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On concircular vector field and Lie derivative in generalized fifth recurrent Finsler space

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Abstract. This paper investigates specific types of concircular motions within a generalized fifth recurrent Finsler space, focusing on Cartan's fourth curvature tensor K^i_{jkh} in sense of Berwald. We established a new definition for the concircular vector field σ_h and studied the direction of the force acting on this field using the Lie derivative. In addition, we proved that the concircular vector field σ_h and the recurrence vector λ_m are equal under certain condition. The mathematical formulas for the Lie-derivative of recurrence vector λ_m and Lie-derivative of the product of two concircular vector fields within this space have been obtained. In conclusion, we have provided applications and practical examples that illustrate some of the results.

Keywords: Concircular motion, Concircular vector field σ_h, Recurrence vector λ_m, Lie-derivative L_v, Generalized $\mathfrak{B}K$-fifth recurrent Finsler space.

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1. Introduction

A Finsler space is an extension of Riemannian space, Riemann studied the distance between points in $n-$dimensions using only positional coordinates, while Finsler generalized Riemann's idea and studied the distance between points in $n-$dimensions by using two coordinates, positional and directional. The basic concepts of Finsler geometry are carefully studied and systematically compared with those of Riemannian geometry, highlighting both the foundational similarities and the key differences between these two geometric frameworks. Furthermore, recent studies and advancements in the field, as presented by [2, 12, 13].

The concepts of concircular vector fields and the Lie derivative provide powerful tools to analyze geometric transformations and invariant properties of manifolds, contributing significantly to both pure mathematical theory and applications in theoretical physics.

The Lie - derivative evaluate the rate of change of a vector field or a tensor field along the flow of another vector or tensor field, it is named after Souphus Lie who introduced it in the 19^{th} century. Some remarks on the Lie-derivative introduced by Gouin [9]. Several identities on Lie-recurrent Finsler space introduced by authors [14, 16]. Further, Opondo [11] studied Lie-derivative in recurrent and bi-recurrent Finsler space.

Al-Qashbari and et al. ([3]–[7]) have significantly advanced our understanding of generalized $\mathfrak{B}K$-fifth recurrent Finsler space ($G\mathfrak{B}K - 5RF_n$), especially through the lens of projective transformations, Lie derivatives, and the inheritance properties of various curvature tensors, including the Kulkarni–Nomizu product and M-projective tensors.

Additional studies have explored related curvature structures such as W_9-curvature tensor within the framework of Lorentzian para-Sasakian manifold by Singh and et al. [15]. Verma [17] established some transformations in Finsler space and special concircural affine motion. The generalized Finsler spaces of different orders for various curvature tensors and the decomposition of them in Berwald sense discussed by [8, 10].

The objective of this paper is obtaining mathematical formulas that represent the concircular motions and describe the direction of the force acting on a field with the directional and extensional changes that occur to this field during the force's influence in generalized fifth recurrent Finsler space. This helps us understand the movement of planets around a star in an elliptical orbit.

2. Preliminaries

This section introduces important equations that serve as the foundation for the main findings. The generalized $\mathfrak{B}K-$fifth recurrent Finsler space that denoted as $G\mathfrak{B}K-5RF_n$, and satisfies the following relation [7]

$$\begin{aligned}&\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m K^i_{jkh} = a_{sqlnm}K^i_{jkh} + b_{sqlnm}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) \\ &- c_{sqlnm}\left(\delta^i_h C_{jkn} - \delta^i_k\, C_{jhn}\right) - d_{sqlnm}\left(\delta^i_h C_{jkl} - \delta^i_k\, C_{jhl}\right) \\ &- e_{sqlnm}\left(\delta^i_h C_{jkq} - \delta^i_k\, C_{jhq}\right) - 2b_{qlnm}y^r\mathfrak{B}_r\left(\delta^i_h C_{jks} - \delta^i_k\, C_{jhs}\right),\end{aligned} \tag{2.1}$$

where $K^i_{jkh} \neq 0$.

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m H^i_{kh} = a_{sqlnm}H^i_{kh} + b_{sqlnm}\left(\delta^i_h y_k - \delta^i_k y_h\right). \tag{2.2}$$

The Cartan's fourth curvature tensor K^i_{jkh} in recurrent Finsler space is defined as

$$\mathfrak{B}_m K^i_{jkh} = \lambda_m K^i_{jkh}, \tag{2.3}$$

where the non-zero vector λ_m is called recurrence vector .
The non-zero metric tensor g_{ij} and Kronecker delta δ^i_h are satisfying the relations:

$$g_{ij}g^{ik} = \delta^k_j = \begin{cases} 1 \text{ if } j = k, \\ 0 \text{ if } j \neq k. \end{cases} \tag{2.4}$$

The Berwald covariant derivatives of the vectors y^i, y_i are vanishing, i.e.

$$\begin{cases} a)\ \mathfrak{B}_k y^i = 0 \\ b)\ \mathfrak{B}_k y_i = 0. \end{cases} \tag{2.5}$$

The curvature tensor K^i_{jkh}, $h(v)$-torsion tensor H^i_{kh}, deviation tensor H^i_h, Ricci tensor K_{jk}, curvature vector H_k, curvature scalar H and vector y^j satisfy the following relations

$$\begin{cases} a)\ K^i_{jkh}\, y^j = H^i_{kh}. \\ b)\ K^i_{jkh} = R^i_{jkh} - C^i_{js}H^s_{kh}. \\ c)\ H^r_{kr} = H_k. \\ d)\ H^r_r = (n-1)H. \\ e)\ H^i_{kh}y^k = H^i_h. \\ f)\ K^r_{jkr} = K_{jk}. \\ g)\ K_{jk}y^j = H_k. \\ h)\ K_{jk}y^j = R_{jk}y^j. \\ i)\ y^j = g^{jk}y_k. \\ j)\ R_{jk}\, g^{jk} = R. \\ k)\ H^i_{kh}y_i = 0. \end{cases} \tag{2.6}$$

A Lie - derivative evaluate the rate of change of a vector field or a tensor field along the smooth vector field $v^i(x)$. The Lie-derivative of a general mixed tensor field $T^i_{jkh}(x,\dot{x})$ expressed in the form [3]

$$L_v T^i_{jkh} = v^m \mathfrak{B}_m T^i_{jkh} - T^m_{jkh}\mathfrak{B}_m v^i + T^i_{mkh}\mathfrak{B}_j v^m + T^i_{jmh}\mathfrak{B}_k v^m \tag{2.7}$$
$$+T^i_{jkm}\mathfrak{B}_h v^m + \dot{\partial}_m T^i_{jkh}\mathfrak{B}_r v^m y^r,$$

where $v^i(x) \neq 0$ is a contravariant vector field independent of directional argument and dependent on positional coordinates x^i only and $\mathfrak{B}_j v^m = 0$. The Lie - derivative of the metric tensors g_{ij} is vanishing, i.e.

$$L_v g_{ij} = 0. \tag{2.8}$$

The concircular motion it is the motion of a particle in a way that makes it follow curved paths that maintain the form of generalized circles around a central point.the sufficient condition for motion to become concircular is [17]

$$\mathfrak{B}_m \sigma_h = \sigma_h\ \sigma_m + L_v g_{hm}, \tag{2.9}$$

where the non-zreo vector field $\sigma_h = \sigma_h(x^i)$ is called concircular vector field. Transvecting (2.3) by y^j, then using [(2.5)a] and [(2.6)a] in result equation, we get

$$\mathfrak{B}_m H^i_{kh} = \lambda_m H^i_{kh}. \tag{2.10}$$

The above equation means that the $h(v)$-torsion tensor H^i_{hk} behaves as recurrent in Finsler space.

3. **Main Results**

In this section, we examine several theorems of concircular motions in generalized fifth recurrent Finsler space. We explore varoius identities for the concircular vector field σ_h. Multiplying (2.3) by the concircular vector field σ_h, we get

$$\mathfrak{B}_m(\sigma_h K^i_{jkh}) - K^i_{jkh}(\mathfrak{B}_m \sigma_h) = \sigma_h \lambda_m K^i_{jkh}.$$

Using (2.8) and (2.9) in above equation, we get

$$\mathfrak{B}_m(\sigma_h K^i_{jkh}) - K^i_{jkh}\sigma_h\sigma_m = \sigma_h \lambda_m K^i_{jkh}.$$

Which can be written as

$$K^i_{jkh} = \frac{1}{\sigma_h \lambda_m}\Big[\mathfrak{B}_m(\sigma_h K^i_{jkh}) - K^i_{jkh}\sigma_h\sigma_m\Big].$$

Using above equation in right side of (2.1), then using (2.4) in result equation, we get

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m K^i_{jkh} = \frac{a_{sqlnm}}{\sigma_h \lambda_m}\Big[\mathfrak{B}_m(\sigma_h K^i_{jkh}) - K^i_{jkh}\sigma_h\sigma_m\Big]. \tag{3.1}$$

Multiplying (3.1) by y^j, using [(2.5)a], [(2.6)a], we get

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m H^i_{kh} = \frac{a_{sqlnm}}{\sigma_h\lambda_m}\Big[\mathfrak{B}_m(\sigma_h H^i_{kh}) - H^i_{kh}\sigma_h\sigma_m\Big]. \tag{3.2}$$

Multiplying above equation by y^K, using [(2.5)a],[(2.6)e], we get

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m H^i_{h} = \frac{a_{sqlnm}}{\sigma_h\lambda_m}\Big[\mathfrak{B}_m(\sigma_h H^i_{h}) - H^i_{h}\sigma_h\sigma_m\Big]. \tag{3.3}$$

Contracting the indices i and h in (3.2) and using [(2.6)c], we get

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m H_{k} = \frac{a_{sqlnm}}{\sigma_h\lambda_m}\Big[\mathfrak{B}_m(\sigma_h H_{k}) - H_{k}\sigma_h\sigma_m\Big]. \tag{3.4}$$

Contracting the indices i and h in (3.3) and using [(2.6)d], we get

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m H = \frac{a_{sqlnm}}{\sigma_h\lambda_m}\Big[\mathfrak{B}_m(\sigma_h H) - H\sigma_h\sigma_m\Big]. \tag{3.5}$$

Thus, we conclude:

Theorem 3.1. *In $G\mathfrak{B}K-5RF_n$, the Berwald covariant derivative of fifth-order for the Cartan's fourth curvature tensor K^i_{jkh}, $h(v)$-torsion tensor H^i_{kh}, deviation tensor H^i_h, curvature vector H_k and curvature scalar H are given by* (3.1), (3.2), (3.3), (3.4) *and* (3.5) *respectively, represent concircular motions.*

In next theorem we obtain new relations for the concircular vector field σ_h and specific types of the same concircular motions for certain curvature tensors. Using (2.4) in (2.1), we get

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m K^i_{jkh} = a_{sqlnm}K^i_{jkh}.$$

Using above equation in (3.1), we get

$$\sigma_h = \frac{1}{\lambda_m K^i_{jkh}}\Big[\mathfrak{B}_m(\sigma_h K^i_{jkh}) - K^i_{jkh}\sigma_h\sigma_m\Big].$$

Which can be written as

$$\sigma_h = \frac{\mathfrak{B}_m(\sigma_h K^i_{jkh})}{(\lambda_m+\sigma_m)K^i_{jkh}}. \tag{3.6}$$

Thus, we conclude

Theorem 3.2. *In $G\mathfrak{B}K-5RF_n$, the concircular vector field is given by* (3.6).

Contracting the indices i and h in (3.1) and using [(2.6)f], we get

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m K_{jk} = \frac{a_{sqlnm}}{\sigma_h\lambda_m}\Big[\mathfrak{B}_m(\sigma_h K_{jk}) - K_{jk}\sigma_h\sigma_m\Big].$$

Multiplying above equation by y^j and using [(2.5)a], we get

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m (K_{jk}y^j) = \frac{a_{sqlnm}}{\sigma_h\lambda_m}\Big[\mathfrak{B}_m(\sigma_h K_{jk}y^j) - K_{jk}y^j\sigma_h\sigma_m\Big].$$

Using [(2.6)g] in the right side of above equation, we get

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m (K_{jk}y^j) = \frac{a_{sqlnm}}{\sigma_h\lambda_m}\Big[\mathfrak{B}_m(\sigma_h H_{k}) - H_{k}\sigma_h\sigma_m\Big].$$

In view of above equation and (3.4), we get

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m H_k = \mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m (K_{jk}y^j).$$

Using [(2.6)h,i,j] in above equation, we get

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m H_k = \mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m (Ry_k). \tag{3.7}$$

Thus, we conclude

Theorem 3.3. *In $G\mathfrak{B}K-5RF_n$, the Berwald covariant derivative of fifth-order for the curvature vector H_k and tensor (Ry_k) represent the same concircular motion.*

Using [(2.6)b] in the left side of (3.1), we get

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m (R^i_{jkh} - C^i_{js}H^s_{kh}) = \frac{a_{sqlnm}}{\sigma_h\lambda_m}[\mathfrak{B}_m(\sigma_h K^i_{jkh}) - K^i_{jkh}\sigma_h\sigma_m].$$

Which can be written as

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m R^i_{jkh} = \frac{a_{sqlnm}}{\sigma_h\lambda_m}\Big[\mathfrak{B}_m(\sigma_h K^i_{jkh}) - K^i_{jkh}\sigma_h\sigma_m\Big], \tag{3.8}$$

if

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m (C^i_{js}H^s_{kh}) = 0. \tag{3.9}$$

In view of (3.1) and (3.8), we get

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m K^i_{jkh} = \mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m R^i_{jkh}.$$

Thus, we conclude

Theorem 3.4. *In $G\mathfrak{B}K-5RF_n$, the Berwald covariant derivative of fifth-order for the Cartan's fourth curvature tensor K^i_{jkh} and Cartan's third curvature tensor R^i_{jkh} represent the same concircular motion [provided (3.9) holds].*

Using (2.4) in (2.2), then using the result equation in (3.2), we get

$$\sigma_h\lambda_m H^i_{kh} = \Big[\mathfrak{B}_m(\sigma_h H^i_{kh}) - H^i_{kh}\sigma_h\sigma_m\Big].$$

Using (3.6) and [(2.6)a] in left side of above equation, we get

$$y^j[\mathfrak{B}_m(\sigma_h K^i_{jkh}) - K^i_{jkh}\sigma_h\sigma_m] = \Big[\mathfrak{B}_m(\sigma_h H^i_{kh}) - H^i_{kh}\sigma_h\sigma_m\Big].$$

Multiplying above equation by y_i, using [(2.5)b] and [(2.6)k], we get

$$y^j y_i\Big[\mathfrak{B}_m(\sigma_h K^i_{jkh}) - K^i_{jkh}\sigma_h\sigma_m\Big] = 0.$$

Which can be written as

$$\Big[K^i_{jkh}\mathfrak{B}_m\sigma_h + \sigma_h\mathfrak{B}_m K^i_{jkh} - K^i_{jkh}\sigma_h\sigma_m\Big] = 0.$$

Using (2.3) in above equation, we get

$$\left[K^i_{jkh}\mathfrak{B}_m\sigma_h + \sigma_h\lambda_m K^i_{jkh} - K^i_{jkh}\sigma_h\sigma_m\right] = 0.$$

Using (2.8) and (2.9) in above equation, then taking the Lie-derivative of both sides of result equation, we get

$$\lambda_m K^i_{jkh} L_v\sigma_h + \sigma_h K^i_{jkh} L_v\lambda_m + \sigma_h\lambda_m L_v K^i_{jkh} = 0.$$

Using (2.7) in above equation, we get

$$\lambda_m K^i_{jkh} L_v\sigma_h = -\sigma_h K^i_{jkh} v^m \mathfrak{B}_m\lambda_m - \sigma_h\lambda_m v^m \mathfrak{B}_m K^i_{jkh}.$$

Using (2.3) in above equation, we get

$$L_v\sigma_h = -\sigma_h\tau^m(\mathfrak{B}_m\lambda_m + \lambda_m^2), \tag{3.10}$$

where $\tau^m = \frac{v^m}{\lambda_m}$. Thus, we conclude

Theorem 3.5. *In $G\mathfrak{B}K-5RF_n$, the force acting on the flow of concircular vector field σ_h is opposite to the direction of that field's flow and given by (3.10).*

Using (2.4) in (2.1), we get

$$a_{sqlnm} = \frac{\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m K^i_{jkh}}{K^i_{jkh}}.$$

Using above equation in (3.2), we get

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m H^i_{kh} = \frac{\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m K^i_{jkh}}{\sigma_h\lambda_m K^i_{jkh}}\left[\mathfrak{B}_m(\sigma_h H^i_{kh}) - H^i_{kh}\sigma_h\sigma_m\right].$$

Using [(2.6)a] and [(2.5)a] in left side of above equation, we get

$$y^j\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m K^i_{jkh} = \frac{\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m K^i_{jkh}}{\sigma_h\lambda_m K^i_{jkh}}\left[\mathfrak{B}_m(\sigma_h H^i_{kh}) - H^i_{kh}\sigma_h\sigma_m\right].$$

Which can be written as

$$\sigma_h\lambda_m K^i_{jkh} y^j = \left[\mathfrak{B}_m(\sigma_h H^i_{kh}) - H^i_{kh}\sigma_h\sigma_m\right].$$

Using [(2.6)a] in above equation, we get

$$\sigma_h\lambda_m H^i_{kh} = \left[\mathfrak{B}_m(\sigma_h H^i_{kh}) - H^i_{kh}\sigma_h\sigma_m\right].$$

Or

$$\sigma_h\lambda_m H^i_{kh} = \left[H^i_{kh}\mathfrak{B}_m\sigma_h + \sigma_h\mathfrak{B}_m H^i_{kh} - H^i_{kh}\sigma_h\sigma_m\right].$$

Using (2.10) in above equation, we get

$$H^i_{kh}\sigma_h\sigma_m = H^i_{kh}\mathfrak{B}_m\sigma_h.$$

Now, if the concircular vector field σ_h behaves as recurrent, then above equation can be written as

$$\sigma_m = \lambda_m. \tag{3.11}$$

Thus, we conclude

Theorem 3.6. *In $G\mathfrak{B}K-5RF_n$, the concircular vector field σ_m and the recurrence vector λ_m have the same extension and direction if this concircular vector field with other lower components behaves as recurrent.*

Applying (2.7) to the concircular vector field σ_h, we get

$$L_v\sigma_h = v^m\mathfrak{B}_m\sigma_h + \sigma_m\mathfrak{B}_h v^m + \dot{\partial}_m\sigma_h\mathfrak{B}_r v^m y^r,$$

Since $\mathfrak{B}_j v^m = 0$, then above equation can be written as

$$L_v\sigma_h = v^m\mathfrak{B}_m\sigma_h.$$

Using above equation in (3.10), we get

$$v^m\mathfrak{B}_m\sigma_h = -\sigma_h\tau^m(\mathfrak{B}_m\lambda_m + \lambda_m^2).$$

Which can be written as

$$v^m\mathfrak{B}_m\sigma_h + \sigma_h\tau^m\lambda_m^2 = -\sigma_h\tau^m(\mathfrak{B}_m\lambda_m).$$

If the concircular vector field σ_h behaves as recurrent, then above equation can be written as

$$\lambda_m\sigma_h(v^m + \tau^m\lambda_m) = -\sigma_h\tau^m(\mathfrak{B}_m\lambda_m).$$

Substituting the value of the tensor τ^m in above equation, we get

$$v^m(\mathfrak{B}_m\lambda_m) = -2\lambda_m^2 v^m.$$

Applying (2.7) to the recurrence vector λ_m and using the result equation in above equation, we get

$$L_v\lambda_m = -2\lambda_m^2 v^m. \tag{3.12}$$

Thus, we conclude

Theorem 3.7. *In $G\mathfrak{B}K-5RF_n$, the force acting on the flow of recurrence vector λ_m is opposite to the direction of that vector's flow and given by (3.12) if the concircular vector field σ_h behaves as recurrent.*

Using (2.7) in (3.12), then using (3.11) in the left side of result equation, we get

$$\mathfrak{B}_m\sigma_m = -2\lambda_m^2.$$

Multiplying above equation by the concircular vector field σ_h, we get

$$\mathfrak{B}_m(\sigma_h\sigma_m) - \sigma_m\mathfrak{B}_m\sigma_h = -2\sigma_h\lambda_m^2.$$

Using (2.8) and (2.9) in above equation, we get

$$\mathfrak{B}_m(\sigma_h\sigma_m) = \sigma_m^2\sigma_h - 2\sigma_h\lambda_m^2$$

Multiplying above equation by the contravariant vector field v^m, we get

$$v^m\mathfrak{B}_m(\sigma_h\sigma_m) = v^m\sigma_m^2\sigma_h - 2v^m\sigma_h\lambda_m^2.$$

Using (3.11) in the right side of above equation, we get

$$v^m \mathfrak{B}_m(\sigma_h \sigma_m) = -v^m \sigma_m^2 \sigma_h.$$

Applying (2.7) to the tensor $(\sigma_h \sigma_m)$ and using the result equation in above equation, we get

$$L_v(\sigma_h \sigma_m) = -v^m \sigma_m^2 \sigma_h. \tag{3.13}$$

Thus, we conclude

Theorem 3.8. *In $G\mathfrak{B}K - 5RF_n$, the Lie - derivative of the product of two concircular vector fields is given by* (3.13).

4. **Applications**

In the motion of a satellite around the Earth, the fundamental force acting is the gravitational force between the Earth and the satellite, and the direction of this force is always towards the Earth. Concircular motion around the Earth can be considered a special case of elliptical orbits, where the radius is not constant. During the satellite's orbit in its elliptical path, the direction of its motion changes continuously. The Earth's gravitational force is responsible for this change in direction, as it constantly pulls the satellite towards the Earth, causing its path to curve.

Let the fifth slope of the Cartan's fourth curvature tensor K^i_{jkh} represents an elliptical path during the satellite's orbit around the Earth, in this path, the satellite does not move at a constant speed, when it is closer to the Earth, its speed is greater and the gravity is stronger, and when it is farther away, its speed is lower and the gravity is weaker. This represents a change in the magnitude of its velocity (expansion and contraction), and the extent and direction of this path are the same as the extent and direction of the fifth slope of the Cartan's third curvature tensor R^i_{jkh} when condition (3.9) is met. Now we mention some practical examples to illustrate the results.

- **Example 1:** If $v^m = \lambda_m$ and $\sigma_m = c$, where c is constant then the Lie-derivative of the concircular vector field σ_h given by

  $$L_v \sigma_h = -C\sigma_h,$$

  where $C = c^2$.
  By [Theorem (3.5) and Theorem (3.6)].
- **Example 2:** The value of $L_v(\sigma_h \sigma_m) = 1$, when $L_v \sigma_h = \frac{-1}{\sigma_m}$.
  By [Theorem (3.8)].
- **Example 3:** If $\tau^m = n$, then the Lie-derivative of the recurrence vector λ_m given by

  $$L_v \lambda_m = -2n\lambda_m^3.$$

  By [Theorem (3.7)].

5. Conclusions

This research paper has yielded several equations representing concircular motions across various curvature tensors in generalized fifth recurrent Finsler space. We have successfully defined the concircular vector field σ_h and elucidated how its direction changes under the influence of an applied force by using Lie-derivative. Furthemore, we found an equality relation between the concircular vector field σ_h and the recurrence vector λ_m under specific conditions. and derived a new relations representing the Lie-derivative of the recurrence vector λ_m and the Lie-derivative of the product of two concircular vector fields in $G\mathfrak{B}K - 5RF_n$.

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