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Study of some curvatures with Z. Shen's square metric

Sachin Kumar* , Abhishek Singh, P. K. Dwivedi, C. K. Mishra and Uma Shanker

Department of Mathematics and Statistics
Dr. Rammanohar Lohia Avadh University, Ayodhya(U.P.), India

E-mail: skumar17011997@gmail.com
E-mail: abhi.rmlau@gmail.com
E-mail: drpkdwivedi@yahoo.co.in
E-mail: chayankumarmishra@gmial.com
E-mail: ussmlk@gmail.com

Abstract. In this research paper, we have studied the Z-Shen square metric under the condition that the 1-form β is a Killing form of constant length. We have derived the explicit expressions for the Ricci and Riemann curvatures associated with this metric. Furthermore, we have investigated the special characteristics of projectively flat Z-Shen square metrics that possess isotropic S-curvature.

Keywords: Z-Shen's square metric, Projectively flat Finsler space, S-curvature, Ricci curvature, Riemann curvature.

1. Introduction

A Finsler space is fundamentally defined by a generating function or Finsler metric function $\mathcal{F}(x, y)$, which is specified on the tangent bundle $\mathcal{TM}$ of a differentiable manifold $\mathcal{M}$. The function $\mathcal{F}$ is positively homogeneous of degree

*Corresponding Author
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one in the directional variable y. The concept of Finsler geometry was originally introduced by Paul Finsler in 1918. In 1854, Bernhard Riemann proposed the Riemannian metric in the form $ds^2 = g_{ij}dx^i dx^j$, which quantifies the infinitesimal distance between nearby points x and $x + y$. The Finsler generating function $\mathcal{F}(x, y)$ is required to satisfy the following essential properties:
(i) $\mathcal{F}$ is continuous on $\mathcal{TM}$ and smooth on slit tangent bundle $\tilde{\mathcal{TM}} = \mathcal{TM} \setminus \{(x, y) \in \mathcal{TM} \,|\, \mathcal{F}(x, y) = 0\}$.
(ii) $\mathcal{F}$ is positively homogeneous of degree one in its second argument: $\mathcal{F}(x, ky) = k\mathcal{F}(x, y)$ for all $k > 0$.
(iii) For each point $x \in \mathcal{M}$, the associated fundamental metric tensor

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 \mathcal{F}^2}{\partial y^i \partial y^j}$$

is non-singular, where indices $i, j = 0, 1, 2, \ldots, n - 1$.

A pair $(\mathcal{M}, \mathcal{F})$ defines a Finsler manifold, where the bilinear form $g = g_{ij}(x, y)dx^i dx^j$ is known as the Finsler metric tensor, when the function $\mathcal{F}$ satisfies the above criteria, it is referred as a regular Finsler metric.

A Finsler metric function $\mathcal{F}(x, y)$ on an n-dimensional manifold $\mathcal{M}^n$ is referred to as an (α, β)-metric [3], denoted by $\mathcal{F}(\alpha, \beta)$, in terms of α and β, where $\alpha^2 = a_{ij}y^i y^j$ represents a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on $\mathcal{M}^n$. Z. Shen introduced a special class of (α, β)-metrics, defined by the formula $\mathcal{F} = \frac{(\alpha+\beta)^2}{\alpha}$ [2], [8]. Studying the Z. Shen's square metric with such a 1-form is useful because it allows mathematicians to explore geometries with high semmetry and explicit structure, leading to potential applications in physics and differential geometry (e.g. geodesics, curvature properties, integrable systems). The Z. Shen's square metric repersent a significant class of Finsler metrics with deep geometric structure. Investigating its behaviour under the presence of a Killing 1-form of constant length enables the analysis of highly semmetric Finsler spaces, offerenig insights in to curvature praperties and links to classical Riemannian results. This work contributes to ongoing efforts to understand the geometric and topological implications of such metrics.

Ricci curvature plays a crucial role in the geometry of Finsler manifolds and is defined as the trace of the Riemann curvature on each tangent space. In addition, Z. Shen [4] introduced the concept of S-curvature, a non-Riemannian invariant that measures the rate of change of the volume form along geodesics in a Finsler space. The S-curvature vanishes in Berwald spaces, including Riemannian manifolds.

Several researchers [1], [2], [5], [6], [8], [9] have explored various properties of (α, β)-metrics and achieved significant results, particularly in the study of projectively flat Finsler spaces, S-curvature, and Ricci curvature. The Riemannian curvature is a family of linear maps $R_y = R^i_k \frac{\partial}{\partial x^i} \times dx^k : T_x\mathcal{M}^n \to T_x\mathcal{M}^n$, defined by

$$R^i_k = 2\frac{\partial \mathcal{G}^i}{\partial x^k} - y^j \frac{\partial^2 \mathcal{G}^i}{\partial x^j \partial y^k} + 2\mathcal{G}^j \frac{\partial^2 \mathcal{G}^i}{\partial y^j \partial y^k} - \frac{\partial \mathcal{G}^i}{\partial y^j}\frac{\partial \mathcal{G}^i}{\partial y^k}. \tag{1.1}$$

The Ricci curvature and Ricci scalar are defined by

$$Ric = R^i_i, \qquad R = \frac{1}{n-1} Ric. \tag{1.2}$$

The $\mathcal{S}$-curvature of a Finsler space $F^n = (\mathcal{M}^n, \mathcal{F})$ is a scalar function $\mathcal{S} : T\mathcal{M}^n \to R$ defined by [4]

$$\mathcal{S} = \frac{\partial \mathcal{G}^m}{\partial y^m} - y^m \frac{\partial (ln\sigma_{\mathcal{F}})}{\partial x^m} \tag{1.3}$$

where

$$\sigma_{\mathcal{F}} = \frac{Vol(B^n)}{Vol\{(y^i) \in R^n : \mathcal{F}(x, y^i \frac{\partial}{\partial x^i}) : x < 1\}}.$$

A Finsler space $\mathcal{F}^n = (\mathcal{M}^n, \mathcal{F})$ is said to have isotropic $\mathcal{S}$-curvature if there exists a smooth function $c = c(x)$ on $\mathcal{M}^n$ such that

$$\mathcal{S} = (n+1)c\mathcal{F}.$$

The E-curvature of a Finsler space $\mathcal{F}^n = (\mathcal{M}^n, \mathcal{F})$ is a scalar function

$$E : T_p\mathcal{M}^n \times T_p\mathcal{M}^n \to \mathbb{R}$$

defined by [10] as

$$E_{ij} = \frac{1}{2}\mathcal{S}_{y^i y^j}.$$

A Finsler space is also said to have isotropic E-curvature if there exists a smooth function $c = c(x)$ on $\mathcal{M}^n$ such that

$$E = (n+1)c\mathcal{F}_{y^i y^j}.$$

2. Riemann curvature and Ricci- curvature of Z-Shen's square metric

For the Z. Shen's square metric $\mathcal{F} = \frac{(\alpha+\beta)^2}{\alpha}$ on an n- dimensional manifold M^n, the geodesic coefficients $\mathcal{G}^i$ of $\mathcal{F}$ are related to the coefficients ${}^\alpha\mathcal{G}^i$ of α by

$$\mathcal{G}^i =^\alpha \mathcal{G}^i + \mathcal{P}y^i + \mathcal{Q}^i, \tag{2.1}$$

where

$$\mathcal{P} = -\frac{1}{b^2}(s + \mathcal{F}^{-1} r_{00}), \tag{2.2}$$

$$\mathcal{Q}^i = -\frac{1}{2}\Big[(\mathcal{F} s^i_0 - \frac{1}{b^2}(\mathcal{F} s_0 + r_{00}) b^i\Big], \tag{2.3}$$

In this research paper, we consider that β is a Killing form of constant length which is satisfies

$$r_{ij} = 0, \quad b^j b_{j;k} = 0, \tag{2.4}$$

where ";" represents the covariant differentiation with respect to the Levi-Civita connection of $\mathbb{R}^n$.

The above equation implies that

$$s_{ij} = b_{i;j}, \quad s_j = b^i s_{ij} = 0, \quad b^i s^i_j = b^i s_{ri}, \quad a^{jr} = -b^i s_{ir} a^{jr} = 0. \tag{2.5}$$

The above relations implies that $\mathcal{P} = 0$, equation (2.1) reduces to

$$\mathcal{G}^i =^\alpha \mathcal{G}^i + \mathcal{Q}^i, \tag{2.6}$$

where

$$\mathcal{Q}^i = -\frac{1}{2}\mathcal{F} s^i_0. \tag{2.7}$$

From equations (1.1) and (2.1), we have obtained the Ricci curvature as:

$$R^i_k =^\alpha \mathcal{R}^i_k + \Big\{2\mathcal{Q}^i_{;k} - y^j(\mathcal{Q}^i_{;j})_{y^k} - (\mathcal{Q}^i)_{y^j}(\mathcal{Q}^j)_{y^k} + 2\mathcal{Q}^j(\mathcal{Q}^i)_{y^j y^k}\Big\} \tag{2.8}$$

where

$$(\mathcal{Q}^i)_{y^k} = \frac{\partial \mathcal{Q}^i}{\partial y^k}$$

Since $\alpha_{;k} = 0$ and $y_{;k} = 0$, we have

$$F_{;k} = \Big(\frac{(\alpha+\beta)^2}{\alpha}\Big)_{;k} = -\frac{2(\alpha+\beta)}{\alpha} s_{0k},$$
$$F_{y^k} = \Big(\frac{(\alpha+\beta)^2}{\alpha}\Big)_{y^k} = \frac{1}{\beta}\Big(2y_k - \frac{(\alpha+\beta)^2}{\alpha}\Big)b_k.$$

Therefore from equation (2.7), we get

$$\mathcal{Q}^i_{;k} = -\frac{(\alpha+\beta)}{\alpha} s^i_0 s_{0k} - \frac{(\alpha+\beta)^2}{2\alpha} s^i_{0;k} \tag{2.9}$$

$$y^j(\mathcal{Q}^i_{;j})_{y^k} = -\frac{(\alpha+\beta)}{\alpha} s_{k0} s^i_0 - \frac{(\alpha+\beta)}{2\alpha^2}\Big\{(1-\frac{\beta}{\alpha})y_k + 2\alpha b_k\Big\} s^i_{0;0} - \frac{(\alpha+\beta)^2}{2\alpha} s^i_{k;0} \tag{2.10}$$

$$(\mathcal{Q}^i)_{y^j}(\mathcal{Q}^j)_{y^k} = -\frac{(\alpha+\beta)^3}{4\alpha^3}\Big\{(1-\frac{\beta}{\alpha})y_k + 2\alpha b_k\Big\} s^i_j s^j_0 + \frac{(\alpha+\beta)^3}{4\alpha^3}(1-\frac{\beta}{\alpha})y_j s^j_k s^i_0 + \frac{(\alpha+\beta)^4}{4\alpha^2} s^i_j s^j_k. \tag{2.11}$$

$$\mathcal{Q}^j(\mathcal{Q}^i)_{y^j y^k} = \frac{(\alpha+\beta)^3}{4\alpha^2}\Big\{(1+\frac{\beta}{\alpha})y_k - 2\alpha b_k\Big\} s^i_j s^j_0 + (\alpha+\beta) s^i_0 s_{k0}. \tag{2.12}$$

Substituting these values into equation (2.8), we obtain the following result

$$
\begin{aligned}
R^i_k =^\alpha \mathcal{R}^i_k - 2\frac{(\alpha+\beta)}{\alpha}s^i_0 s_{0k} - \frac{(\alpha+\beta)^2}{\alpha}s^i_{0;k} + \frac{(\alpha+\beta)}{\alpha}s_{k0}s^i_0 \\
+\frac{(\alpha+\beta)}{2\alpha^2}\Big\{(1-\frac{\beta}{\alpha})y_k + 2\alpha b_k\Big\}s^i_{0;0} - \frac{(\alpha+\beta)^2}{2\alpha}s^i_{k;0} \\
+\frac{(\alpha+\beta)^3}{4\alpha^3}\Big\{(1-\frac{\beta}{\alpha})y_k + 2\alpha b_k\Big\}s^i_j s^j_0 + \frac{(\alpha+\beta)^3}{4\alpha^3}(1-\frac{\beta}{\alpha})y_j s^j_k s^i_0 \\
+\frac{(\alpha+\beta)^4}{4\alpha^2}s^i_j s^j_k + \frac{(\alpha+\beta)^3}{2\alpha^2}\Bigg[\Big\{(1+\frac{\beta}{\alpha})y_k - 2\alpha b_k\Big\}s^i_j s^j_0 \\
+2(\alpha+\beta)s^i_0 s_{k0}\Bigg].
\end{aligned} \tag{2.13}
$$

After simplification, we find

$$
\begin{aligned}
R^i_k =^\alpha \mathcal{R}^i_k - 2\frac{(\alpha+\beta)}{\alpha}s_{0k}s^i_0 - \frac{(\alpha+\beta)^2}{\alpha}s^i_{0;k} + \frac{(\alpha+\beta)}{\alpha}s_{k0}s^i_0 \\
+\frac{(\alpha+\beta)}{2\alpha^2}(1-\frac{\beta}{\alpha})y_k s^i_{0;0} - \frac{(\alpha+\beta)}{\alpha}b_k s^i_{0;0} - \frac{(\alpha+\beta)^2}{2\alpha}s^i_{k;0} \\
+\frac{(\alpha+\beta)}{4\alpha^3}(1-\frac{\beta}{\alpha})y_j s^j_k s^i_0 + \frac{(\alpha+\beta)^4}{4\alpha^2}s^i_j s^j_k + \frac{(\alpha+\beta)^3}{2\alpha^2}(1+\frac{\beta}{\alpha})y_k s^i_j s^j_0 \\
-\frac{(\alpha+\beta)^3}{\alpha}b_k s^i_j s^j_0 + (\alpha+\beta)s^i_0 s_{k0}.
\end{aligned} \tag{2.14}
$$

Since $s_{k0} = -s_{0k}$, $y_j s^j_k = s_{0k}$, then the equation (2.14), can be written as

$$
\begin{aligned}
R^i_k =^\alpha \mathcal{R}^i_k - \frac{(\alpha+\beta)^2}{\alpha}s^i_{0;k} + \frac{(\alpha+\beta)}{2\alpha^2}(1-\frac{\beta}{\alpha})y_k s^i_{0;0} \\
-\frac{(\alpha+\beta)}{\alpha}b_k s^i_{0;0} - \frac{(\alpha+\beta)^2}{2\alpha}s^i_{k;0} + \frac{(\alpha+\beta)^4}{4\alpha^2}s^i_j s^j_k \\
+\frac{(\alpha+\beta)^4}{2\alpha^3}y_k s^i_j s^j_0 - \frac{(\alpha+\beta)^3}{\alpha}b_k s^i_j s^j_0.
\end{aligned} \tag{2.15}
$$

Now, equation (2.15), can be written as

$$
R^i_k =^\alpha \mathcal{R}^i_k + \frac{(\alpha+\beta)}{2\alpha}\Big[\mathcal{A} + \mathcal{B}\alpha^{-2} + \mathcal{C}\alpha^{-1}\Big]. \tag{2.16}
$$

where

$$
\begin{aligned}
\mathcal{A} &= -2(\alpha+\beta)s^i_{0;k} - 2b_k s^i_{0;0} - (\alpha+\beta)s^i_{k;0} - 2(\alpha+\beta)^2 b_k s^i_j s^j_0, \\
\mathcal{B} &= (\alpha-\beta)y_k s^i_{0;0} + (\alpha+\beta)^3 y_k s^i_j s^j_0, \\
\mathcal{C} &= \frac{(\alpha+\beta)^3}{2}s^i_j s^j_k.
\end{aligned}
$$

Taking the trace of R^i_k, in equation (2.16) and using the relations $y_k s^k_{0;0} = 0$, and $b_k s^k_{0;0} = -s_{k0}s^k_0$, we obtain

$$
\mathcal{R}ic =^\alpha \mathcal{R}ic - \frac{(\alpha+\beta)^2}{\alpha}s^i_{0;k} + \frac{(\alpha+\beta)^4}{4\alpha^2}s^i_j s^j_k. \tag{2.17}
$$

Zhang and Shen [7] recently established a relationship between the Ricci curvature Ric of the Finsler metric $\mathcal{F}$ and the Ricci curvature ${}^{\alpha}$Ric of the Riemannian metric α. This relationship simplifies to equation (2.17) when β is a Killing form of constant length. By the equation (2.17), we immediatly obtain the relation

$$\mathcal{R}ic = c\mathcal{F}^2 \tag{2.18}$$

where

$$c = \frac{1}{4}s^j{}_k s^k{}_j. \tag{2.19}$$

if and only if

$${}^{\alpha}\mathcal{R}ic = \frac{(\alpha+\beta)^2}{\alpha}s^i_{0;k}. \tag{2.20}$$

Equation (2.20) is linear in y^i. Therefore, for $n > 2$, there does not exist a scalar function λ such that $s^k_{0;k} = \lambda\beta$. As a result, equation (2.20) cannot be expressed in the form ${}^{\alpha}\mathcal{R}ic = \lambda\alpha^2$, since β is a Killing form of constant length. This implies that α does not define an Einstein metric. Hence, we conclude the following.

Theorem 2.1. *A Z-Shen square metric $F = \frac{(\alpha+\beta)^2}{\alpha}$, where β is a Killing 1-form of constant length on an n-dimensional manifold M^n $(n > 2)$, is not form an Einstein metric because the Riemannian metric α is not an Einstein metric.*

Let us consider a local orthonormal frame b_i on M^n with respect to the Riemannian metric α, and define $e_i = (y, b_i)_{i=1}^n$ as the corresponding local orthonormal frame on the pulled-back bundle π^*TM^n, determined by b_i. The form of equation (2.15) remains valid for the components of the Riemann curvature with respect to the frame $\{e_i\}_{i=1}^n$. We then have

$$\mathcal{Q}^i = -\frac{1}{2}\mathcal{F}s^i_p y^p = -\frac{1}{2}\mathcal{F}b_{i:p}y^p \tag{2.21}$$

Therefore equation (2.15) takes a form

$$\begin{aligned}\mathcal{R}^i_k =& {}^{\alpha}\mathcal{R}ic^i_k - \mathcal{F}b_{i;j;k}y^j + \frac{(\alpha+\beta)}{\alpha}\Big\{(\frac{\alpha-\beta}{2\alpha^2})y_k - b_k\Big\}b_{i;j;p}y^j y^p - \frac{\mathcal{F}}{2}b_{i;k;j}y^j \\ &+ \frac{\mathcal{F}^2}{4}b_{i;j}b_{j;k} + \frac{\mathcal{F}^2}{2\alpha}y_k b_{i;j}b_{j;p}y^p - \mathcal{F}(\alpha+\beta)b_k b_{i;j}b_{j;p}y^p.\end{aligned} \tag{2.22}$$

Using the Bianchi identities, we get

$$b_{i;j;k} - b_{i;k;j} = b_m{}^{\alpha}\mathcal{R}^m_{ijk}, \tag{2.23}$$

we have from [5]

$$b_{i;j;k}y^j = b_{i;k;j}y^j + b_m{}^{\alpha}\mathcal{R}^m_{ijk}y^j$$

and

$$\begin{aligned} b_{i;j;k}y^jy^k &= -b_{j;i;k}y^jy^k \\ &= -(b_{j;k;i}+b_m{}^\alpha\mathcal{R}^m_{ijk})y^jy^k \\ &= b_m{}^\alpha\mathcal{R}^m_i. \end{aligned}$$

Substituting these values into (2.22), we get

$$\begin{aligned} \mathcal{R}^i_k =^\alpha \mathcal{R}ic^i_k - \mathcal{F}(b_{i;k;j}+b^\alpha_m\mathcal{R}^m_{ijk})y^j - \frac{(\alpha+\beta)}{\alpha}\Big\{\frac{(\alpha-\beta)}{2\alpha^2}y_k - b_k\Big\}b^\alpha_m\mathcal{R}^m_i \\ -\frac{\mathcal{F}}{2}b_{i;k;j}y^j + \frac{\mathcal{F}^2}{2\alpha}y_kb_{i;j}b_{j;p}y^p - \frac{\mathcal{F}^2}{4}\Big\{b_kb_{i;j}b_{j;p}y^p - b_{i;j}b_{j;k}\Big\} \end{aligned} \tag{2.24}$$

Since α is Riemannian metric,

$$^\alpha\mathcal{R}^h_{ihk} = -^\alpha\mathcal{R}^h_{hik} = -\frac{(n-1)}{2}{}^\alpha\mathcal{R}_{y^iy^k}.$$

Contracting equation (2.24) with respect to i and k and using (1.2), we obtain an equation of Ricci curvature $\mathcal{R}ic$ of $\mathcal{F}$ as follows:

$$\begin{aligned} \mathcal{R}ic =^\alpha \mathcal{R}ic - \frac{(n-1)}{2\beta}\frac{(\alpha+\beta)^2}{\alpha}b_m{}^\alpha R_{y^my^r}y^r + \frac{(\alpha+\beta)^2}{2\alpha^2}b_m{}^\alpha R^m_ib_i \\ -\frac{(\alpha+\beta)^4}{4\alpha^2}(b_{i;j})^2, \end{aligned} \tag{2.25}$$

where $^\alpha\mathcal{R}ic$ is the Ricci curvature of α.

3. The S- curvature and projectively flat Z Shen square metric

In this section, we will focus on S curvature of Z Shen square metric $F = \frac{(\alpha+\beta)^2}{\alpha}$

Differentiating equtions (2.1), (2.2) and (2.3) with respect to y^i, we obtain

$$\mathcal{G}^m_{y^m} =^\alpha \mathcal{G}^m_{y^m} + (\mathcal{P}y_m)_{y^m} + \mathcal{Q}^m_{y^m}, \tag{3.1}$$

$$(\mathcal{P}y_m)_{y^m} = (n+1)\mathcal{P}, \tag{3.2}$$

and

$$\mathcal{Q}^m_{y^m} = [\frac{(\alpha+\beta)}{\alpha} - \frac{1}{b^2}]s_0 - \frac{1}{b^2}r_{0m}b^m - \frac{1}{2}\mathcal{F}\mathcal{S}^m_m, \tag{3.3}$$

where

$$\mathcal{G}^m_{y^m} = \frac{\partial\mathcal{G}^m}{\partial y^m}.$$

Using (3.1), (3.2) and (3.3) in (1.3), we get

$$\begin{aligned} \mathcal{S} =^\alpha \mathcal{G}^m_{y^m} - (n+1)\frac{-1}{b^2}(s_0+\mathcal{F}^{-1}r_{00}) + (\frac{(\alpha+\beta)^2}{\alpha^2} - \frac{1}{b^2})s_0 \\ -\frac{\mathcal{F}}{2}\mathcal{S}^m_m - \frac{1}{b^2}r_{0m}b^m - y^m(\ln\sigma_{\mathcal{F}})_{x^m}, \end{aligned} \tag{3.4}$$

where

$$(\ln \sigma_{\mathcal{F}})_{x^m} = \frac{\partial(\ln \sigma_{\mathcal{F}})}{\partial x^m}.$$

We note that [1]

$$\sigma_{\mathcal{F}}(x) = \rho^{n+1}\sigma_\alpha(x), \tag{3.5}$$

where $\rho = \rho(x)$.

For a Riemannian metric α, we have

$$y^m(\ln \sigma_{\mathcal{F}})_{x^m} =^\alpha \mathcal{G}^m_{y^m}. \tag{3.6}$$

Putting (3.5) and (3.6) in (3.4), we have

$$\begin{aligned}\mathcal{S} = -\frac{1}{b^2}[(n+2)s_0 + (n+1)\mathcal{F}^{-1}r_{00} + r_{0m}b^m] + \frac{(\alpha+\beta)^2}{\alpha^2}s_0 \\ -\frac{\mathcal{F}}{2}\mathcal{S}^m_m - (n+1)\rho^{-1}\rho_{x^m}y^m.\end{aligned} \tag{3.7}$$

Suppose β is a Killing form that is $r_{ij} = 0$ and $\mathcal{S} = (n+1)c\mathcal{F}$, where

$$c = -\frac{1}{2(n+1)}\mathcal{S}^m_m.$$

Then from (3.7), we have

$$-[(n+2)\alpha - b^2(\alpha+\beta)]s_0 - (n+1)b^2\alpha\rho^{-1}\rho_{x^m}y^m = 0. \tag{3.8}$$

The above equation may be expressed as

$$(c_{lm}s_n + d_l y_m \rho_{x^n})y^l y^m y^n = 0, \tag{3.9}$$

where,

$$c_{lm} = ((n+2)a_l - b^2 a_{lm}) \tag{3.10}$$

and

$$d_{lm} = -(n+1)b^2 a_l \rho^{-1}. \tag{3.11}$$

Differentiating (3.9) successively witth respect to y^i, y^j and y^k, we obtain

$$c_{ij}s_k + d_{ij}\rho_{x^k} + (ijk) = 0, \tag{3.12}$$

where $+(ijk)$ denote the cyclic interchange of i, j, K and summation. Conversely, suppose (3.12) holds. Then multiplying (3.12) by $y^i y^j y^k$, we obtain (3.8). Putting (3.8) in (3.7), we get $\mathcal{S} = (n+1)c\mathcal{F}$, where $c = c(x)$, some function of x that is $\mathcal{F}$ has isotropic $\mathcal{S}$-currvature. thus we have

Theorem 3.1. *The $\mathcal{S}$-curvature of a Z-Shen square metric $\mathcal{F} = \frac{(\alpha+\beta)^2}{\alpha}$ is given by equation (3.7). If β is a Killing form, then the following two statements are equivalent:*
(1) The metric $\mathcal{F}$ has isotropic $\mathcal{S}$-curvature.
(2) The condition $c_{ij}s_k + d_{ij}\rho_{x^k} = 0$ holds, where the coefficients c_{ij} and d_{ij} are defined in equations (3.10) and (3.11), respectively.

4. **Geometrical and Physical Significance of the Results**

(i) The S-curvature is a measure of how the volume changes along geodesics in Finsler geometry. For Z. Shen's square metric, having isotropic S-curvature implies uniform volumetric distortion, which can be interpreted as a generalized form of constant divergence in the manifold.

(ii) The condition that β is a Killing 1-form of constant length implies that it preserves the metric (infinitesimal isometry) and has no variation in norm. This restricts the geometry to be highly symmetric, akin to constant curvature in Riemannian geometry.

(iii) In contexts like general relativity or optics, where Finsler geometry is sometimes applied, isotropic S-curvature can be related to models with uniform entropy production or energy dispersion.

If β is a Killing form of constant length then from (3.7), it follows that $\mathcal{S} = (n+1)c\mathcal{F}$, where $c = c(x)$, some function of x.
Conversely, suppose $\mathcal{F}$ has isotropic $\mathcal{S}$-curvature, then from (3.8), we have

$$[(n+2)\alpha + (\alpha+\beta)b^2]s_0 - (n+1)b^2\alpha\rho^{-1}\rho_{x^m}y^m = b^2\alpha^2 s_0. \tag{4.1}$$

Comparing the coefficients of α^2 on both sides, we get $s_0 = 0$. This implies $s_i = 0$, i.e. β is of constant length. Thus we have

Theorem 4.1. *The $\mathcal{S}$-curvature of a Z-Shen square metric $\mathcal{F} = \frac{(\alpha+\beta)^2}{\alpha}$ is given by equation (3.7). If β is a Killing form, the following statements are equivalent:*
(1) β is of constant length.
(2) The metric $\mathcal{F}$ has isotropic $\mathcal{S}$-curvature.

Combining above theorems we have

Theorem 4.2. *The $\mathcal{S}$-curvature of a Z-Shen square metric $\mathcal{F} = \frac{(\alpha+\beta)^2}{\alpha}$ is given by equation (3.7). If β is a Killing form, the following conditions are equivalent:*
(1) β is of constant length.
(2) The metric $\mathcal{F}$ has isotropic $\mathcal{S}$-curvature.
(3) The relation $c_{ij}s_k + d_{ij}\rho_{x^k} = 0$ holds, where c_{ij} and d_{ij} are defined in equations (3.10) and (3.11), respectively.

Next, we consider projectively flat Z. shen square metric. Let the Z shen square metric $\mathcal{F} = \frac{(\alpha+\beta)^2}{\alpha}$ be projectively flat and β is a Killing form of constant length. Then $\mathcal{G}^i = \mathcal{P}y^i$ and $r_{ij} = 0, s_i = 0$. therefore (2.1) and (2.2) yields

$$\mathcal{P}y^i =^{\alpha} \mathcal{G}^i + \mathcal{Q}^i, \tag{4.2}$$

$$(n+1)\mathcal{P}y^i =^\alpha \mathcal{G}^m_{y^m}y^i + \mathcal{Q}^m_{y^m}y^i, \tag{4.3}$$

From (3.144) and (3.15), we have

$$\mathcal{Q}^m_{y^m}y^i - (n+1)\mathcal{Q}^i =^\alpha \mathcal{G}^m_{y^m}y^i -^\alpha \mathcal{G}^i, \tag{4.4}$$

Using (2.3) and (3.3) in (3.16), we have

$$(\alpha+\beta)^2(\mathcal{S}^m_m y^i - (n+1)\mathcal{S}^i_0) = 2\alpha({}^\alpha\mathcal{G}^i -^\alpha \mathcal{G}^m_{y^m}y^i). \tag{4.5}$$

Since $({}^\alpha\mathcal{G}^i -^\alpha \mathcal{G}^m_{y^m}y^i)$ is quadratic in y^i, both sides are identically zero. i.e. α is projectivel flat and

$$\mathcal{S}^m_m y^i - (n+1)\mathcal{S}^i_0 = 0. \tag{4.6}$$

Differentiating equation (3.18) with respect to y^i, we obtain $\mathcal{S}^m m = 0$. Substituting this result into (3.18), we get $\mathcal{S}^i 0 = 0$. Therefore, β must be closed.

Conversely, suppose β is closed and α is projectively flat. Then, from equation (2.1), it follows that $\mathcal{F}$ is projectively flat. Thus, we conclude that

Theorem 4.3. *A Z-Shen square metric $\mathcal{F} = \frac{(\alpha+\beta)^2}{\alpha}$, where β is a Killing form of constant length, is projectively flat if and only if β is closed and α is projectively flat.*

From theorems (4.2) and (4.3), we have

Theorem 4.4. *For a Z-Shen square metric $\mathcal{F} = \frac{(\alpha+\beta)^2}{\alpha}$, where β is a Killing form, the following conditions are equivalent:*
(1) The metric $\mathcal{F}$ has isotropic $\mathcal{S}$-curvature.
(2) The metric $\mathcal{F}$ is projectively flat.
(3) The Riemannian metric α is projectively flat, and β is closed.

4.1. **Geometrical and Physical significance of above theorems (3.2), (3.3), (3.4) and (3.5).** (i) Isotropic S-curvature implies uniformity of volume change in geodesic flow, crucial in classifying Finsler spaces akin to constant curvature in Riemannian geometry.

(ii) Killing forms represent symmetries or invariants; constant length further constrains the geometry to highly symmetric structures.
(iii) Projective flatness implies that geodesics are straight lines in some coordinate system; it reflects maximal simplicity of the geodesic structure.

(iv) In general relativity and spacetime geometry, Finsler structures extend Riemannian models to account for direction-dependent behavior; isotropic S-curvature relates to conserved or uniformly distributed physical quantities like energy density.

(v) Killing forms of constant length relate to conserved momentum or angular momentum via Noether-type results.

(vi) Projective flatness can model scenarios where particles or light rays move in free-fall along straight trajectories, akin to gravitational vacuum solutions.

If β is a Killing form, then from (3.7) and (1.4), we have

$$E_{ij} = \frac{1}{2} + \frac{\mathcal{F}^2}{4} b_{i;j} b_{j;k} \Big[\frac{1}{\alpha}(\mathcal{F}_{y^j} s_i + \mathcal{F}_{y^i} s_j) + \frac{1}{\alpha^2}(\mathcal{F}_{y^j} a_i - \mathcal{F}_{y^i} a_j) s_0 + \mathcal{F}_{y^i y^j} \frac{1}{\alpha} s_0 + \mathcal{F}(2\alpha^{-3} a_i a_j s_0 - \frac{1}{\alpha^2}(a_i s_j - a_j s_i)) - \frac{1}{2} \mathcal{S}^m_m \mathcal{F}_{y^i y^j} \Big]. \tag{4.7}$$

From (3.19), it follows that $E_{ij} = (n+1)c\mathcal{F}$, where $c = -\frac{1}{4(n+1)}$ if and only if

$$\frac{1}{\alpha}(\mathcal{F}_{y^j} s_i + \mathcal{F}_{y^i} s_j) + \frac{1}{\alpha^2}(\mathcal{F}_{y^j} a_i - \mathcal{F}_{y^i} a_j) s_0 + \mathcal{F}_{y^i y^j} \frac{1}{\alpha} s_0 + \mathcal{F}(2\alpha^{-3} a_i a_j s_0 - \frac{1}{\alpha^2}(a_i s_j - a_j s_i)) - \frac{1}{2} \mathcal{S}^m_m \mathcal{F}_{y^i y^j}. \tag{4.8}$$

But (3.20) holds if and only if $s_0 = 0$. Thus we have

Theorem 4.5. *A Z-Shen square metric $\mathcal{F} = \frac{(\alpha+\beta)^2}{\alpha}$, where β is a Killing form, has isotropic E-curvature if and only if $s_0 = 0$.*

4.2. **Geometrical and Physical significance of theorem (3.6).** (i) The condition $s_0 = 0$ characterizes when the Z-Shen square metric $\mathcal{F} = \frac{(\alpha+\beta)^2}{\alpha}$ possesses isotropic E-curvature, meaning the E-curvature is directionally invariant. β being a Killing form implies that it generates infinitesimal isometries with respect to the Riemannian metric α, and thus contributes symmetries to the geometry.

(ii) This leads to strong constraints on the geometry, potentially simplifying curvature computations and revealing deeper geometric structure. (iii) In physical theories, especially in Finslerian extensions of general relativity or modified gravity models, curvature quantities relate to the behavior of particles and fields.

(iv) Isotropic E-curvature could be interpreted as a measure of uniformity in the deviation of geodesics, akin to an isotropic force field.

(v) If $s_0 = 0$, this uniformity (isotropy) in curvature suggests physical laws might be the same in all directions within that space, a desirable symmetry in many physical models.

References

1. S.S. Chern and Z. Shen, *Riemann- Finsler Geometry*, World Scientific, Singapore, (2005).
2. V. K. Kropina, *On Projective Two dimensional Finsler spaces with special metric* , Trudy Sem. Vector Tenzor Anal. **11**(1961), 277-292.
3. M. Matsumoto, *Foundation of Finsler geometry and special Finsler spaces*, Kaiseisha Press, Saikawa, Otsu, 520, Japan, 1986
4. Z. Shen, *Volume comparision and its applications in Riemann-Finsler geometry*, Adv. Math. **128**(1997), 306-328.
5. Z. Shen , *Lectures on Finsler Geometry*, World Scientific, Singapore, (2001).
6. Z. Shen, *Finsler metrics with* $K = 0$ *an* $S = 0$, Can. J. Math. **55**(1) (2003), 112-132.
7. X. Zhang and Y. B. Shen, *On Einstein Kropina Metrics*, arXiv:1207.1944v1[math. DG](2012).
8. Amr Soleiman and Salah Gomaa Elgendi, *On Generalized Shen's Square metric*, Korean J. Math. **32**(3) (2024), 467-484.
9. P. K. Dwivedi, C.K. Mishra and Sachin Kumar, *On Projectively-flat Finsler space with n-power* (α, β)*-metric*, J. Finsler Geom. Appl. **5**(2) (2024), 14-24.
10. H. G. Nagaraja, *On Kropina metrics* , J. African. Math. Union, (2013).

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