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On quasi-Einstein Kropina metrics

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Abstract. In this paper, we consider weakly quasi-Einstein Finsler metrics, which is extension of Einstein conception. In fact, we investigate quasi-Einstein Kropina metrics in both regular and singular case and we find the necessary and sufficient conditions of quasi-Ricci flat kropina metrics.

Keywords: Kropina metrics, quasi-Einstein, quasi-Ricci flat.

1. Introduction

In Finsler geometry, the Ricci curvature plays an important role. It is a natural extension of the Ricci curvature in Riemannian geometry and defined as the trace of the Riemman curvature. A Finsler metric F is called an Einstein metric on an n-dimensional manifold M if it satisfies

$$Ric = (n-1)c(x)F^2, \tag{1.1}$$

where $c = c(x)$ is a scalar function [6][7]. Finsler metric F is said Ricci constant if F satisfies (1.1) where c is constant . Especially when $c = 0$, F is called Ricci flat. There is another quantity which is determined by the Busemann-Hausdorff volume form, that is the so-called distortion τ which the

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horizontal covariant derivative of τ gives a non-Riemannian quantity the S-curvature.

In Finsler geometry, so-called (α, β)-metrics are those Finsler metrics which can be expressed in the form $F = \alpha\phi(s)$, where $\alpha := \alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta := \beta(y) = b_i(x)y^i$ is a 1-form on M. In the past several years, we have witnessed a rapid development in Finsler geometry. This is partly because of the research on the (α, β)-metrics [2]. When $\phi(s) = \frac{1}{s}$, the Finsler metric $F = \frac{\alpha^2}{\beta}$ is called Kropina which was introduced by Berwald [8]. These metrics are called regular Finsler metrics if $\phi(s)$ is a smooth function on $(-b_0, b_0)$ satisfying

$$\phi(s) > 0, \quad \phi(s) - (s\phi'(s) + b^2 - s^2)\phi''(s) > 0, \quad |s| < b < b_0. \tag{1.2}$$

and β satisfies $||\beta||_\alpha < b_0$ (see [23]). If ϕ does not apply condition (1.2), then Finsler metrics have been called singular. Singular Finsler metrics is introduced by Z. Shen [3, 4]. In recent years, many scholars have conducted a great deal of research on them. Cheng-Shen-Tian proved that the polynomial (α, β)-metric is an Einstein metric if and only if it is Ricci-flat [5]. In 2012, Zhang and Shen specified the condition of Einstein Kropina metric . They proved a non-Riemannian Kropina metric $F = \alpha^2/\beta$ with constant Killing from β on a manifold M with dimensional $n \geq 2$, is an Einstein metric if and only if Reimannian metric α is an Einstein metric [16]. In Reimannian geometry, J. Case, Y .Shu and G .Wei studied m-quasi-Einstein which is a generalization of Einstein metrics [20, 21, 22, 10]. The Ricci-curvature and S-curvature have important and fundamental topic in Finsler geometry [15, 18]. Recently, Ohta introduced a definition of N-Ricci curvature in Finsler geometry [11]. This concept is generalized by H.Zhu, who characterize quasi-Einstein metrics. He found the structure of quasi-Ricci flat square metric which is the famous Berwalds metric [14].

Finsler manifold (M, F) is called N- weakly quasi-Einstein if it satisfies

$$Ric + \dot{\mathbf{S}} - \frac{\mathbf{S}^2}{N-n} = (n-1)\left(c + \frac{3\theta}{F}\right)F^2,$$

where $\dot{S}$ is the covariant derivative of S along a geodesic of F and $c = c(x)$ is scalar function and θ is a 1- form on M. If $\theta = 0$ and $N = \infty$, then Finsler metric F is called quasi-Einstein and if $c = 0$ is said quasi-Ricci flat.

In this paper, we are going to study Kropina metrics of quasi-Einstein and quasi-Ricci flat cases. In fact, the main theorem is as follows:

Theorem 1.1. *Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on n-dimensional manifold M with volume form $dV_F = e^{-f}dV_\alpha$. Then F is quasi-Einstein if and only if*

$$s^i_j s^j_i = -2\big[2c(n-1) + s^i s_i\big], \tag{1.3}$$

***Case I**: Assume $n \neq 2$*

- *if F be regular then*

$$Ric_\alpha = \frac{1}{B^2}\Big[(n-2)\big(s_0^2-\sigma^2\beta^2\big)-2(n-2)\sigma s_0\beta\Big] -\frac{1}{B}\Big[2s_{0|0}-2f_0s_0\Big]-f_{0|0}+\eta\alpha^2; \tag{1.4}$$

- *if F be singular then*

$$Ric_\alpha = (n-2)\bigg(s_0^2-\sigma^2\beta^2\bigg)-2(n-2)\sigma s_0\beta -2s_{0|0}+2f_0s_0-f_{0|0}+\eta\alpha^2, \tag{1.5}$$

Case II: *Assume $n=2$*

- *if F be regular then*

$$Ric_\alpha = -\frac{1}{B}\Bigg[2s_{0|0}-2f_0s_0\Bigg]-f_{0|0}+\eta\alpha^2; \tag{1.6}$$

- *if F be singular then $Ric_\alpha=-2s_{0|0}+2f_0s_0-f_{0|0}+\eta\alpha^2$,*

where $\eta=\eta(x)$ is function on M.

2. Preliminaries

In 1918, Finsler metrics studied by P.Finsler's [9]. Let F be a Finsler metric on manifold M, a spray is a smooth vector field G on TM_0 which is expressed by

$$G(x,y):=y^i\frac{\partial}{\partial x^i}-2G^i(x,y)\frac{\partial}{\partial y^i},$$

where geodesic coefficients defined by

$$G^i:=\frac{1}{4}g^{il}\Bigg[[F^2]_{x^my^l}y^m-[F^2]_{x^l}\Bigg],$$

and $G^i(x,\lambda y)=\lambda^2G^i(x,y)$, $\lambda>0$.

For Finsler metric F on manifold M, the Riemann curvature $R_y=R^i_k(y)\frac{\partial G^i}{\partial x^i}\otimes dx^k$ of F is defined by

$$R^i_k:=2\frac{\partial G^i}{\partial x^k}-y^j\frac{\partial^2G^i}{\partial x^j\partial y^k}+2G^j\frac{\partial^2G^i}{\partial y^j\partial y^k}-\frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{2.1}$$

Ricci curvature is the trace of the Riemann curvature, which is called by

$$Ric:=R^m_m. \tag{2.2}$$

For a Finsler metric F, let

$$R^i_k=cF^2\big(\delta^i_k-F^{-1}F_{y^k}y^i\big). \tag{2.3}$$

Then F is called of scalar curvature, where $c=c(x,y)$ is function on TM.

The Busemann-Hausdorff volume form $dV_{BH}(x) := \sigma_{BH} dx^1 \wedge ...... \wedge dx^n$ on Finsler space (M, F) is defined by

$$\sigma_{BH}(x) := \frac{w_n}{\big\{Vol(y^i) \in R^n | F\big(x, y^i \frac{\partial}{\partial x^i}|_x\big)\big\}},$$

where $Vol\{.\}$ denotes the Euclidean volume function and $w_n := Vol(B^n(1))$ denotes the unit ball in R^n. There is the scalar function $\tau = \tau(x, y)$ on TM_0 associated with the Busemann-Hausdorff volume form $dV_{BH} := \sigma_{BH}(x) dx^1 \wedge \cdots \wedge dx^n$ is called the distortion and is as following

$$\tau(x, y) := ln\Big[\frac{\sqrt{det\big(g_{ij}(x, y)\big)}}{\sigma_{BH}(x)}\Big].$$

The S-curvature is given by

$$\mathbf{S}(x, y) := \frac{d}{dt}\Big[\tau\big(c(t), c^{\cdot}(t)\big)\Big]|_0,$$

here $c(t)$ is the geodesic with $c(0) = x$ and $c^{\cdot}(x) = y$.

For Finsler metric F on manifold M, the S-curvature is defined by

$$\mathbf{S}(x, y) := \frac{\partial G^l}{\partial y^l} - \frac{y^l}{\sigma_{BH}} \frac{\partial(\sigma_{BH})}{\partial x^l}. \tag{2.4}$$

Let F be metric Finsler with volume form $dV_F = e^{-f} dV_{BH}$ on TM_0. Then quasi-Ricci curvature is called by

$$Qric := Ric + \dot{\mathbf{S}}, \tag{2.5}$$

where $\dot{S}$ is the covariant derivative of S along geodesic of F [14, 19]. The (α, β)-metric can be expressed by the form

$$F := \alpha\phi(s), \qquad s = \frac{\beta}{\alpha}. \tag{2.6}$$

It is known that is positive and strongly convex on TM_0 if and only if

$$\phi(s) - s\phi'(s) + \big(B - s^2\big)\phi''(s), \tag{2.7}$$

where $B := a^{ij} b_i b_j = ||B||\alpha^2$.

The spray coefficients of (α, β)-metrics are given by [13]

$$G^i = G^i_\alpha + Q^i, \tag{2.8}$$

where

$$Q^i := \alpha Q s_0^i + \theta\Big(r_{00} - 2\alpha Q s_0\Big)\frac{y^i}{\alpha} + \psi\Big(r_{00} - 2\alpha Q s_0\Big)b^i, \tag{2.9}$$

$$Q = \frac{\phi'}{\phi - s\phi'}, \quad \theta = \frac{\big(\phi - s\phi'\big)\phi' - s\phi'\phi''}{2\phi\Big[\phi - s\phi' + (B - s^2)\phi''\Big]} \tag{2.10}$$

$$\psi = \frac{\phi''}{2\Big[\phi - s\phi' + (B - s^2)\phi''\Big]}, \tag{2.11}$$

$$G^i_\alpha = \frac{1}{4}a^{ij}\Big[[\alpha^2]_{x^l y^j} y^k - [\alpha^2]_{x^j}\Big]. \tag{2.12}$$

are the spray coefficients of the Riemannian metric α. The spray coefficients of $F = \frac{\alpha^2}{\beta}$ are given by

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \theta(r_{00} - 2\alpha Q s_0)\frac{y^i}{\alpha} + \Psi(r_{00} - 2\alpha Q s_0)b^i, \tag{2.13}$$

where

$$\begin{aligned} Q &= -\tfrac{1}{2s}, \\ \psi &= \tfrac{1}{2B}, \\ \theta &= -\tfrac{s}{B}. \end{aligned}$$

The S-curvature for (α, β)-metric is given by

$$\mathbf{S} := 2\psi\big(r_0 + s_0\big) + \Big((n+1)\theta + \psi_s\big(B - s^2\big)\Big)\big(r_{00} - 2\alpha Q s_0\big)\frac{1}{\alpha} + f_0, \tag{2.14}$$

where $f_0 := f_{x^i} y^i$.

We use some notations for (α, β)-metrics as follows,

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}), \quad r_{00} = r_{ij}y^i y^j, \quad s^i_0 = a^{ij}s_{jk}y^k,$$
$$r_i = b^i r_{ji}, \quad s_i = b^j r_{ji}, \quad s_0 = s_i y^i, \quad r^i = a^{ij} r_j, \quad s^i = a^{ij} s_j, \quad r = b^i r_i,$$

where "$|$" denotes the covariant derivative with respect to Levi-Civita connection of α, $(a^{ij}) := (a_{ij})^{-1}$ and $b^i := a^{ij} b_j$.

The Riemann curvature of Kropina metric as follows

$$Ric_F = Ric_\alpha + T^n_n, \tag{2.15}$$

where

$$\begin{aligned} T^n_n: \quad = \quad & -F^2\Big[\frac{1}{2B}s_is^i+\frac{1}{4}s^i{}_js^j{}_i\Big]+F\Big[\frac{2n-3}{2B}s_is^i{}_0-\frac{1}{B^2}rs_0 \\ & -\frac{1}{2B}\big(2r_is^i{}_0+b^is_{i|0}-2s_{0|b}-2s_0r^i_i+3s_ir^i_0\big)-s^i{}_{0|i}\Big] \\ & +\frac{1}{B}\Big[\big(r^i_ir_{00}-b^ir_{i0|0}-r_{0i}r^i_0\big)+\big(2n-1\big)r_{0i}s^i{}_0+\big(n-2\big)s_{0|0}\Big] \\ & -\frac{1}{B^2}\Big[\big(n-2\big)s^2_0+\big(r_{00}r-r^2_0\big)+2\big(2n-3\big)r_0s_0\Big] \\ & +\frac{n-1}{FB^2}\Big[2r_{00}s_0-4r_{00}r_0+Br_{00|0}\Big]+\frac{3(n-1)}{B^2F^2}r^2_{00}. \end{aligned} \tag{2.16}$$

Now we obtain $\dot{\mathbf{S}}$ for Kropina metric as following

$$\begin{aligned} \dot{\mathbf{S}} \quad := \quad & \frac{F}{B}\Big[-\frac{rs_0}{B}-ns_is^i_0+r_is^i_0+Bf_{x^i}s^i_0-s_0f_b\Big] \\ & -\frac{1}{B^2}\Big[r_{00}r+2r^2_0-B\Big(2(2n+1)r_0s_0-2(n+1)r_{0i}s^i{}_0 \\ & -ns_{0|0}+r_{0|0}-f_br_{00}+s_02f_0\Big)-B^2f_{0|0}\Big] \\ & +\frac{1}{B^2F}\Big[-\frac{4(n+1)}{F}r^2_{00}-2(2n+1)r_{00}s_0 \\ & +B\Big(2(2n+3)r_{00}r_0-(n+1)r_{00|0}+2f_0r_{00}\Big)\Big]. \end{aligned} \tag{2.17}$$

Lemma 2.1. *Let $F=\frac{\alpha^2}{\beta}$ be a Kropina metric an n-dimensional manifold M with volume form $dV=e^{-f}dV_\alpha$. Then quasi-Ricci curvature of F is given by*

$$\begin{aligned} Ric+\dot{\mathbf{S}} \quad = \quad & Ric_\alpha+f_{0|0}-\frac{s^i_js^j_iF^2}{4}+F\big(f_{x^i}s^i_0-s^i_{0|i}\big)-\frac{1}{B^2}\big((n-2)s^2_0 \\ & +2r_{00}r+r^2_0-8r_0s_0+2rs_0F\big)+\frac{1}{B}\Big[r^i_ir_{00}+r_{00|b}+r_{i0|0}b^i \\ & -r_{0i}r^i_0-3r_{0i}s^i_0-2s_{0|0}+r_{0|0}-f_br_{00}+2f_0s_0-F\big[\frac{3}{2}\alpha s_is^i_0 \\ & +r_is^i_0+\frac{1}{2}s_{i|0}b^i-s_{0|b}-s_0r^i_i+\frac{3}{2}r^i_0+\frac{s_is^iF}{2} \\ & -r_is^i_0+f_bs_0\big]\Big]-\frac{(n+7)}{F^2B^2}r^2_{00} \\ & -\frac{1}{F}\Big[2(n+3)r_{00}s_0-\frac{10r_{00}r_0}{B^2}+\frac{2r_{00|0}-2f_0r_{00}}{B}\Big]. \end{aligned} \tag{2.18}$$

Proof. By equation (2.15) direct computation to (2.18). □

Proof of Theorem 1.1: Let F be a quasi-Einstein kropina metric, by means quasi-Einstein and lemma 2.1, we have

$$Ric+\dot{\mathbf{S}}-(n-1)cF^2=0, \tag{2.19}$$

where $c=c(x)$ is a scalar function. Then we can get by

$$\begin{aligned}
0 \;=\; & -\frac{1}{B^2}\Big((n+7)\big[\frac{sr_{00}}{\alpha}\big]^2+2(n+2)\frac{sr_{00}s_0}{\alpha}-10\frac{sr_{00}r_0}{\alpha}\\
&+(n-2)s_0^2+2r_{00}r+r_0^2-8r_0s_0+2\frac{\alpha r s_0}{s}\Big)\\
&-\frac{1}{B}\Big(\frac{2sr_{00|0}}{\alpha}-r^i_ir_{00}-r_{00|b}-b^ir_{i0|0}+r_{0i}r^i_0+3r_{0i}s^i_0\\
&+2s_{0|0}+\frac{3\alpha s_is^i_0}{2s}+\frac{r_is^i_0\alpha}{s}+\frac{b^is_{i|0}\alpha}{2s}-\frac{s_{0|b}\alpha}{s}-\frac{s_0r^i_i\alpha}{s}\\
&+\frac{3s_ir^i_0\alpha}{2s}+\frac{s_is^i\alpha^2}{2s^2}-\frac{r_is^i_0\alpha}{s}-r_{0|0}+f_br_{00}-\frac{2sf_0r_{00}}{\alpha}\\
&+\frac{f_b\alpha s_0}{s}-2f_0s_0\Big)-\frac{1}{s}\Big(\alpha s^i_{0|i}+\frac{s^i_js^j_i\alpha^2}{4s}-f_{x^i}s^i_0\alpha\Big)\\
&+f_{0|0}+Ric_\alpha-(n-1)cF^2.
\end{aligned} \tag{2.20}$$

Now by multiplying (2.20) with $\beta^2\alpha^4$, we can equation mentioned above by α as follows

$$0 \;=\; A_1\alpha^8+A_2\alpha^6+A_3\alpha^4+A_4\alpha^2+A_5, \tag{2.21}$$

where

$$\begin{aligned}
A_1 &= -\frac{1}{2B}s_is^i-\frac{1}{4}s^i_js^j_i-(n-1)c,\\
A_2 &= \beta\Big(\frac{1}{B}\big[-\frac{2}{B}rs_0-\frac{3}{2}s_is^i_0-\frac{1}{2}b^is_{i|0}+s_{0|b}+s_0r^i_i-\frac{1}{2}s_ir^i_0\\
&\quad -Bs^i_{0|i}+f_{x^i}s^i_0-f_bs_0\big]\Big),\\
A_3 &= -\Big[\frac{1}{B}\Big(\big[(n-2)s_0^2+2r_{00}r+r_0^2-8r_0s_0\big]\frac{1}{B}-r^i_ir_{00}\\
&\quad -r_{00|b}-b^ir_{i0|0}+r_{0i}r^i_0+3r_{0i}s^i_0\\
&\quad +2s_{0|0}-r_{0|0}+f_br_{00}\Big)+f_{0|0}+Ric_\alpha\Big]\beta^2,\\
A_4 &= -2\Big(\big[(n+2)s_0-5r_0\big]\frac{1}{B}r_{00}+r_{00|0}-f_0r_{00}\Big)\beta^3,\\
A_5 &= -(n+7)\frac{1}{B^2}r^2_{00}\beta^4.
\end{aligned}$$

By this equation, we conclude that α^2 divides $-(n+7)\frac{1}{B^2}r_{00}^2\beta^4$. This means that there is scalar function $\sigma=\sigma(x)$ on M, where

$$r_{00}=\sigma\alpha^2. \tag{2.22}$$

We only consider the case (2.22). Then, one can obtain the expression of the following quantities

$$\begin{aligned} & r^i_i=n\sigma, \quad r_{00}=\sigma\alpha^2, \quad r_0=\sigma\beta, \quad r_{0i|0}=\sigma_0 y^i, \quad r=\sigma B, \\ & r_{0|0}=\sigma^2\alpha^2+\sigma_0\beta, \quad r_i s^i_0=\sigma s_0, \quad r_{0i}r^i_0=\sigma^2\alpha^2, \\ & r_{0i|0}b^i=\sigma_0\beta, \quad r_{00|0}=\sigma_0\alpha^2. \end{aligned} \tag{2.23}$$

By plugging all the above quantities into (2.21), we can get

$$0 = \alpha^8 A_1+\alpha^6 A_2+\alpha^4 A_3, \tag{2.24}$$

where

$$\begin{aligned} A_1 &= -\frac{1}{2B}\big[s^i s_i+\frac{s^i_j s^j_i}{2}+2(n-1)c\big], \\ A_2 &= \beta\Big[\frac{1}{B}\big[\sigma s_0(n-\frac{7}{2})-\frac{3}{2}s_i s^i_0-\frac{1}{2}b^i s_{i|0}+s_{0|b}-f_b s_0 \\ &\quad +(n+2)\sigma^2\beta-f_b\sigma\beta\big]-s_{0|i}+f_{x^i}s^i_0\Big], \\ A_3 &= \beta^2\big[-\frac{1}{B^2}\big((n-2)s_0^2+(n-2)\sigma^2\beta^2+2(n-2)\sigma s_0\beta\big) \\ &\quad +\frac{1}{B}\big(2s_{0|0}-2f_0 s_0\big)+Ric_\alpha+f_{0|0}\big]. \end{aligned}$$

Case I: Assume $n\neq 2$. In this case, we have

$$\begin{aligned} Ric_\alpha &= \frac{1}{B^2}\big[(n-2)\big(s_0^2-\sigma^2\beta^2\big)-2(n-2)\sigma s_0\beta\big] \\ &\quad -\frac{1}{B}\big[2s_{0|0}-2f_0 s_0\big]-f_{0|0}+\eta\alpha^2, \end{aligned} \tag{2.25}$$

where $\eta=\eta(x)$is a scalar function.

Case II: Assume $n=2$ then, we obtain

$$Ric_\alpha=-\frac{1}{B}\big[2s_{0|0}-2f_0 s_0\big]-f_{0|0}+\eta\alpha^2. \tag{2.26}$$

This completes the proof. □

Finally, we conclude the following.

Corollary 2.2. *Let $F=\frac{\alpha^2}{\beta}$ be a Kropina metric on n-dimensional manifold M with volume form $dV=e^{-f}dV_\alpha$. Suppose F is quasi-Einstein. Then F quasi-Ricci flat if and only if it is satisfy*

$$\begin{aligned} s_i s^i_0 &= \frac{1}{3}\Big[(2n-7)\sigma s_0 - b^i s_{i|o} + 2s_{0|b} - 2f_b s_0 \\ &\quad +2\beta\big(\sigma\big[(n+2)\sigma - f_b\big] - \eta\big)\Big], \qquad (2.27) \\ s^i s_i &= -\frac{1}{2} s^i_j s^j_i. \qquad (2.28) \end{aligned}$$

Proof. Suppose F be quasi-Einstein Kropina metric. By plugging quantities (2.22), (2.23), (2.25) into the following equation

$$Ric_F + \dot{\mathbf{S}} = 0. \tag{2.29}$$

We can get (2.27). Also by plugging quantities (2.22),(2.23) (2.26) into (2.29), we can get (2.28). □

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