Journal of Finsler Geometry and its Applications
Vol. 6, No. 2 (2025), pp 32-40
https://doi.org/10.22098/jfga.2025.17203.1156

Cartan connection for h-Matsumoto change

M. K. Gupta[a], Abha Sahu[*b] and Suman Sharma[c]

[a,b,c]Department of Mathematics, Guru Ghasidas Vishwavidyalaya
Bilaspur (C.G.), India
E-mail: mkgiaps@gmail.com
E-mail: abhasahu118@gmail.com
E-mail: sharma.suman209@gmail.com

Abstract. In the present paper, we have studied the Matsumoto change $\overline{L}(x,y) = \frac{L^2(x,y)}{L(x,y)-\beta(x,y)}$ with an h-vector $b_i(x,y)$. We have derived some fundamental tensors for this transformation. We have also obtained the necessary and sufficient condition for which the Cartan connection coefficients for both the spaces $F^n = (M^n, L)$ and $\overline{F}^n = (M^n, \overline{L})$ are same.

Keywords: Finsler space, Cartan connection, Matsumoto change and h-vector

1. Introduction

Let M be an n-dimensional C^∞ manifold and T_xM denotes the tangent space of M at x. The tangent bundle of M is the union of tangent space $TM := \bigcup\limits_{x\epsilon M} T_xM$. A function $L : TM \to [0,\infty)$ is called Finsler metric function if it has the following properties[12]

(1) L is C^∞ on $TM\backslash\{0\}$,
(2) For each $x\epsilon M$, $L_x := L|_{T_xM}$ is a Minkowski norm on T_xM.

*Corresponding Author
AMS 2020 Mathematics Subject Classification: 53B40, 53C30

The pair (M^n, L) is then called a Finsler space. The normalized supporting element, metric tensor, angular metric tensor and Cartan tensor are defined by $l_i = \dot{\partial}_i L$, $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$, $h_{ij} = L\dot{\partial}_i\dot{\partial}_j L$ and $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ respectively.

The Cartan connection for the Finsler space F^n is given by $(F^i_{jk}, N^i_j, C^i_{jk})$. The h-covariant and v-covariant derivative of the tensor T^i_j with respect to Cartan connection, are respectively given as follows

$$T^i_{j|k} = \delta_k T^i_j + T^r_j F^i_{rk} - T^i_r F^r_{jk}\,,$$
$$T^i_j|_k = \dot{\partial}_k T^i_j + T^r_j C^i_{rk} - T^i_r C^r_{jk}\,,$$

where δ_k is differential operator $\delta_k = \partial_k - N^r_k\dot{\partial}_r$.

In 1984, C. Shibata [13] introduced the change $\overline{L} = f(L, \beta)$ as a generalization of Randers change, where f is positively homogeneous function of degree one in L and $\beta(x, y) = b_i(x)y^i$. This change is called β-change. An important class of β-change is Matsumoto change, given by

$$\overline{L}(x, y) = \frac{L^2}{L - \beta}.$$

If $L(x, y)$ reduces to a Riemannian metric then $\overline{L}(x, y)$ becomes Matsumoto metric. A famous example of Finsler space "A slope measure of a moutain with respect to time measure" was given by M. Matsumoto[11]. Due to his great contribution in Finsler geometry, this metric was named after him.

A. Tayebi et al. [16] and Bankteshwar Tiwari et al. [17] discussed the Kropina change and generalized Kropina change respectively, for the Finsler space with m^{th} root metric. In 2017, A. Tayebi et al.[15] obtained the condition for the Finsler space given by Matsumoto change to be projectively related with the original Finsler space. To provide a more comprehensive overview of the related literature, we have added additional references [5, 6, 7, 8] discussing recent developments in Matsumoto change of Finsler spaces.

The concept of h-vector b_i, was first introduced by H. Izumi [9], which is v-covariant constant with respect to Cartan connection and satisfies $LC^h_{ij}b_h = \rho h_{ij}$, where ρ is a non-zero scalar function. He showed that the scalar ρ depends only on positional coordinates *i.e.* $\dot{\partial}_i\rho = 0$. From the definition of h-vector, it is clear that it depends not only on positional coordinates, but also on directional arguments.

Gupta and Pandey [2, 4], discussed certain properties of Randers change and Kropina change with an h-vector. They[4] showed that *If the h-vector is gradient then the scalar ρ is constant*, *i.e.* $\partial_j\rho = 0$. In 2016, Gupta and Gupta [1, 3] have analyzed Finsler space subjected to h-exponential change.

In the present paper, we have studied a Finsler metric defined by

$$\overline{L}(x, y) = \frac{L^2(x, y)}{L(x, y) - b_i(x, y)y^i}\,, \tag{1.1}$$

where $b_i(x, y)$ is an h-vector in (M^n, L).

The structure of this paper is as follows: In section 2, we have obtained the expressions for different fundamental tensors of the transformed Finsler space. In section 3, we have observed how the Cartan connection coefficients change due to Matsumoto change with an h-vector and also find the necessary and sufficient condition for which both connection coefficients would be same.

Let the Finsler space transformed by the Matsumoto change (1.1) with an h-vector, be denoted by $\overline{F}^n = (M^n, \overline{L})$. If we denote $\beta = b_i(x,y)y^i$, then indicatory property of angular metric tensor yields $\dot{\partial}_j \beta = b_j$. Throughout this paper, we have barred the geometrical objects associated with $\overline{F}^n$.
From (1.1), we get the normalized supporting element as

$$\bar{l}_i = \frac{\tau}{(\tau-1)}\, l_i + \frac{\tau^2}{(\tau-1)^2}\, m_i\,, \tag{1.2}$$

where

$$\tau := \frac{L}{\beta}, \quad m_i := b_i - \frac{1}{\tau} l_i.$$

Remark 1.1. *The covariant vector m_i stratifies the following relations*
(i) $m_i \neq 0$ (ii) $m^i = g^{ij} m_j$ (iii) $m^2 = m_i m^i$ (iv) $m_i y^i = 0$.

Differentiating equation (1.2) with respect to y^j, and using the notation $L_{ij} = \dot{\partial}_j l_i$ we get

$$\overline{L}_{ij} = \frac{\tau(\tau+\rho\tau-2)}{(\tau-1)^2}\, L_{ij} + \frac{2\tau^2}{\beta(\tau-1)^3}\, m_i m_j.$$

Therefore, the angular metric tensor $\overline{h}_{ij}$ is obtained as

$$\overline{h}_{ij} = \frac{\tau^2(\tau+\rho\tau-2)}{(\tau-1)^3}\, h_{ij} + \frac{2\tau^4}{(\tau-1)^4} m_i m_j\,. \tag{1.3}$$

The metric tensor $\overline{g}_{ij} = \overline{h}_{ij} + \bar{l}_i \bar{l}_j$ is given by

$$\overline{g}_{ij} = \frac{\tau^2(\tau+\rho\tau-2)}{(\tau-1)^3}\, g_{ij} + \frac{\tau^2(1-\rho\tau)}{(\tau-1)^3}\, l_i l_j + \frac{\tau^3}{(\tau-1)^3}\,(m_i l_j + m_j l_i) + \frac{3\tau^4}{(\tau-1)^4}\, m_i m_j\,, \tag{1.4}$$

which can be rewritten as

$$\overline{g}_{ij} = p\, g_{ij} + p_1 l_i l_j + p_2(m_i l_j + m_j l_i) + p_3\, m_i m_j\,, \tag{1.5}$$

where

$$p = \frac{\tau^2(\tau+\rho\tau-2)}{(\tau-1)^3}, \quad p_1 = \frac{\tau^2(1-\rho\tau)}{(\tau-1)^3}, \quad p_2 = \frac{\tau^3}{(\tau-1)^3}, \quad p_3 = \frac{3\tau^4}{(\tau-1)^4}\,.$$

The following lemma helps us to compute the inverse of metric tensor $\overline{g}_{ij}$.

Lemma 1.2. [10]: *Let (m_{ij}) be a non-singular matrix and $l_{ij} = m_{ij} + n_i n_j$. The elements l^{ij} of the inverse matrix, and the determinant of the matrix (l_{ij}) are given by*

$$l^{ij} = m^{ij} - (1 + n_k n^k)^{-1} n^i n^j, \quad det(l_{ij}) = (1 + n_k n^k) det(m_{ij})$$

respectively, where m^{ij} are elements of the inverse matrix of (m_{ij}) and $n^k = m^{ki} n_i$.

The inverse metric tensor of $\overline{F}^n$ can be derived as follows:

$$\overline{g}^{ij} = q\, g^{ij} + q_1\, l^i l^j + q_2\, (l^i m^j + m^i l^j) + q_3\, m^i m^j\,, \tag{1.6}$$

where

$$q = \frac{1}{p}\,, \qquad q_1 = -\frac{1}{2}\Big[\frac{p_1 p_3 - p_2^2}{(p_1 + p)p_3 - p_2^2} + \frac{2p^2 p_2^2 p_3}{(3p + 2p_3 m^2)\{(p_1 + p)p_3 - p_2^2\}^2}\Big],$$

$$q_2 = \frac{-2p_2 p_3}{(3p + 2p_3 m^2)\{(p_1 + p)p_3 - p_2^2\}}, \qquad q_3 = \frac{-2p_3}{p(3p + 2p_3 m^2)}\,.$$

The Cartan tensor $\overline{C}_{ijk}$ is obtained by differentiating the equation (1.5) with respect to y^k, as follows:

$$\overline{C}_{ijk} = p\, C_{ijk} + V_{ijk}\,, \tag{1.7}$$

where

$$V_{ijk} = K_1(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + K_2\, m_i m_j m_k$$

and

$$K_1 = \frac{\tau^3(\tau + 3\rho\tau - 4)}{2L(\tau - 1)^4}, \quad K_2 = \frac{6\tau^4}{\beta(\tau - 1)^5}\,.$$

Remark 1.3. *From above we can retrieve relations between the scalars as*

$$\frac{\partial\, p}{\partial\, \tau} = -\frac{2L}{\tau^2}\, K_1\,, \quad \frac{\partial\, p_3}{\partial\, \tau} = -\frac{2L}{\tau^2}\, K_2\,,$$

$$K_1 = \frac{1}{2L}\left\{p_2 + p_3\left(\rho - \frac{1}{\tau}\right)\right\} \quad \text{and} \quad p_1 + p_2\left(\rho - \frac{1}{\tau}\right) = 0\,.$$

From equation (1.6) and (1.7), we get the *(h)hv*-torsion tensor $\overline{C}^i_{jk}$

$$\overline{C}^i_{jk} = C^i_{jk} + M^i_{jk}\,, \tag{1.8}$$

where

$$\begin{aligned} M^i_{jk} &= q\, K_1(m_k h^i_j + m_j h^i_k) + (q_2\, l^i + q_3\, m^i)\left\{2K_1 m_j m_k + \frac{p}{L}\rho\, h_{jk}\right\} \\ &\quad + \left\{q\, m^i + (q_2\, l^i + q_3\, m^i) m^2\right\}\left(K_2 m_j m_k + K_1 h_{jk}\right). \end{aligned}$$

2. Cartan Connection of the space $\overline{F}^n$

The Cartan connection for an n-dimensional Finsler space $\overline{F}^n$ is given by the third $(\overline{F}^i_{jk}, \overline{N}^i_j, \overline{C}^i_{jk})$. The v-connection coefficient $\overline{C}^i_{jk}$ is given by equation (1.8). Now, we are obtaining the h-connection coefficient $\overline{F}^i_{jk}$ and non-linear connection coefficient $\overline{N}^i_j$. First, we will try to find canonical spray of the transformed space $\overline{F}^n$.

Differentiating equation (1.5) with respect to x^k, and using the definition of h-covariant derivative, we obtain

$$\begin{aligned}\partial_k\overline{g}_{ij} =& p\,\partial_k g_{ij} + p_1(l_i l_r F^r_{jk} + l_j l_r F^r_{ik}) + p_2(\rho_k h_{ij} + l_i b_{j|k} + l_j b_{i|k} + m_r F^r_{jk} l_i \\ &+ m_r F^r_{ik} l_j + m_i F^r_{jk} l_r + m_j F^r_{ik} l_r) + p_3(m_i b_{j|k} + m_j b_{i|k} + m_i m_r F^r_{jk} \\ &+ m_j m_r F^r_{ik}) + 2(K_1 h_{ij} + K_2 m_i m_j)(\beta_k + N^r_k m_r) \\ &+ 2K_1(h_{jr} N^r_k m_i + h_{ir} N^r_k m_j)\end{aligned} \tag{2.1}$$

where $\partial_k\rho = \rho_{|k} = \rho_k$ and $\beta_{|k} = \beta_k$.

Applying Christoffel process with respect to indices $i\,, j\,, k$ in above equation, we obtain the coefficient of Christoffel symbol as follows:

$$\begin{aligned}\overline{\gamma}_{ijk} =& p\gamma_{ijk} + \mathfrak{S}_{ijk}\left\{\frac{p_2}{2}\rho_k h_{ij} + (\beta_k + N^r_k m_r)\,B_{ij} + K_1\,(h_{jr} m_i + h_{ir} m_j)\,N^r_k\right\} \\ &+ Q_i F_{jk} + Q_k F_{ji} + Q_j E_{ik} \\ &+ (\overline{g}_{rj} - p\,g_{rj})\Big\{\gamma^r_{ik} + g^{rt}(C_{ikm} N^m_t - C_{tkm} N^m_i - C_{itm} N^m_k)\Big\}\,,\end{aligned} \tag{2.2}$$

where the symbol $\mathfrak{S}_{ijk}$ is defined as

$$\mathfrak{S}_{ijk}\,U_{ijk} = U_{ijk} - U_{jki} + U_{kij}$$

and we have used the notation

$$Q_i = p_2 l_i + p_3 m_i\,, \quad B_{ij} = K_1 h_{ij} + K_2 m_i m_j\,,$$

$$2E_{ij} = b_{i|j} + b_{j|i}\,, \qquad 2F_{ij} = b_{i|j} - b_{j|i}\,.$$

Remark 2.1. *The tensors Q_i and B_{ij} satisfy the following*

(i) $\dot{\partial}_j Q_i = 2B_{ij}$ (ii) $B_{ij} = B_{ji}$ (iii) $B_{ij} y^i = 0\,.$

The Christoffel Symbol of second kind of the Finsler space $\overline{F}^n$ is given by

$$\begin{aligned}\overline{\gamma}^{\,i}_{jk} = \gamma^{\,i}_{jk} + (g^{it} - p\overline{g}^{\,it})(C_{jkm}N^m_t - C_{tkm}N^m_j - C_{jtm}N^m_k) + \overline{g}^{\,is}\Bigg[Q_jF_{sk} \\ +Q_kF_{sj} + Q_sE_{jk} + \mathfrak{S}_{jsk}\Big\{\frac{p_2}{2}\rho_k h_{js} + (\beta_k + N^r_k m_r)B_{sj} \\ +K_1\left(h_{jr}m_i + h_{ir}m_j\right)N^r_k\Big\}\Bigg]. \quad (2.3)\end{aligned}$$

Transvecting equation (2.3) by $y^j y^k$ and using $G^{\,i} = \frac{1}{2}\gamma^{\,i}_{jk}y^j y^k$, we get

$$\overline{G}^{\,i} = G^{\,i} + D^{\,i}\,, \tag{2.4}$$

where

$$D^{\,i} = \frac{1}{2}\,\overline{g}^{\,is}\big[Q_sE_{oo} + 2p_2LF_{so}\big]\,. \tag{2.5}$$

Thus, we have:

Proposition 2.2. *The spray coefficient of the transformed space is given by equation* (2.4).

Remark 2.3. *In the subscript zero 'o' is used to denote the transvection by y^i, i.e. $F_{so} = F_{si}y^i$.*

Differentiating equation (2.4) with respect to y^i and using $\dot{\partial}_j G^i = N^i_j$ and $\dot{\partial}_j\overline{g}^{\,is} = -2\,\overline{g}^{\,ir}\overline{C}^{\,s}_{rj}$, we get

$$\overline{N}^{\,i}_j = N^{\,i}_j + D^{\,i}_j\,, \tag{2.6}$$

where

$$\begin{aligned}D^i_j = \overline{g}^{\,ir}\Big\{-2D^m(\,p\,C_{mrj} + V_{mrj}) + Q_rE_{oj} + E_{oo}B_{rj} + p_2LF_{rj} \\ +Q_jF_{ro} + \frac{p_1}{2}\rho_k y^k h_{rj}\Big\}. \quad (2.7)\end{aligned}$$

Thus, we have:

Proposition 2.4. *The non linear connection coefficient of the transformed space is given by the equation* (2.6).

Now, we are in a position to obtain the Cartan connection coefficient for the space $\overline{F}^{\,n}$. We know that the relation between the Christoffel symbol and Cartan connection coefficient is given by

$$F^i_{jk} = \gamma^i_{jk} + g^{\,is}(C_{jkr}N^r_s - C_{skr}N^r_j - C_{jsr}N^r_k)\,.$$

In view of equation (1.7), (2.3) and (2.6), we have

$$\begin{aligned}\overline{F}^{\,i}_{jk} = \gamma^i_{jk} + (g^{it} - p\overline{g}^{\,it})(C_{ikm}N^m_t - C_{tkm}N^m_i - C_{itm}N^m_k) + \overline{g}^{\,is}\Big\{ Q_j F_{sk} \\ + Q_k F_{sj} + Q_s E_{jk} + \mathfrak{S}_{jsk}\Big(\frac{p_2}{2}\rho_k h_{js} + (\beta_k + N^r_k m_r)B_{sj} \\ + K_1\,(h_{jr}m_i + h_{ir}m_j)\,N^r_k\Big) + (pC_{jkr} + V_{jkr})(N^r_s + D^r_s) \\ -(pC_{skr} + V_{skr})(N^r_j + D^r_j) - (pC_{jsr} + V_{jsr})(N^r_k + D^r_k)\Big\}\end{aligned} \tag{2.8}$$

which can be simplified as

$$\begin{aligned}\overline{F}^{\,i}_{jk} = F^i_{jk} + \overline{g}^{\,is}\Big\{Q_j F_{sk} + Q_k F_{js} + Q_s E_{jk} + \mathfrak{S}_{jsk}\Big(\frac{p_2}{2}\rho_k h_{js} + \beta_k B_{js} \\ -p\,C_{jsr}D^r_k - V_{jsr}D^r_k\Big)\Big\}.\end{aligned}$$

Above equation can be rewritten as

$$\overline{F}^{\,i}_{jk} = F^i_{jk} + D^i_{jk}\,, \tag{2.9}$$

where

$$\begin{aligned}D^i_{jk} = \overline{g}^{\,is}\Big\{Q_j F_{sk} + Q_k F_{sj} + Q_s E_{jk} + \mathfrak{S}_{jsk}\Big(\frac{p_2}{2}\rho_k h_{js} \\ +\beta_k B_{js} - p\,C_{jsr}D^r_k - V_{jsr}D^r_k\Big)\Big\}.\end{aligned} \tag{2.10}$$

Hence, we have the following.

Theorem 2.5. *The relation between the Cartan connection coefficients of F^n and $\overline{F}^{\,n}$ is given by equation (2.9).*

Remark 2.6. *The tensors D^i_{jk}, D^i_j and D^i are related as*
(i) $D^i_{jk}\,y^k = D^i_j\,,$ (ii) $D^i_j\,y^j = 2D^{\,i}\,,$ (iii) $\dot{\partial}_j D^{\,i} = D^i_j.$

Now, we want to find the condition for which the Cartan connection coefficients for both spaces F^n and $\overline{F}^{\,n}$ are same, *i.e.* $\overline{F}^{\,i}_{jk} = F^i_{jk}$ then $D^i_{jk} = 0$, which implies $D^i_j = 0$, then $D^{\,i} = 0$. Therefore the equation (2.5) gives

$$2\,p_1 L F_{io} + E_{oo} Q_i = 0\,,$$

which on transvection by y^i gives $E_{oo} = 0$ and then $F_{io} = 0$. Differentiating $E_{oo} = 0$ partially with respect to y^i gives $E_{io} = 0$. Therefore we have $E_{io} = 0 = F_{io}$, which implies $b_{i|o} = b_{o|i} = \beta_{|i} = 0$. Differentiating $\beta_{|i}$ partially with respect to y^j and using the commutation formula

$$\dot{\partial}_j(\beta_{|i}) - (\dot{\partial}_j\beta)_{|i} = (\dot{\partial}_r\beta)C^r_{ij|o}$$

we get

$$b_{j|i} = -\, b_r C^r_{ij|o}\,. \tag{2.11}$$

This will give us $F_{ij}=0$. Taking h-covariant derivative of $LC^r_{ij}b_r = \rho h_{ij}$ and using $\rho_{|k}=0\,, L_{|k}=0$ and $h_{ij|k}=0$, we get

$$\left(C^r_{ij}b_r\right)_{|k} = \left(\frac{\rho}{L}h_{ij}\right)_{|k} = 0\,,$$

This gives

$$C^r_{sj}b_{r|k} + C^r_{sj|k}b_r = 0\,.$$

Transvecting by y^k and using $b_{r|o}=0$, we get $C^r_{ij|o}b_r = 0$ and then equation (2.11) gives $b_{i|j}=0$, *i.e.* the h-vector b_i is parallel with respect to Cartan connection of F^n.

Conversely, If $b_{i|j}=0$ then we get $E_{ij}=F_{ij}=0$ and $\beta_i = \beta_{|i} = b_{j|i}\,y^j = 0$. Then equation (2.5) reduces to $D^i=0$. From $F_{ij}=0$ we have $\rho_i=0$, which implies $D^i_j=0$. Therefore, from equation (2.10), we get $D^i_{jk}=0$, which gives $\overline{F}{}^i_{jk} = F^i_{jk}$. Thus, we have:

Theorem 2.7. *For the Matsumoto change with an h-vector, the Cartan connection coefficients for both spaces F^n and $\overline{F}{}^n$ are the same if and only if the h-vector b_i is parallel with respect to the Cartan connection of F^n.*

Now, differentiating equation (2.6) with respect to y^k, and using $\dot{\partial}_k N^i_j = G^i_{jk}$, we obtain

$$\overline{G}{}^i_{jk} = G^i_{jk} + \dot{\partial}_k\, D^i_j\,, \tag{2.12}$$

where G^i_{jk} are the Berwald connection coefficients.
Now, if the h-vector b_i is parallel with respect to the Cartan connection of F^n, then by Theorem 2.7, the Cartan connection coefficients for both Finsler space F^n and $\overline{F}{}^n$ are the same, *i.e.* $D^i_{jk}=0$ which implies $D^i_j=0$. Then from equation (2.12), we get $\overline{G}{}^i_{jk} = G^i_{jk}$.

Conversely, if $\overline{G}{}^i_{jk} = G^i_{jk}$ then, from equation (2.12), we have $\dot{\partial}_k\, D^i_j = 0$, which on transvecting by y^j and using Remark 2.6, gives $D^i_k = 0$. Using the same procedure as in Theorem 2.7, we get $b_{i|j}=0$, *i.e.* the h-vector b_i is parallel with respect to Cartan connection of F^n. Thus, we have:

Theorem 2.8. *For the Matsumoto change with an h-vector, the Berwald connection coefficients for both spaces F^n and $\overline{F}{}^n$ are the same if and only if the h-vector b_i is parallel with respect to the Cartan connection of F^n.*

Conclusion

In the present paper, The Cartan connection of the changed Finlser space is discovered and with the condition (h-vector b_i is parallel, *i.e.* $b_{i|j} = 0$), the Cartan connection of both the spaces are same. For this transformation we can also find some geometric properties for the transformed Finsler space like the curvature tensor, torsion tensor, T-Tensor etc.

References

1. M. K. Gupta and A. K. Gupta, *Hypersurface of a Finsler space subjected to an h-exponential change of metric.* Int. J. Geom. Meth. Mod. Phys. **13**(10) (2016), 1650129.
2. M. K. Gupta and P. N. Pandey, *Hypersurfaces of conformally and h-conformally related Finsler spaces.* Acta Mathematica Hungarica; **123**(3) (2009), 257-264.
3. M. K. Gupta and A. K. Gupta, *h-exponential change of Finsler metric.* Facta Univ. Series: Math. Inform. **31**(5) (2016), 1029-1039.
4. M. K. Gupta and P. N. Pandey, *Finsler space subjected to a Kropina change with an h-vector.* Facta Univ. Series: Math. Inform. **30(4)** (2015), 513-525.
5. M. K. Gupta and A. Sahu, *On Douglas Tensor of Generalized Matsumoto Finsler Space.* Lobachevskii Journal of Mathematics **45**(2) (2024), 685-692.
6. M. K. Gupta, A. Sahu, and C. Ozel, *On Douglas tensor of infinite series Finsler space.* Facta Univ. Series: Math. Inform. (2025), 049-055.
7. M. K. Gupta and A. Sahu, *On Matsumoto-Randers change on m-th root Finsler metric*, Palestine J. Math. **14**(1) (2025), 262-268.
8. M. K. Gupta, A. Sahu, and A. Tayebi, *On Conformal-Matsumoto change of mth root Finsler metrics.* Facta Univ. Series: Math. Inform. **38**(5) (2023), 895-903.
9. H. Izumi, *Conformal transformations of Finsler spaces II.* Tensor, N.S. **33**(1980), 337-359.
10. M. Matsumoto, *On C-reducible Finsler spaces*, Tensor, N.S. **24**(1972), 29-37.
11. M. Matsumoto, *A slope of a mountain is a Finsler surface with respect to a time measure*, J. Math. Kyoto. Univ. **29**(1)(1989), 17-25.
12. Z. Shen *Lectures on Finsler geometry.* World Scientific, 2001.
13. C. Shibata, *On invariant tensors of β-changes of Finsler metrics.* J. Math. Kyoto. Univ. **24**(1) (1984), 163-188.
14. H. S. Shukla, O. P. Pandey and K. Mandal, *Matsumoto change of Finsler metric by h-vector.* South Asian Journal of Mathematics, **5**(3) (2015), 109-116.
15. A. Tayebi and M. Shahbazi Nia, *On Matsumoto change of m^{th}-root Finsler metrics.* Publ. Ins. Mathematique, **101**(115) (2017), 183-190.
16. A. Tayebi, T. Tabatabaeifar and E. Peyghan, *On Kropina change for m^{th}-root Finsler metrics.* Ukrainian Mathematical Journal, **66**(1) (2014), 160-164.
17. B. Tiwari and G. K. Prajapati, *On generalized Kropina change of m^{th}-root Finsler metric.* Int. J. Geom. Meth. Mod. Phys. **14**(05) (2017): 1750081.

Received: 15.04.2025
Accepted: 18.05.2025