A}^k_{j\gamma}}{\delta x^i} - \widetilde{\Gamma}^\lambda_{i\gamma}\hat{A}^k_{j\lambda} + \hat{A}^l_{j\gamma}\Gamma^k_{l\ i} - \Gamma^l_{i\ j}\hat{A}^k_{l\gamma}\Big)\partial_k + \Big(\hat{A}^l_{j\gamma}C^k_{il} \\ -\frac{\partial\hat{C}^k_{i\gamma}}{\partial y^j} - \hat{C}^l_{i\gamma}C^k_{lj}\Big)\delta^*_k, \\ (17)\mathcal{R}(\partial_i,\partial_j)\delta^*_\gamma = \Big(\frac{\partial\hat{C}^k_{j\gamma}}{\partial y^i} - \frac{\partial\hat{C}^k_{i\gamma}}{\partial y^j} + \hat{C}^l_{j\gamma}C^k_{li} - \hat{C}^l_{i\gamma}C^k_{lj}\Big)\delta^*_k, \\ (18)\mathcal{R}(\partial_i,\partial_j)\partial_\gamma = \Big(\frac{\partial\hat{A}^k_{j\gamma}}{\partial y^i} - \frac{\partial\hat{A}^k_{i\gamma}}{\partial y^j} + \hat{A}^l_{j\gamma}A^k_{li} - \hat{A}^l_{i\gamma}A^k_{lj}\Big)\delta^*_k \end{cases} \tag{3.16}$$

$$
\begin{cases}
(19)\mathcal{R}(\delta^*_\alpha,\delta^*_\beta)\delta^*_k = \Big(P^\lambda_{\beta k}\widetilde{\Gamma}^l_{\alpha\ \lambda} - P^\lambda_{\alpha k}\widetilde{\Gamma}^l_{\beta\ \lambda}\Big)\partial_l + \Big(\widetilde{\Gamma}^\lambda_{\alpha\ k}\widetilde{\Gamma}^l_{\beta\ \lambda} - \widetilde{\Gamma}^\lambda_{\beta\ k}\widetilde{\Gamma}^l_{\alpha\ \lambda}\Big)\delta^*_l \\
+\Big(\frac{\delta^* P^\mu_{\beta k}}{\delta u^\alpha} - \frac{\delta^* P^\mu_{\alpha k}}{\delta u^\beta} + P^\lambda_{\alpha k}\overline{\Gamma}^\mu_{\beta\ \lambda} - P^\lambda_{\beta k}\overline{\Gamma}^\mu_{\alpha\ \lambda} + \widetilde{\Gamma}^\lambda_{\beta\ k}Q^\mu_{\lambda\alpha} - \widetilde{\Gamma}^\lambda_{\alpha\ k}Q^\mu_{\lambda\beta}\Big)\partial_\mu \\
\Big(P^\lambda_{\beta k}D^\mu_{\lambda\alpha} - P^\lambda_{\alpha k}D^\mu_{\lambda\beta} + \widetilde{\Gamma}^\lambda_{\beta\ k}\overline{\Gamma}^\mu_{\lambda\ \alpha} - \widetilde{\Gamma}^\lambda_{\alpha\ k}\overline{\Gamma}^\mu_{\lambda\ \beta}\Big)\delta^*_\mu, \\
(20)\mathcal{R}(\delta^*_\alpha,\delta^*_\beta)\partial_k = \Big(\widetilde{\Gamma}^\lambda_{\beta\ k}\widetilde{\Gamma}^l_{\lambda\ \alpha} - \widetilde{\Gamma}^\lambda_{\alpha\ k}\widetilde{\Gamma}^l_{\lambda\ \beta}\Big)\partial_l + \Big(\hat{D}^\lambda_{\beta k}\widetilde{\Gamma}^l_{\alpha\ \lambda} - \hat{D}^\lambda_{\alpha k}\widetilde{\Gamma}^l_{\beta\ \lambda}\Big)\delta^*_l \\
+\Big(\widetilde{\Gamma}^\lambda_{\beta\ k}\overline{\Gamma}^\mu_{\lambda\ \alpha} - \widetilde{\Gamma}^\lambda_{\alpha\ k}\overline{\Gamma}^\mu_{\lambda\ \beta} + \hat{D}^\lambda_{\beta k}Q^\mu_{\lambda\alpha} - \hat{D}^\lambda_{\alpha k}Q^\mu_{\lambda\beta} + \overline{R}^{\lambda}_{\alpha\beta 0}\hat{B}^\mu_{\lambda k}\Big)\partial_\mu \\
+\Big(\frac{\delta^*\hat{D}^\mu_{\beta k}}{\delta u^\alpha} - \frac{\delta^*\hat{D}^\mu_{\alpha k}}{\delta u^\beta} + \widetilde{\Gamma}^\lambda_{\beta\ k}D^\mu_{\lambda\alpha} - \widetilde{\Gamma}^\lambda_{\alpha\ k}D^\mu_{\lambda\beta} + \hat{D}^\lambda_{\beta k}\overline{\Gamma}^\mu_{\lambda\ \alpha} - \hat{D}^\lambda_{\alpha k}\overline{\Gamma}^\mu_{\lambda\ \beta}\Big)\delta^*_\mu, \\
(21)\mathcal{R}(\delta^*_\alpha,\partial_\beta)\delta^*_k = \Big(\hat{D}^\lambda_{\beta k}Q^\mu_{\lambda\alpha} - P^\lambda_{\alpha k}B^\mu_{\lambda\beta} - \frac{\partial P^\mu_{\alpha k}}{\partial v^\beta}\Big)\partial_\mu + \Big(\hat{D}^\lambda_{\beta k}\widetilde{\Gamma}^l_{\lambda\ \alpha} \\
-\widetilde{\Gamma}^p_{\alpha\ \beta}C^l_{pk}\Big)\delta^*_l + \Big(\frac{\delta^*\hat{D}^\mu_{\beta k}}{\delta u^\alpha} + \hat{D}^\lambda_{\beta k}\overline{\Gamma}^\mu_{\lambda\ \alpha} - \widetilde{\Gamma}^\lambda_{\alpha\ k}D^\mu_{\lambda\beta} - \widetilde{\Gamma}^\lambda_{\alpha\ \beta}\hat{D}^\mu_{\lambda k}\Big)\delta^*_\mu, \\
(22)\mathcal{R}(\delta^*_\alpha,\partial_\beta)\partial_k = \Big(\hat{B}^\lambda_{\beta k}\widetilde{\Gamma}^l_{\lambda\ \alpha} - \widetilde{\Gamma}^p_{\alpha\ \beta}A^l_{pk} - \overline{\Gamma}^{\lambda}_{\alpha\ \beta}\hat{A}^l_{\lambda k}\Big)\partial_l \\
+\Big(\frac{\delta^*\hat{B}^\mu_{\beta k}}{\delta u^\alpha} + \hat{B}^\lambda_{\beta k}\overline{\Gamma}^\mu_{\lambda\ \alpha} - \widetilde{\Gamma}^\lambda_{\alpha\ k}B^\mu_{\lambda\beta}\Big)\partial_\mu + \Big(\hat{B}^\lambda_{\beta k}D^\mu_{\lambda\alpha} - \frac{\partial\hat{D}^\mu_{\alpha k}}{\partial v^\beta} - \hat{D}^\lambda_{\alpha k}D^\mu_{\lambda\beta}\Big)\delta^*_\mu, \\
(23)\mathcal{R}(\partial_\alpha,\partial_\beta)\delta^*_k = \Big(\frac{\partial\hat{D}^\mu_{\beta k}}{\partial v^\alpha} - \frac{\partial\hat{D}^\mu_{\alpha k}}{\partial v^\beta} + \hat{D}^\lambda_{\beta k}D^\mu_{\lambda\alpha} - \hat{D}^\lambda_{\alpha k}D^\mu_{\lambda\beta}\Big)\delta^*_\mu, \\
(24)\mathcal{R}(\partial_\alpha,\partial_\beta)\partial_k = \Big(\frac{\partial\hat{B}^\mu_{\beta k}}{\partial v^\alpha} - \frac{\partial\hat{B}^\mu_{\alpha k}}{\partial v^\beta} + \hat{B}^\lambda_{\beta k}B^\mu_{\lambda\alpha} - \hat{B}^\lambda_{\alpha k}B^\mu_{\lambda\beta}\Big)\partial_\mu
\end{cases}
\tag{3.17}
$$

$$
\begin{cases}
(25)\mathcal{R}(\delta^*_i,\delta^*_\beta)\delta^*_k = \Big(\widetilde{\Gamma}^\lambda_{\beta\ k}F^l_{\lambda i} - \frac{\delta^* E^l_{jk}}{\delta u^\beta}\Big)\partial_l + \Big(\frac{\delta^* P^\mu_{\beta k}}{\partial x^i} + P^\lambda_{\beta k}\widetilde{\Gamma}^\mu_{\lambda\ i} - E^l_{ik}\widetilde{\Gamma}^\mu_{l\ \beta} \\
-\Gamma^l_{i\ k}P^\mu_{l\beta}\Big)\partial_\mu + \Big(P^\lambda_{\beta k}\hat{C}^l_{\lambda i} + \widetilde{R}^j_{i\beta\lambda}C^l_{jk}v^\lambda\Big)\delta^*_l + \Big(\frac{\delta^*\widetilde{\Gamma}^\mu_{\beta\ k}}{\delta x^i} + \widetilde{\Gamma}^\lambda_{\beta\ k}\widetilde{\Gamma}^\mu_{i\ \lambda} - E^j_{ik}\hat{D}^\mu_{j\beta} \\
-\Gamma^j_{i\ k}\widetilde{\Gamma}^\mu_{\beta\ j} + \widetilde{R}^\lambda_{i\beta j}y^j\hat{D}^\mu_{\lambda k}\Big)\delta^*_\mu, \\
(26)\mathcal{R}(\delta^*_i,\delta^*_\beta)\partial_k = \Big(\hat{D}^\lambda_{\beta k}F^l_{i\lambda} + \widetilde{R}^j_{i\beta\lambda}v^\lambda A^l_{jk}\Big)\partial_l + \Big(\widetilde{\Gamma}^\lambda_{\beta\ k}\hat{C}^l_{i\lambda} - \frac{\delta^* C^l_{ik}}{\delta u^\beta}\Big)\delta^*_l \\
+\Big(\frac{\delta^*\widetilde{\Gamma}^\mu_{\beta\ k}}{\delta x^i} + \widetilde{\Gamma}^\lambda_{\beta\ k}\widetilde{\Gamma}^\mu_{i\ \lambda} - \Gamma^j_{i\ k}\widetilde{\Gamma}^\mu_{j\ \beta} - C^j_{ik}P^\mu_{j\beta} + \widetilde{R}^\lambda_{i\beta j}y^j\hat{B}^\mu_{\lambda k}\Big)\partial_\mu \\
+\Big(\frac{\delta^*\hat{D}^\mu_{\beta k}}{\delta x^i} + \hat{D}^\lambda_{\beta k}\widetilde{\Gamma}^\mu_{i\ \lambda} - \Gamma^j_{i\ k}\hat{D}^\mu_{\beta j} - C^j_{ik}\widetilde{\Gamma}^\mu_{\beta\ j}\Big)\delta^*_\mu, \\
(27)\mathcal{R}(\delta^*_i,\partial_\beta)\delta^*_k = \Big(\hat{D}^\lambda_{\beta k}F^l_{i\lambda} - \frac{\partial E^l_{ik}}{\partial v^\beta}\Big)\partial_l - E^j_{ik}\hat{B}^\mu_{j\beta}\partial_\mu \\
+\Big(\frac{\delta^*\hat{D}^\mu_{\beta k}}{\delta x^i} + \hat{D}^\lambda_{\beta k}\widetilde{\Gamma}^\mu_{\lambda\ i} - \Gamma^j_{i\ k}\hat{D}^\mu_{\beta j} - \widetilde{\Gamma}^\lambda_{i\ \beta}\hat{D}^\mu_{\lambda k}\Big)\delta^*_\mu, \\
(28)\mathcal{R}(\delta^*_i,\partial_\beta)\partial_k = \hat{B}^\lambda_{\beta k}\hat{C}^l_{i\lambda}\delta^*_l + \Big(\frac{\delta^*\hat{B}^\mu_{\beta k}}{\delta x^i} + \hat{B}^\lambda_{\beta k}\widetilde{\Gamma}^\mu_{\lambda\ i} - \Gamma^j_{i\ k}\hat{B}^\mu_{\beta j} - \widetilde{\Gamma}^\lambda_{i\ \beta}B^\mu_{\lambda k}\Big)\partial_\mu \\
-C^j_{ik}\hat{D}^\mu_{\beta j}\delta^*_\mu, \\
(29)\mathcal{R}(\partial_i,\delta^*_\beta)\delta^*_k = P^\lambda_{\beta k}\hat{A}^l_{i\lambda}\partial_l + \Big(\widetilde{\Gamma}^\lambda_{\beta k}\hat{C}^l_{i\lambda} - \frac{\delta^* C^l_{ik}}{\delta v^\beta}\Big)\delta^*_l + \Big(\frac{\partial P^\mu_{\beta k}}{\partial y^i} - C^j_{ik}P^\mu_{\beta j}\Big)\partial_\mu \\
+\Big(\frac{\partial\widetilde{\Gamma}^\mu_{\beta\ k}}{\partial y^i} - C^j_{ik}\widetilde{\Gamma}^\mu_{\beta\ j} - \widetilde{\Gamma}^\lambda_{\beta\ i}\hat{D}^\mu_{\lambda k}\Big)\delta^*_\mu, \\
(30)\mathcal{R}(\partial_i,\delta^*_\beta)\partial_k = \Big(\widetilde{\Gamma}^\lambda_{\beta\ k}\hat{A}^l_{\lambda i} - \frac{\delta^* A^l_{jk}}{\delta u^\beta}\Big)\partial_l + \hat{D}^\lambda_{\beta k}\hat{C}^l_{\lambda i}\delta^*_l - \Big(A^j_{ik}\widetilde{\Gamma}^\mu_{\beta\ j} \\
-\widetilde{\Gamma}^\lambda_{\beta\ i}\hat{B}^\mu_{\lambda k}\Big)\partial_\mu + \Big(\frac{\partial\hat{D}^\mu_{\beta k}}{\partial y^i} - A^j_{ik}\hat{D}^\mu_{\beta j}\Big)\delta^*_\mu, \\
(31)\mathcal{R}(\delta^*_i,\delta^*_\beta)\delta^*_k = \Big(\hat{D}^\lambda_{\beta k}\hat{C}^l_{\lambda i} - \frac{\partial C^l_{ik}}{\partial v^\beta}\Big)\delta^*_l + \Big(\frac{\partial\hat{D}^\mu_{\beta k}}{\partial y^i} - C^l_{ik}\hat{D}^\mu_{l\beta}\Big)\delta^*_\mu, \\
(32)\mathcal{R}(\delta^*_i,\delta^*_\beta)\partial_k = \Big(\hat{B}^\lambda_{\beta k}\hat{A}^l_{\lambda i} - \frac{\partial A^l_{ik}}{\partial v^\beta}\Big)\partial_l + \Big(\frac{\partial\hat{B}^\mu_{\beta k}}{\partial y^i} - A^l_{ik}\hat{B}^\mu_{l\beta}\Big)\partial_\mu,
\end{cases}
\tag{3.18}
$$

$$\left\{\begin{array}{l}
(33)\mathcal{R}(\delta^*_\alpha,\partial_j)\delta^*_\gamma=\Big(\frac{\delta^* F^k_{j\gamma}}{\delta u^\alpha}-\widetilde{\Gamma}^l_{\alpha\ \gamma}E^k_{lj}-\overline{\Gamma}^\lambda_{\alpha\ \gamma}F^k_{j\lambda}\Big)\partial_k\\
+\Big(\widetilde{\Gamma}^\lambda_{j\ \gamma}\widetilde{\Gamma}^k_{\alpha\ \lambda}-Q^\lambda_{\alpha\gamma}\hat{C}^k_{j\lambda}-\frac{\delta^*\widetilde{\Gamma}^k_{\alpha\ \gamma}}{\delta x^j}-\widetilde{\Gamma}^l_{\alpha\ \gamma}\Gamma^k_{j\ l}-\widetilde{R}^l_{\alpha j\lambda}v^\lambda\hat{C}^k_{l\gamma}\Big)\delta^*_k\\
+\Big(F^l_{j\gamma}\widetilde{\Gamma}^\mu_{\alpha\ l}+\widetilde{\Gamma}^\lambda_{j\ \gamma}Q^\mu_{\alpha\lambda}-\frac{\delta^* Q^\mu_{\alpha\gamma}}{\delta x^j}-Q^\lambda_{\alpha\gamma}\widetilde{\Gamma}^\mu_{\lambda\ j}\Big)\partial_\mu\\
+\Big(F^l_{j\gamma}\hat{D}^\mu_{\alpha l}+\widetilde{\Gamma}^\lambda_{j\ \gamma}\overline{\Gamma}^\mu_{\alpha\ \lambda}+\frac{\delta^*\widetilde{\Gamma}^\mu_{j\ \gamma}}{\delta u^\alpha}-\frac{\delta^*\overline{\Gamma}^\mu_{\alpha\ \gamma}}{\delta x^j}-\overline{\Gamma}^\lambda_{\alpha\ \gamma}\widetilde{\Gamma}^\mu_{j\ \lambda}+\widetilde{R}^\lambda_{\alpha jh}y^h D^\mu_{\lambda\gamma}\Big)\delta^*_\mu,\\
(34)\mathcal{R}(\delta^*_i,\partial_j)\partial_\gamma=\Big(\widetilde{\Gamma}^\lambda_{j\ \gamma}\widetilde{\Gamma}^k_{\alpha\ \lambda}-\widetilde{\Gamma}^l_{\alpha\ \gamma}\Gamma^k_{j\ l}-\frac{\delta^*\widetilde{\Gamma}^k_{\alpha\ \gamma}}{\delta x^j}-D^\lambda_{\alpha\gamma}F^k_{j\lambda}\\
+\widetilde{R}^l_{\alpha j\lambda}v^\lambda\hat{A}^k_{l\gamma}\Big)\partial_k+\Big(\frac{\delta^*\hat{C}^k_{j\gamma}}{\delta u^\alpha}-\widetilde{\Gamma}^l_{\alpha\ \gamma}C^k_{jl}\Big)\delta^*_k+\Big(\frac{\delta^*\widetilde{\Gamma}^\mu_{j\ \gamma}}{\delta u^\alpha}+\widetilde{\Gamma}^\lambda_{j\ \gamma}\overline{\Gamma}^\mu_{\alpha\ \lambda}+\hat{C}^l_{j\gamma}P^\mu_{\alpha l}\\
+\widetilde{R}^\lambda_{\alpha jh}y^h B^\mu_{\lambda\gamma}\Big)\partial_\mu+\Big(\widetilde{\Gamma}^\lambda_{j\ \gamma}D^\mu_{\alpha\lambda}+\hat{C}^l_{j\gamma}\widetilde{\Gamma}^\mu_{\alpha\ l}-\frac{\delta^* D^\mu_{\alpha\gamma}}{\delta x^j}-D^\lambda_{\alpha\gamma}\widetilde{\Gamma}^\mu_{j\ \lambda}\Big)\delta^*_\mu,\\
(35)\mathcal{R}(\delta^*_\alpha,\partial_j)\delta^*_\gamma=\Big(-\widetilde{\Gamma}^l_{\alpha\ \gamma}A^k_{jl}-\overline{\Gamma}^\lambda_{\alpha\ \gamma}\hat{A}^k_{\lambda j}\Big)\partial_k+\Big(\frac{\delta^*\hat{C}^k_{j\gamma}}{\delta u^\alpha}-D^\lambda_{\alpha\gamma}\hat{C}^k_{\lambda j}\Big)\delta^*_k\\
+\hat{C}^l_{j\gamma}P^\mu_{l\gamma}\partial_\mu+\Big(\hat{C}^l_{j\gamma}\widetilde{\Gamma}^\mu_{l\ \gamma}-\widetilde{\Gamma}^\lambda_{j\ \alpha}D^\mu_{\lambda\gamma}\Big)\delta^*_\mu,\\
(36)\mathcal{R}(\delta^*_\alpha,\partial_j)\partial_\gamma=\Big(\frac{\delta^*\hat{A}^k_{j\gamma}}{\delta u^\alpha}-\widetilde{\Gamma}^l_{\alpha\ \gamma}A^k_{jl}-\overline{\Gamma}^\lambda_{\alpha\ \gamma}\hat{A}^k_{\lambda j}\Big)\partial_k-D^\lambda_{\alpha\gamma}\hat{C}^k_{\lambda j}\delta^*_k\\
+\Big(-\widetilde{\Gamma}^\lambda_{\alpha\ j}B^\mu_{\lambda\gamma}+\hat{A}^l_{j\gamma}\widetilde{\Gamma}^\mu_{\alpha\ l}\Big)\partial_\mu+\hat{A}^l_{j\gamma}\hat{D}^\mu_{l\alpha}\delta^*_\mu,\\
(37)\mathcal{R}(\partial_\alpha,\delta^*_j)\delta^*_\gamma=\Big(\frac{\partial F^k_{j\gamma}}{\partial v^\alpha}-D^\lambda_{\alpha\gamma}F^k_{j\lambda}\Big)\partial_k+F^l_{j\gamma}\hat{B}^\mu_{l\alpha}\partial_\mu\\
+\Big(\frac{\partial\widetilde{\Gamma}^\mu_{j\ \gamma}}{\partial v^\alpha}-\frac{\delta^* D^\mu_{\alpha\gamma}}{\delta x^j}+\widetilde{\Gamma}^\lambda_{j\ \gamma}D^\mu_{\lambda\alpha}-D^\lambda_{\alpha\gamma}\widetilde{\Gamma}^\mu_{j\ \lambda}+\widetilde{\Gamma}^\lambda_{\alpha\ j}D^\mu_{\lambda\gamma}\Big)\delta^*_\mu,\\
(38)\mathcal{R}(\partial_\alpha,\delta^*_j)\partial_\gamma=\Big(\frac{\partial\hat{C}^k_{j\gamma}}{\partial v^\alpha}-B^\lambda_{\alpha\gamma}\hat{C}^k_{\lambda j}\Big)\delta^*_k\\
+\Big(\frac{\partial\widetilde{\Gamma}^\mu_{j\ \gamma}}{\partial v^\alpha}+\widetilde{\Gamma}^\lambda_{j\ \gamma}B^\mu_{\alpha\lambda}-\frac{\delta^* B^\mu_{\alpha\gamma}}{\delta x^j}-B^\lambda_{\alpha\gamma}\widetilde{\Gamma}^\mu_{j\ \lambda}+\widetilde{\Gamma}^\lambda_{\alpha\ j}B^\mu_{\lambda\gamma}\Big)\partial_\mu+\hat{C}^l_{j\gamma}\hat{D}^\mu_{\alpha l}\delta^*_\mu,\\
(39)\mathcal{R}(\partial_\alpha,\partial_j)\delta^*_\gamma=\Big(\frac{\partial\hat{C}^k_{j\gamma}}{\partial v^\alpha}-D^\lambda_{\alpha\gamma}\hat{C}^k_{j\lambda}\Big)\delta^*_k+\Big(\hat{C}^l_{j\gamma}\hat{D}^\mu_{\alpha l}-\frac{\partial D^\mu_{\alpha\gamma}}{\partial y^j}\Big)\delta^*_\mu,\\
(40)\mathcal{R}(\partial_\alpha,\partial_j)\partial_\gamma=\Big(\frac{\partial\hat{A}^k_{j\gamma}}{\partial v^\alpha}-B^\lambda_{\alpha\gamma}\hat{A}^k_{j\lambda}\Big)\partial_k+\Big(\hat{A}^l_{j\gamma}\hat{B}^\mu_{\alpha l}-\frac{\partial B^\mu_{\alpha\gamma}}{\partial y^j}\Big)\partial_\mu,
\end{array}\right. \tag{3.19}$$

where, we put

$$\left\{\begin{array}{l}
\nabla^{(1)}_{\delta^*_i}T^l_{jk}:=\frac{\delta^* T^l_{jk}}{\delta x^i}+\Gamma^l_{ih}T^h_{jk}-\Gamma^h_{ij}T^l_{hk}-\Gamma^h_{ik}T^l_{jh}\\
\nabla^{(2)}_{\delta^*_\alpha}\mathbf{T}^\lambda_{\beta\gamma}:=\frac{\delta^*\mathbf{T}^\lambda_{\beta\gamma}}{\delta u^\alpha}+\overline{\Gamma}^\lambda_{\alpha\ \mu}\mathbf{T}^\mu_{\beta\gamma}-\overline{\Gamma}^\mu_{\alpha\ \beta}\mathbf{T}^\lambda_{\mu\gamma}-\overline{\Gamma}^\mu_{\alpha\ \gamma}\mathbf{T}^\lambda_{\beta\mu},
\end{array}\right.$$

for every 3-tensors T^l_{jk} and $\mathbf{T}^\lambda_{\beta\gamma}$. Here, $\delta^*_i:=\frac{\delta^*}{\delta x^i}$, $\delta^*_\alpha:=\frac{\delta^*}{\delta u^\alpha}$, $\partial_i:=\frac{\partial}{\partial y^i}$ and $\partial_\alpha:=\frac{\partial}{\partial v^\alpha}$.

The **warped Ricci curvature** tensor, is a contractions of the warped Riemann curvature tensor $\mathcal{R}$ (with respect to warped pseudo-Riemannian metric G), and denoted by $\widetilde{\mathbf{Ric}}$. It is defined as

$$\widetilde{\mathbf{Ric}}(G)(X,Y):=trace(Z\mapsto\mathcal{R}(Z,X)Y)$$

for every $X,Y,Z\in\Gamma(T\widetilde{M})$.

Corollary 3.5. *From the Proposition 3.4 and also the relationships (3.14), (3.15), (3.16), (3.17), (3.18) and (3.19), we get the warped Ricci curvature tensor as follows*

$$\begin{cases} (1)\widetilde{\mathbf{Ric}}(\delta_i^*,\delta_j^*) = 2(\mathbf{Ric}(\widetilde{g}))_{ij} + \frac{\partial^2 f}{\partial x^i \partial x^j} - C^l_{hj}E^h_{li} + E^l_{ij}\hat{D}^{\alpha}_{l\alpha}, \\ (2)\widetilde{\mathbf{Ric}}(\delta_i^*,\partial_j) = 0, \\ (3)\widetilde{\mathbf{Ric}}(\partial_i,\partial_j) = A^l_{ij}C^k_{lk} - \frac{\partial C^k_{ij}}{\partial y^k} - C^l_{hj}C^h_{li} - \frac{\partial \hat{D}^{\alpha}_{\alpha j}}{\partial y^i} + A^l_{ij}\hat{D}^{\alpha}_{l\alpha} \\ +A^l_{ij}A^h_{lh} - A^l_{hj}A^h_{li} + \frac{\partial A^h_{ih}}{\partial y^j} - \frac{\partial A^h_{jh}}{\partial y^i}, \end{cases} \tag{3.20}$$

$$\begin{cases} (1)\widetilde{\mathbf{Ric}}(\delta_\alpha^*,\delta_\beta^*) = 2(\mathbf{Ric}(\widetilde{g}))_{\alpha\beta} - \frac{\partial (e^{2f})^k}{\partial x^k}\overline{g}_{\alpha\beta} - Q^{\lambda}_{\alpha\beta}\hat{C}^k_{k\lambda} - \frac{\partial Q^{\gamma}_{\alpha\beta}}{\partial v^{\gamma}} \\ +D^{\lambda}_{\alpha\beta}Q^{\gamma}_{\lambda\gamma} - Q^{\lambda}_{\alpha\beta}B^{\gamma}_{\lambda\gamma} + Q^{\lambda}_{\beta\gamma}D^{\gamma}_{\lambda\alpha} - Q^{\lambda}_{\alpha\beta}D^{\gamma}_{\lambda\gamma} - \widetilde{\Gamma}^l_{\alpha\ \beta}A^h_{th} - \overline{\Gamma}^{\lambda}_{\alpha\ \beta}\hat{A}^h_{\lambda h}, \\ (2)\widetilde{\mathbf{Ric}}(\delta_\alpha^*,\partial_\beta) = 0, \\ (3)\widetilde{\mathbf{Ric}}(\partial_\alpha,\partial_\beta) = -\frac{\partial \hat{C}^k_{k\beta}}{\partial v^{\alpha}} + B^{\lambda}_{\alpha\beta}\hat{C}^k_{\lambda k} - \frac{\partial \hat{A}^h_{h\beta}}{\partial v^{\alpha}} + B^{\lambda}_{\alpha\beta}\hat{A}^k_{\lambda k} - \frac{\partial D^{\mu}_{\beta\mu}}{\partial v^{\alpha}} \\ +B^{\lambda}_{\mu\beta}B^{\mu}_{\lambda\alpha} - B^{\lambda}_{\alpha\beta}B^{\mu}_{\lambda\mu} + B^{\lambda}_{\alpha\beta}D^{\mu}_{\lambda\mu} - D^{\lambda}_{\mu\beta}D^{\mu}_{\lambda\alpha} + \frac{\partial B^{\mu}_{\beta\mu}}{\partial v^{\alpha}} - \frac{\partial B^{\mu}_{\alpha\beta}}{\partial v^{\mu}}, \end{cases} \tag{3.21}$$

$$\begin{cases} (1)\widetilde{\mathbf{Ric}}(\delta_i^*,\delta_\beta^*) = F^l_{i\beta}C^k_{lk} - F^l_{k\beta}C^k_{li} + \widetilde{R}^l_{ik0}\hat{C}^k_{l\beta} \\ -\hat{C}^l_{h\beta}E^h_{li} - F^l_{i\beta}\hat{B}^{\mu}_{l\mu} + \frac{\partial F^h_{i\beta}}{\partial y^h} + F^l_{i\beta}A^h_{lh}, \\ (2)\widetilde{\mathbf{Ric}}(\delta_i^*,\partial_\beta) = 0, \\ (3)\widetilde{\mathbf{Ric}}(\partial_i,\delta_\beta^*) = 0, \\ (4)\widetilde{\mathbf{Ric}}(\partial_i,\partial_\beta) = \hat{A}^l_{i\beta}C^k_{lk} - \frac{\partial \hat{C}^k_{k\beta}}{\partial y^i} + \hat{A}^l_{i\beta}A^h_{lh} - \hat{A}^l_{h\beta}A^h_{li} \\ +\hat{A}^l_{i\beta}\hat{B}^{\mu}_{l\mu} - \frac{\partial B^{\mu}_{\beta\mu}}{\partial y^i} + \hat{A}^l_{i\beta}\hat{D}^{\mu}_{l\mu} + \frac{\partial \hat{A}^h_{i\beta}}{\partial y^h} - \frac{\partial \hat{A}^h_{h\beta}}{\partial y^i}. \end{cases} \tag{3.22}$$

4. **The Warped Lagrange Geometry**

In this section, we find some interesting geometric properties of the tangent bundle $T\widetilde{M} = TM \oplus T\overline{M}$ by using a Lagrangian function defined as integral of a real smooth function $\varphi : [0,\infty) \to \mathbb{R}$ such that depending on the warped kinetic energy only, i.e.

$$L := \int \varphi(t)dt \tag{4.1}$$

where,

$$t = t(\widetilde{t},\overline{t}) = \widetilde{t} + e^{2f}\overline{t}.$$

We have $t \in [0,\infty)$ for all $(y,v) \in T\widetilde{M}$. We suppose $\varphi(t) > 0$ and $\varphi'(t) > 0$ for every $t \geq 0$, therefore, the Lagrangian function L is regular [4]. Now, by using relation 3.2, we consider the symmetric $\widetilde{M}$-tensor field on $T\widetilde{M}$, defined by the components

$$G_{ij} = \frac{\partial^2 L}{\partial y^i \partial y^j} = \varphi' g_{i0} g_{j0} + \varphi g_{ij}, \tag{4.2}$$

$$G_{\alpha\beta} = \frac{\partial^2 L}{\partial v^\alpha \partial v^\beta} = e^{4f} \varphi' \overline{g}_{\alpha 0} \overline{g}_{\beta 0} + e^{2f} \varphi \overline{g}_{\alpha\beta}, \tag{4.3}$$

$$G_{\alpha i} = \frac{\partial^2 L}{\partial y^i \partial v^\beta} = \varphi' e^{2f} g_{i0} \overline{g}_{\alpha 0}. \tag{4.4}$$

As usual in the warped Lagrange geometry, a regular Lagrangian L defines a warped non-linear connection $\mathbf{N}_a^b$ on the bundle $T\widetilde{M}$ given by the warped horizontal distribution $\overset{\circ}{H}(T\widetilde{M})$ spanned by $\left\{ \widetilde{\left(\frac{\delta}{\delta x^i}\right)}, \widetilde{\left(\frac{\delta}{\delta u^\alpha}\right)} \right\}$ where

$$\widetilde{\left(\frac{\delta}{\delta x^i}\right)} := \frac{\partial}{\partial x^i} - \mathbf{N}_i^j(x,u,y,v)\frac{\partial}{\partial y^j} - \mathbf{N}_i^\beta(x,u,y,v)\frac{\partial}{\partial v^\beta}$$

$$\widetilde{\left(\frac{\delta}{\delta u^\alpha}\right)} := \frac{\partial}{\partial u^\alpha} - \mathbf{N}_\alpha^j(x,u,y,v)\frac{\partial}{\partial y^j} - \mathbf{N}_\alpha^\beta(x,u,y,v)\frac{\partial}{\partial v^\beta}$$

and

$$\mathbf{N}_a^b = (\mathbf{N}_i^j, \mathbf{N}_i^\beta, \mathbf{N}_\alpha^j, \mathbf{N}_\alpha^\beta).$$

Here

$$\mathbf{N}_i^j := \frac{\partial \mathbf{N}^j}{\partial y^i}, \mathbf{N}_i^\beta := \frac{\partial \dot{\mathbf{N}}^\beta}{\partial y^i}, \mathbf{N}_\alpha^j := \frac{\partial \mathbf{N}^j}{\partial v^\alpha}, \mathbf{N}_\alpha^\beta := \frac{\partial \dot{\mathbf{N}}^\beta}{\partial v^\alpha}, \tag{4.5}$$

where

$$\begin{aligned} 2\mathbf{N}^i := {} & \widetilde{H}^{ij}\Big(\frac{\partial^2 L}{\partial y^j \partial x^k} y^k + \frac{\partial^2 L}{\partial y^j \partial u^\gamma} v^\gamma - \frac{\partial L}{\partial x^j}\Big) \\ & + \widetilde{H}^{i\beta}\Big(\frac{\partial^2 L}{\partial v^\beta \partial x^k} y^k + \frac{\partial^2 L}{\partial v^\beta \partial u^\gamma} v^\gamma - \frac{\partial L}{\partial u^\beta}\Big), \end{aligned} \tag{4.6}$$

$$\begin{aligned} 2\dot{\mathbf{N}}^\alpha := {} & \widetilde{H}^{\alpha\beta}\Big(\frac{\partial^2 L}{\partial v^\beta \partial u^\gamma} v^\gamma + \frac{\partial^2 L}{\partial v^\beta \partial x^k} y^k - \frac{\partial L}{\partial u^\beta}\Big) \\ & + \widetilde{H}^{\alpha j}\Big(\frac{\partial^2 L}{\partial y^j \partial u^\gamma} v^\gamma + \frac{\partial^2 L}{\partial y^j \partial x^k} y^k - \frac{\partial L}{\partial x^j}\Big). \end{aligned} \tag{4.7}$$

Using (4.1), (4.5), (4.6), (4.7) and by straightforward computation the following theorem obtained.

Proposition 4.1. *Given the regular Lagrangian defined by (4.1), we have* $\overset{\circ}{H}(T\widetilde{M}) = H(T\widetilde{M})$.

Corollary 4.2. *(1) Taking into account the Proposition (4.1) and the condition $\psi = \varphi'$ in the expression (3.3) of G, it follows that, the warped pseudo Riemannian metric G defined on the tangent bundle $T\widetilde{M} = TM \oplus T\overline{M}$, is the complete lift of the quadratic form*

$$\mathbf{h} := G_{ij}dx^i \otimes dx^j + G_{i\beta}dx^i \otimes du^\beta + G_{\alpha j}du^\alpha \otimes dx^i + G_{\alpha\beta}du^\alpha \otimes du^\beta. \quad (4.8)$$

(2) From relation (3.13) it easily follows that

$$\begin{aligned}
&\frac{\partial A^h_{ih}}{\partial y^j} - \frac{\partial A^h_{jh}}{\partial y^i} = 0,\\
&A^k_{ij}A^h_{kh} - A^k_{hj}A^h_{ki} = 0,\\
&\frac{\partial B^\mu_{\beta\mu}}{\partial v^\alpha} - \frac{\partial B^\mu_{\alpha\beta}}{\partial v^\mu} = 0,\\
&B^\tau_{\mu\beta}B^\mu_{\tau\alpha} - B^\tau_{\alpha\beta}B^\mu_{\tau\mu} = 0,\\
&\frac{\partial \hat{A}^h_{i\beta}}{\partial y^h} - \frac{\partial \hat{A}^h_{h\beta}}{\partial y^i} = 0,\\
&\hat{A}^l_{i\beta}A^h_{lh} - \hat{A}^l_{h\beta}A^h_{li} = 0.
\end{aligned}$$

Now, let us consider $\psi = \varphi'$ then, from relations (3.20), (3.21) and (3.22) it follows that

$$\begin{cases}
(1)\widetilde{\mathbf{Ric}}(\delta^*_i, \delta^*_j) = 2(\mathbf{Ric}(\widetilde{g}))_{ij} + \frac{\partial^2 f}{\partial x^i \partial x^j},\\
(2)\widetilde{\mathbf{Ric}}(\delta^*_i, \partial_j) = 0,\\
(3)\widetilde{\mathbf{Ric}}(\partial_i, \partial_j) = 0,
\end{cases} \quad (4.9)$$

$$\begin{cases}
(1)\widetilde{\mathbf{Ric}}(\delta^*_\alpha, \delta^*_\beta) = 2(\mathbf{Ric}(\widetilde{g}))_{\alpha\beta} - \frac{\partial (e^{2f})^k}{\partial x^k}\overline{g}_{\alpha\beta}\\
-\frac{\partial Q^\gamma_{\alpha\beta}}{\partial v^\gamma} - Q^\lambda_{\alpha\beta}B^\gamma_{\lambda\gamma} - \widetilde{\Gamma}^l_{\alpha\ \beta}A^h_{lh} - \overline{\Gamma}^\lambda_{\alpha\ \beta}\hat{A}^h_{\lambda h},\\
(2)\widetilde{\mathbf{Ric}}(\delta^*_\alpha, \partial_\beta) = 0,\\
(3)\widetilde{\mathbf{Ric}}(\partial_\alpha, \partial_\beta) = -\frac{\partial \hat{A}^h_{h\beta}}{\partial v^\alpha} + B^\lambda_{\alpha\beta}\hat{A}^k_{\lambda k},
\end{cases} \quad (4.10)$$

$$\begin{cases}
(1)\widetilde{\mathbf{Ric}}(\delta^*_i, \delta^*_\beta) = -F^l_{i\beta}\hat{B}^\mu_{l\mu} + \frac{\partial F^h_{i\beta}}{\partial y^h} + F^l_{i\beta}A^h_{lh},\\
(2)\widetilde{\mathbf{Ric}}(\delta^*_i, \partial_\beta) = 0,\\
(3)\widetilde{\mathbf{Ric}}(\partial_i, \delta^*_\beta) = 0,\\
(4)\widetilde{\mathbf{Ric}}(\partial_i, \partial_\beta) = \hat{A}^l_{i\beta}\hat{B}^\mu_{l\mu} - \frac{\partial B^\mu_{\beta\mu}}{\partial y^i}.
\end{cases} \quad (4.11)$$

5. **Main Results**

Using the Corollary (4.2), we obtain the main results of this paper as follows.

Theorem 5.1. *Suppose, $(T\widetilde{M}, G)$ is the warped pseudo Riemannian manifold, where G is defined by (3.3). Assume the warping function $f : M \to \mathbb{R}$ holds in the following conditions:*

$$\nabla^2 f = 0, \quad (e^{2f})^i = \frac{\partial e^{2f}}{\partial x^k} g^{ik} = 0. \tag{5.1}$$

If

$$\begin{cases} \frac{\partial \hat{A}^h_{h\beta}}{\partial v^\alpha} - B^\lambda_{\alpha\beta}\hat{A}^k_{\lambda k} = 0, \quad \hat{A}^l_{i\beta}\hat{B}^\mu_{l\mu} - \frac{\partial B^\mu_{\beta\mu}}{\partial y^i} = 0 \\ \frac{\partial Q^\gamma_{\alpha\beta}}{\partial v^\gamma} + Q^\lambda_{\alpha\beta}B^\gamma_{\lambda\gamma} + \overline{\Gamma}^\lambda_{\alpha\ \beta}\hat{A}^h_{\lambda h} = 0, \\ -F^l_{i\beta}\hat{B}^\mu_{l\mu} + \frac{\partial F^h_{i\beta}}{\partial y^h} + F^l_{i\beta}A^h_{lh} = 0, \end{cases} \tag{5.2}$$

then, the following assertions are equivalent:

- **(1):** *The warped pseudo Riemannian manifold $(T\widetilde{M}, G)$ is Ricci flat.*
- **(2):** *The warped Riemannian manifold $(\widetilde{M} = M \times_f \overline{M}, \widetilde{g} = g + e^{2f}\overline{g})$ is Ricci flat, and the functions $\varphi(t)$ and $\psi(t)$ are related by the condition $\varphi' = \psi$.*
- **(3):** *The warped Riemannian manifold $(\widetilde{M} = M \times_f \overline{M}, \widetilde{g} = g + e^{2f}\overline{g})$ is Ricci flat and the warped pseudo Riemannian metric G is the complete lift of the quadratic form* **h** *in the expression (4.8), where the components G_{ij}, $G_{\alpha\beta}$ and $G_{\alpha j}$ are defined by (4.2), (4.3) and (4.4).*

Theorem 5.2. *Suppose, the warped Riemannian manifold $(M \times_f \overline{M}, g + e^{2f}\overline{g})$ is Ricci flat, and the warped pseudo Riemannian metric G is complete lift of the quadratic form* **h**. *Whenever relations (5.1) and (5.2) are established then, for every function $F \in C^2(M \times_f \overline{M})$, a triple $(M \times_f \overline{M}, G, \nabla^G F)$ is the steady gradient Ricci soliton.*

Proof. By using (3.4) and relation $(HessF)(X,Y) = (\nabla^G)^2_{X,Y}F$ we have

$$\nabla^G_{\partial_i}\nabla^G_{\delta^*_j}F = -C^k_{ij}\frac{\partial F}{\partial x^k},$$

$$\nabla^G_{\delta^*_i}\nabla^G_{\partial_j}F = -C^k_{ij}\frac{\partial F}{\partial x^k}$$

$$\nabla^G_{\partial_\alpha}\nabla^G_{\delta^*_\beta}F = -D^\gamma_{\alpha\beta}\frac{\partial F}{\partial u^\gamma},$$

$$\nabla^G_{\delta^*_\alpha}\nabla^G_{\partial_\beta}F = -D^\gamma_{\alpha\beta}\frac{\partial F}{\partial u^\gamma}$$

$$\nabla^G_{\partial_i}\nabla^G_{\delta^*_\beta}F = -\hat{C}^k_{i\beta}\frac{\partial F}{\partial x^k},$$

$$\nabla^G_{\delta^*_\beta}\nabla^G_{\partial_i}F = -\hat{D}^\gamma_{i\beta}\frac{\partial F}{\partial u^\gamma}$$
$$\nabla^G_{\partial_\alpha}\nabla^G_{\delta^*_j}F = -\hat{D}^\gamma_{\alpha j}\frac{\partial F}{\partial u^\gamma},$$
$$\nabla^G_{\delta^*_i}\nabla^G_{\partial_\alpha}F = -\hat{C}^k_{i\alpha}\frac{\partial F}{\partial x^k}$$

So the condition $\varphi' = \psi$ in the Theorem (5.1) and Proposition (3.4) requires that

$$C^k_{ij} = 0,\ \hat{C}^k_{i\alpha} = 0,\ D^\gamma_{\alpha\beta} = 0,\ \hat{D}^\gamma_{i\beta} = 0,$$

therefore, $HessF \equiv 0$. □

A warped product Riemannian manifolds $(\widetilde{M} = M \times_f \overline{M}, \widetilde{g} = g + e^{2f}\overline{g})$ can be considered. The tangent bundle $T\widetilde{M} = TM \oplus T\overline{M}$ carries important global vertical field

$$\mathbb{L} := y^k\frac{\partial}{\partial y^k} + v^\gamma\frac{\partial}{\partial v^\gamma},$$

which dose not vanish on the manifold $T\widetilde{M}^\circ = T\widetilde{M}\setminus\{0\}$, and is independent of any Riemannian metric on the base manifold $\widetilde{M} = M \times_f \overline{M}$. It is called the *warped Liouville vector field.*

Theorem 5.3. *Let the warped Riemannian manifold $(\widetilde{M} = M \times_f \overline{M}, g + e^{2f}\overline{g})$ be a Ricci flat. Whenever relations (5.1) and (5.2) are established then, the tangent bundle $T\widetilde{M}$ carries a 1-parametric family of shrinking Ricci solitons $(G_\varepsilon, \mathbb{L}, \varepsilon)$ for $\varepsilon < 0$.*

Proof. We have

$$\mathbb{L}(G_{ij}) = 2\widetilde{t}(\varphi' g_{ij} + \varphi'' g_{i0}g_{j0}) + 2\varphi' g_{i0}g_{j0} + 2e^{2f}\overline{t}(\varphi' g_{ij} + \varphi'' g_{i0}g_{j0}), \quad (5.3)$$

$$\mathbb{L}(G_{i\alpha}) = 2e^{2f}\overline{g}_{\alpha 0}g_{i0}(\varphi''(\widetilde{t} + e^{2f}\overline{t}) + \varphi'), \quad (5.4)$$

$$\mathbb{L}(G_{\alpha\beta}) = 2e^{2f}\widetilde{t}(\varphi'\overline{g}_{\alpha\beta} + e^{2f}\varphi''\overline{g}_{\alpha 0}\overline{g}_{\beta 0}) + e^{4f}(\varphi'' + 2\varphi')\overline{g}_{\alpha 0}\overline{g}_{\beta 0}$$
$$+2\overline{t}e^{4f}(\varphi'\overline{g}_{\alpha\beta} + \varphi'' e^{2f}\overline{g}_{\alpha 0}\overline{g}_{\beta 0}). \quad (5.5)$$

Then, by using (1.1), (5.3), (5.4) and (5.5), for the warped Liouville vector field $\mathbb{L}$ the following cases occur:

1: $\widetilde{\mathbf{Ric}(G)}$ on $(\frac{\partial}{\partial y^i}, \frac{\delta^*}{\delta x^j})$ yields

$$\mathbb{L}(G_{ij}) + (2\varepsilon + 1)G_{ij} = 0,$$

which is equal to

$$\Big(2t\varphi' + (2\varepsilon + 1)\varphi\Big)g_{ij} + \Big(2(\varphi' + t\varphi'') + (2\varepsilon + 1)\varphi'\Big)g_{i0}g_{j0} = 0.$$

2: $\widetilde{\mathbf{Ric}(G)}$ on $(\frac{\partial}{\partial y^i}, \frac{\delta^*}{\delta u^\alpha})$ (or $(\frac{\partial}{\partial v^\alpha}, \frac{\delta^*}{\delta x^i})$) yields

$$\mathbb{L}(G_{i\alpha}) + (2\varepsilon+1)G_{i\alpha} = 0,$$

which is equal to

$$e^{2f}\Big(2t\varphi'' + (2\varepsilon+3)\varphi'\Big)g_{i0}\overline{g}_{\alpha 0} = 0.$$

3: $\widetilde{\mathbf{Ric}(G)}$ on $(\frac{\partial}{\partial v^\alpha}, \frac{\delta^*}{\delta u^\beta})$ yields

$$\mathbb{L}(G_{\alpha\beta}) + (2\varepsilon+1)G_{\alpha\beta} = 0,$$

which is equal to

$$\Big(2t\varphi' + (2\varepsilon+1)\varphi\Big)\overline{g}_{\alpha\beta} + \Big(2(\varphi' + t\varphi'') + (2\varepsilon+1)\varphi'\Big)\overline{g}_{\alpha 0}\overline{g}_{\beta 0} = 0.$$

Applying Lemma 1 of [6] the three previous cases are equivalent with

$$2t\varphi' = -(2\varepsilon+1)\varphi.$$

Hence, $\varphi(t) = t^{-(2\varepsilon+1)}$ and $\varphi(t) > 0$. Also, the condition $\varphi(t) + 2t\varphi'(t) > 0$ is equivalent with $\varepsilon < 0$ and therefore, the triplet $(G_\varepsilon, \mathbb{L}, \varepsilon)$ is shrinking Ricci solitons. □

6. Conclusion and Future Directions

In this work, we established a geometric framework for studying Ricci-flat metric on warped product manifolds $\widetilde{M} = M \times_f \overline{M}$ and their tangent bundles $T\widetilde{M}$. By introducing a Sasaki-Matsumoto lift, we constructed a pseudo-Riemannian metric G on $T\widetilde{M}$ and derived explicit conditions (Theorem 5.1-5.3) under which $(T\widetilde{M}, G)$ inherits Ricci-flatness or admits shrinking Ricci solitons.

Key takeaways include:

- The equivalence between Ricci-flat structures on $\widetilde{M}$ and $T\widetilde{M}$ hinges on the warped function f satisfying $\nabla^2 f = 0$ and the vanishing of specific tensor fields (Eq. 5.2).
- The Liouville vector field $\mathbb{L}$ naturally induces a family of shrinking solitons $(G_\epsilon, \mathbb{L}, \epsilon)$ for $\epsilon < 0$, governed by the ODE $\varphi(t) = t^{-(2\epsilon+1)}$.

Future directions:

- Extend this work to Lorentzian warped products, relevant in general relativity.
- Investigate the role of $\mathcal{L}_X g$ for non-Killing vector fields X.
- Explore applications in geometric mechanics, where $T\widetilde{M}$ models phase spaces.

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