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Causal automorphisms of two-dimensional Minkowski spacetime and homeomorphisms between its Cauchy surfaces

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Abstract. In this paper, we show that for two-dimensional Minkowski spacetime $\mathbb{R}^2_1$ with a non-compact Cauchy surface Σ, every compact and connected subset of Σ is a future and past causally admissible subset and it means that the set of all the future causally admissible subset of $\mathbb{R}^2_1$ with respect to Σ is equal to the set of all the set of all the past causally admissible subset of $\mathbb{R}^2_1$ with respect to Σ. Moreover it has been shown that for every spacelike Cauchy surfaces Σ, $\Sigma^{'}$ of the globally hyperbolic spactime $\mathbb{R}^2_1$, every bijection $f : \Sigma \to \Sigma^{'}$ can be consider as a homeomorphism or (future, past) causally admissible function.

Keywords: Lorentzian geometry, Globally hyperbolic, Order-isomorphism, Vietoris topology, Causally admissible system.

1. **Introduction**

The study of causal automorphisms on spasetime is very important because the existence of causal automorphisms on spasetime $\mathcal{M}$, implies the existence of some kind of symmetry on $\mathcal{M}$.

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In 1964, Zeeman introduced a standard form of causal automorphism on Minkowski spacetime $\mathbb{R}^n_1$ for $n \geqslant 3$ [1]. Zeeman showed that any causal automorphism can be represented by a composite of orthochronous transformation, translation and dilatation and he classified all forms of causal automorphisms on $\mathbb{R}^n_1$ with $n \geqslant 3$. In[1] Zeeman showed that the group of all causal automorphisms on $\mathbb{R}^n_1$ is of finite dimensional when $n \geqslant 3$. Moreover, this result applies only when $n \geqslant 3$, and for two-dimensional Minkowski spacetime $\mathbb{R}^2_1$ the standard form of causal automorphism is not yet known. Recently, in [2] it is shown that the group of all homeomorphisms on $\mathbb{R}$ is a subgroup of the group of all causal automorphisms on $\mathbb{R}^2_1$, and thus the dimensional of group of all causal automorphisms on $\mathbb{R}^2_1$ is infinite. This result is different from the case of $n \geqslant 3$.

The causally admissible system which has been developed in [4], is the main tool to introduce the standard form of causal automorphism on $\mathbb{R}^2_1$, that is given by Kim in [3]. He has shown that, in causal theoretic viewpoint, $\mathbb{R}^2_1$ has much more symmetry than $\mathbb{R}^n_1$ has for $n \geqslant 3$. The standard form of causal automorphism on $\mathbb{R}^2_1$ is stated in Theorem 4.4 in [3] as follows:

> Let $F : \mathbb{R}^2_1 \to \mathbb{R}^2_1$ be a causal automorphism. Then, there exist a continuous function $g : \mathbb{R} \to \mathbb{R}$ and a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ which satisfy $sup(g \pm f) = \infty$, $inf(g \pm f) = -\infty$ and
>
> $$\Big|\frac{g(t+\delta t)-g(t)}{f(t+\delta t)-f(t)}\Big| < 1$$
>
> for all t and δt, such that if f is increasing, then F is given by
>
> $$F(x,y) = \left(\tfrac{f(x-y)+f(x+y)}{2} + \tfrac{g(x+y)-g(x-y)}{2}, \tfrac{f(x+y)-f(x-y)}{2} + \tfrac{g(x+y)+g(x-y)}{2}\right)$$
>
> and if f is decreasing, then we have
>
> $$F(x,y) = \left(\tfrac{f(x+y)+f(x-y)}{2} + \tfrac{g(x-y)-g(x+y)}{2}, \tfrac{f(x-y)-f(x+y)}{2} + \tfrac{g(x+y)+g(x-y)}{2}\right).$$
>
> Conversely, for any functions f and g which satisfy the above conditions, the function $F : \mathbb{R}^2_1 \to \mathbb{R}^2_1$ defined as above is a causal automorphism.

In this paper, we show that every spacelike Cauchy surface Σ of 2-dimensional Minkowski spacetime $\mathbb{R}^2_1$ can consider as a graph of some continuous function $f : \mathbb{R} \to \mathbb{R}$. In Proposition 4.2 and Proposition 4.3 we reconstruct the causal relation " $\leqslant$ " on two-dimensional Minkowski spacetime $\mathbb{R}^2_1$ in a new manner only by using the usual order relation " $\leqslant$ " on $\mathbb{R}$ and the absolute value of real numbers . In view of these results we show that every compact and connected subset of Σ is both future causally admissible subset and past causally admissible subset of $\mathbb{R}^2_1$. Therefore, the set of all compact and connected subset of Σ is equal to the causally admissible system C on spacelike Cauchy surface Σ (see Theorem 4.13). Finally, in Theorem 4.14 we prove that (future or past) causally admissible function and homeomorphism between two spacelike Cauchy surfaces coincide for two-dimensional Minkowski spacetime $\mathbb{R}^2_1$.

2. **Basics on causality theory**

In this section we introduce some basic notation and facts about causality theory on spacetimes, some good references are [7], [8] and [16].

A spacetime $\mathcal{M}$, is a smooth, connected, Housdorff, time-orientable n-dimensional Lorentzian manifold with signature $(-,+,...,+)$. Every $v \in T_p\mathcal{M}$ is called timelike (null, spacelike, resp.) if its inner product with itself is less than (equal to, greater than, resp.) zero. Let $\gamma : I \to \mathcal{M}$ be a smooth curve in $\mathcal{M}$. γ is said to be timelike (spacelike, null, causal) if its tangent is everywhere timelike (spacelike, null, causal, resp.). Since $\mathcal{M}$ is time-orientable, then it admits a smooth timlike vector field X. A timlike (resp. causal) curve $\gamma : I \to \mathcal{M}$ is said to be future directed provided each tangent vector $\gamma^{'}(t)$, is future directed, for all $t \in I$ (i.e. $\langle X_{\gamma(t)}, \gamma^{'}(t)\rangle < 0$). Past-directed timelike and causal curves are defined in a time-dual manner. If there exists a future-directed timelike curve in $\mathcal{M}$ from p to q, we write $p \ll q$ and say that q lies in the chronological future of p or p lies in the chronological past of q. Moreover, $p < q$ means there exists a future-directed causal curve from p to q, and we say that q lies in the causal future of p or p lies in the causal past of q. We shall use the notation $p \leqslant q$ to mean $p = q$ or $p < q$. The relation $p \leqslant q$ but not $p \ll q$ is written as $p \to q$ and is termed as horismos. A future (past, resp.) directed causal curve γ is said to be future (past, resp.) inextendible if it has no future (past, resp.) endpoint. A subset $\mathcal{S} \subset \mathcal{M}$ is achronal (acausal) provided no two points in $\mathcal{S}$ are chronologically (causally) related. Now we will state some of the basic properties of causal relations.

Proposition 2.1. *Let $p, q, r \in \mathcal{M}$;*
(i) If $p \leqslant q$ and $q \ll r$, then $p \ll r$.
(ii) If $p \ll q$ and $q \leqslant r$, then $p \ll r$.

Proof. See [16], Proposition 2.18. □

Definition 1. *Given any point p in a spacetime $\mathcal{M}$, the timelike (chronological) future and causal future of p, denoted $I^+(p)$ and $J^+(p)$, respectively are defined as $I^+(p) = \{q \in \mathcal{M} : p \ll q\}$ and $J^+(p) = \{q \in \mathcal{M} : p \leqslant q\}$. The timelike (chronological) past and causal past of p, denoted by $I^-(p)$ and $J^-(p)$, respectively are defined in a time-dual manner in terms of past directed timelike and causal curves. The chronological or causal future of any subset $\mathcal{S} \subset \mathcal{M}$ is* *defined by*

$$I^{+}(\mathcal{S})=\bigcup_{p\in\mathcal{S}}I^{+}(p),\qquad J^{+}(\mathcal{S})=\bigcup_{p\in\mathcal{S}}J^{+}(p),$$

respectively. $I^{-}(\mathcal{S})$ *and* $J^{-}(\mathcal{S})$ *are defined in a time-dual manner.*

It is known that for any $\mathcal{S}\subset\mathcal{M}$, $I^{+}(\mathcal{S})$ is always open. A number of results in this paper, require some of causality conditions such as follows.

A spacetime $\mathcal{M}$ is said to be strongly causal at p, if p has an arbitrarily small neighborhood U such that no causal curve intersects U in a disconnected set.A spacetime $\mathcal{M}$ is said to be strongly causal if strong causality holds at all p in $\mathcal{M}$. There is an interesting connection between strong causality and the so called Alexandrov topology. Since $I^{+}(p)$ is open, $I^{+}(p)\cap I^{-}(q)$ is open for any p and q in $\mathcal{M}$. The sets of the form $I^{+}(p)\cap I^{-}(q)$ define a basis for a topology on $\mathcal{M}$, which is called the Alexandrov topology of $\mathcal{M}$. This topology is in general more coarse than the manifold topology of $\mathcal{M}$. However It can be shown that the Alexandrov topology agrees with the given manifold topology if and only if the spacetime $\mathcal{M}$ is strongly causal .

There is a fundamental causality condition for a spacetime which is called the globally hyperbolicity and it is very important for us in this paper. Mathematically, global hyperbolicity plays a role analogous to geodesic completeness in Riemannian geometry, that any pair of causally related points can be joined by a causal geodesic with maximal length.

Definition 2. *A spacetime* $\mathcal{M}$ *is said to be globally hyperbolic provided* $\mathcal{M}$ *is strongly causal and the sets* $J^{+}(p)\cap J^{-}(q)$ *are compact for any* p *and* q *in* $\mathcal{M}$.

A hypersurface H in $\mathcal{M}$ is an embedded topological submanifold without boundary of codimension 1 in $\mathcal{M}$. We can regard H as a subset of $\mathcal{M}$ and, then, H will be closed if it is a closed subset of $\mathcal{M}$. A spacelike hypersurface is an embadded smooth hypersurface such that its tangent space at each point is spacelike. A Cauchy surface in $\mathcal{M}$ is a subset Σ that is met exactly once by every inextendible timelike curve in $\mathcal{M}$. Then, Σ will be a closed achronal connected topological hypersurface and it is intersected by any inextendible causal curve (see [8], Lemma 14.29). About Cauchy surfaces we state the following facts.

Proposition 2.2. *Let* Σ *be a Cauchy surface in spacetime* $\mathcal{M}$ *and let* γ *be an inextendible causal curve in* $\mathcal{M}$ *such that* $t_1<t_2$ *and* $\gamma(t_1),\gamma(t_2)\in\Sigma$ *for some real numbers* t_1,t_2. *Then, for each* $t_1<t<t_2$, $\gamma(t)\in\Sigma$.

Proof. Suppose $\gamma(t)\in\Sigma$ fails for some $t_1<t<t_2$. Then, $\gamma(t)\in I^{+}(\Sigma)$ or $\gamma(t)\in I^{-}(\Sigma)$. If $\gamma(t)\in I^{+}(\Sigma)$ then there exists $p\in\Sigma$ such that, $p\ll\gamma(t)$ and since $\gamma(t)\leqslant\gamma(t_2)$, by proposition 2.1 (i), we imply that $p\ll\gamma(t_2)$. This is a contradiction because Σ is achronal. If $\gamma(t)\in I^{-}(\Sigma)$ then there exists $q\in\Sigma$ such that, $\gamma(t)\ll q$ and since $\gamma(t_1)\leqslant\gamma(t)$, by proposition 2.1 (i), we imply

that $\gamma(t_1) \ll q$. This is a contradiction because Σ is achronal.
These contradictions seem from the assumption that $\gamma(t) \in \Sigma$ fails for some $t_1 < t < t_2$. Hence, $\gamma(t) \in \Sigma$ for each $t_1 < t < t_2$. □

In view of proposition 2.2, we note that the intersection of a Cauchy surface Σ with an inextendible causal curve in $\mathcal{M}$ may be a closed geodesic segment instead a single point.

Proposition 2.3. *Let Σ be a spacelike Cauchy surface in spacetime $\mathcal{M}$. Then, Σ is met exactly once by every inextendible causal curve in $\mathcal{M}$. In particular, Σ is a causal.*

Proof. Let γ be an inextendible causal curve in $\mathcal{M}$. Then, by Lemma 14.29 in [8], γ intersect Σ. Now, suppose for some real numbers t_1 and t_2, we have $t_1 < t_2$ and $\gamma(t_1), \gamma(t_2) \in \Sigma$. Then, by proposition 2.2, for each $t \in [t_1, t_2]$, $\gamma(t) \in \Sigma$. Since Σ is achronal, $\gamma|_{[t_1,t_2]}$ is a null curve segment. This is a contradiction because Σ is spacelike Cauchy surface. Therefore, Σ is met exactly once by γ and the proof is complete. □

It is known that a spacetime $\mathcal{M}$ is globally hyperbolic if and only if there exists an spacelike Cauchy surface Σ on $\mathcal{M}$ and then $\mathcal{M}$ is diffeomorphic to $\mathbb{R} \times \Sigma$, where Σ is a spacelike Cauchy surface in $\mathcal{M}$ [11]. Also, any two Cauchy surfaces in $\mathcal{M}$ are homeomorphic (see [8], Corollary 14.32). Furthermore, any two spacelike Cauchy surfaces in globally hyperbolic spacetime $\mathcal{M}$ are diffeomorphic (see [11], Lemma 2.2).

3. Causally admissible systems

Throughout this paper we assume that $\mathcal{M}$ is a globally hyperbolic spacetime with a non-compact, smooth, spacelike Cauchy surface Σ. Let C^+ and C^- be respectively the sets of all future and past causally admissible subsets of $\mathcal{M}$ with respect to Σ. That is

$$C^+ = \{S_p^+ = J^-(p) \cap \Sigma : p \in J^+(\Sigma)\}$$

and

$$C^- = \{S_p^- = J^+(p) \cap \Sigma : p \in J^-(\Sigma)\}$$

and they are called future and past admissible systems respectively. We note that S_p^+ and S_q^- are compact, connected subsets of Σ for each $p \in J^+(\Sigma)$ and each $q \in J^-(\Sigma)$. Let $C = (C^+, C^-)$. It is called causally admissible system on Σ.

Some important properties of the causally admissible subsets are the following (see [2]):

Theorem 3.1. *Let Σ is non-compact Cauchy surface of $\mathcal{M}$;*
(i) If $p, q \in J^+(\Sigma)$, then $p \leqslant q$ if and only if $S_p^+ \subseteq S_q^+$.

(ii) If $p, q \in J^-(\Sigma)$, then $p \leqslant q$ if and only if $S_p^- \supseteq S_q^-$.
(iii) if $p \in J^-(\Sigma)$ and $q \in J^+(\Sigma)$, then $p \leqslant q$ if and only if $S_p^- \cap S_q^+ \neq \phi$.

In the following proposition we review some known results about the causally admissible subsets.

Proposition 3.2. *For a spacetime $\mathcal{M}$ with a non-compact Cauchy surface Σ;*
(i) If $p, q \in J^+(\Sigma)$, then $S_p^+ = S_q^+$ if and only if $p = q$.
(ii) If $p, q \in J^-(\Sigma)$, then $S_p^- = S_q^-$ if and only if $p = q$.

Proof. See [4]. □

Some of the most important results in this paper, are about the causal or chronological isomorphisms between two spacetimes. Thus we are lead to introduce them as follows.

Definition 3. *A bijective function $f : \mathcal{M} \to \mathcal{M}'$ between two spacetimes is called a causal isomorphism if $p \leqslant q \Leftrightarrow f(p) \leqslant f(q)$ and a chronological isomorphism if $p \ll q \Leftrightarrow f(p) \ll f(q)$. If there exists a causal isomorphism (chronological isomorphism, resp.) between $\mathcal{M}$ and $\mathcal{M}'$ then we say that $\mathcal{M}$ and $\mathcal{M}'$ are causally isomorphic (chronologically isomorphic, resp.).*

In the following we will state some results about the causal isomorphisms which can be found in [13], [14] and [15].

Theorem 3.3. *For a bijection $f : \mathcal{M} \to \mathcal{M}'$ between two chronological spacetimes, we have the following properties.*
(i) f is a causal isomorphism if and only if f is a chronological isomorphism.
(ii) If f is a causal isomorphism, then f is a smooth conformal diffeomorphism.

Suppose that $\mathcal{M}$ and $\mathcal{M}'$ are globally hyperbolic spacetimes with non-compact Cauchy surfaces Σ and Σ' , respectively. Let C^+ and C'^+ be the corresponding future admissible systems for Σ and Σ' respectively, and we denote these by (Σ, C^+) and (Σ', C'^+). Then, since the causal relation is encoded into C through the relation of inclusion, it is not difficult to see the following theorem.

Theorem 3.4. *Two spacetimes $\mathcal{M}$ and $\mathcal{M}'$ with non-compact Cauchy surfaces are causally isomorphic if and only if there exists a causally admissible function $f : (\Sigma, C) \to (\Sigma', C')$ between the corresponding causally admissible systems.*

Proof. See [4], Theorem 5.4. □

Since $\mathbb{R}_1^2$ is globally hyperbolic with the non-compact Cauchy surface, we can apply the theory of a causally admissible system to analyze causal automorphisms on $\mathbb{R}_1^2$. This is the main tool for our goal in this paper.

In the next section, we use the following theorem to assert our main results.

Theorem 3.5. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2_1$ given by $t \to (f(t), g(t))$ be an injective, continuous curve in $\mathbb{R}^2_1$. Then, $\gamma(\mathbb{R})$ is an acausal Cauchy surface if and only if f is a homeomorphism, $sup(g \pm f) = \infty$, $inf(g \pm f) = -\infty$ and $|\frac{g(t+\delta t)-g(t)}{f(t+\delta t)-f(t)}| < 1$ for all t and $\delta t \neq 0$.*

Proof. See [3], Theorem 4.3. □

Example 3.6. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2_1$ given by $t \to (f(t), g(t))$ where $f(t) = t$ and $g(t) = \alpha \cos t$ for all $t \in \mathbb{R}$ and $0 < \alpha < 1$. Then, the curve γ is an injective and continuous curve in $\mathbb{R}^2_1$ and the component functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ of the curve γ have the following properties. The component function f is a homeomorphism, $sup(g \pm f) = \infty$ and $inf(g \pm f) = -\infty$. By using the mean value theorem of calculus we have*

$$|\frac{g(t+\delta t)-g(t)}{f(t+\delta t)-f(t)}| < 1$$

for all t and $\delta t \neq 0$. In view of Theorem 3.5, $\gamma(\mathbb{R})$ (the graph of the cosine function) is an acausal Cauchy surface of $\mathbb{R}^2_1$.

4. Main Results

In this section we assume that Σ is a non-compact spacelike Cauchy surface of two-dimensional Minkowski spacetime $\mathbb{R}^2_1$. Also, we assume that $\mathcal{A}$ is the set of all compact and connected subsets of Σ. We use y as the time coordinate of $\mathbb{R}^2_1$ and we suppose that the future direction on $\mathbb{R}^2_1$ is the positive direction on y axis. For all $r \in \mathbb{R}^2_1$ and $m \in \mathbb{R}$ we set $\ell_{r,m}$ as the line that passes through the point r and has slope m.

Proposition 4.1. *Let Σ be a non-compact spacelike Cauchy surface of two-dimensional Minkowski spacetime $\mathbb{R}^2_1$. Then, there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $\Sigma = \{(t, f(t)) : t \in \mathbb{R}\}$.*

Proof. Let $\pi_2 : \mathbb{R}^2_1 \to \mathbb{R}$ be the projection map defined by $\pi_2(x, y) = y$ and let for each $t \in \mathbb{R}$, $\gamma_t : \mathbb{R} \to \mathbb{R}^2_1$ be the timelike curve in $\mathbb{R}^2_1$ defined by $\gamma_t(y) = (t, y)$. We can define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(t) = \pi_2(\gamma_t(\mathbb{R}) \cap \Sigma)$ the function f is well-defined, because Σ is Cauchy surface and $\gamma_t(\mathbb{R})$ is the graph of the timelike curve γ_t and γ_t intersects Σ exactly once. In the following, we will prove that f is continuous and $\Sigma = \{(t, f(t)) : t \in \mathbb{R}\}$. Let $\mathbb{R}_{y_0} = \{(x, y) \in \mathbb{R}^2_1 : y = y_0\}$ (we know that $\mathbb{R}_{y_0}$ is a Cauchy surface of $\mathbb{R}^2_1$). By Corollary 14.32 in [8], there exists a homeomorphism $F : \mathbb{R}_{y_0} \to \Sigma$ given by $(x, y_0) \to (g(x, y_0), h(x, y_0))$. We know that the map $\iota : \mathbb{R} \to \mathbb{R}_{y_0}$ defined by $\iota(x) = (x, y_0)$ is a homeomorphism and it implies that the map $Fo\iota : \mathbb{R} \to \Sigma$ defined by $Fo\iota(x) = (go\iota(x), ho\iota(x))$ is a homeomorphism and $Fo\iota(\mathbb{R}) = \Sigma$. Then, by theorem 3.5, the function $go\iota : \mathbb{R} \to \mathbb{R}$ is a homeomorphism and the map $(Fo\iota)o(go\iota)^{-1} : \mathbb{R} \to \Sigma$ is a homeomorohism such that $(Fo\iota)o(go\iota)^{-1}(t) = (t, (ho\iota)o(go\iota)^{-1}(t))$. Therefore, the function $(ho\iota)o(go\iota)^{-1} : \mathbb{R} \to \mathbb{R}$ is continuous. We note that for all $t \in \mathbb{R}$

the points $(t, (ho\iota)o(go\iota)^{-1}(t))$ and $(t, f(t))$ are on the Cauchy surface Σ and they are also on the timelike curve γ_t. Since Σ is a Cauchy surface, we must have $f(t) = (ho\iota)o(go\iota)^{-1}(t)$ and it means that $f = (ho\iota)o(go\iota)^{-1}$. This prove that the function f is continuous and

$$\Sigma = (Fo\iota)o(go\iota)^{-1}(\mathbb{R}) = \{(t, (ho\iota)o(go\iota)^{-1}(t)) : t \in \mathbb{R}\} = \{(t, f(t)) : t \in \mathbb{R}\}.$$

□

Recall that on a spacetime $\mathcal{M}$ the causal relation " $\leqslant$ " has been defined as follows:
For all $p, q \in \mathcal{M}$, $p \leqslant q$ if and only if there exists a future directed causal curve from p to q or $p = q$. In the two-dimensional Minkowski spacetime $\mathbb{R}^2_1$, we can state the causal relation " $\leqslant$ " as the following proposition.

Proposition 4.2. *Let $(a, b), (x, y) \in \mathbb{R}^2_1$. Then, $(x, y) \in J^+(a, b)$ if and only if (x, y) satisfies in one of the following conditions*
(i) $(x, y) = (a, b)$,
(ii) $b < y$ and $|x - a| \leqslant y - b$.

Proof. If $(x, y) = (a, b)$, then $(x, y) \in J^+(a, b)$ (by definition of $J^+(a, b)$). Let us assume that $(x, y) \neq (a, b)$. By the future direction on $\mathbb{R}^2_1$ we know that if $b > y$ then $(x, y) \notin J^+(a, b)$. Let $x \neq a$ and $b < y$. Define $f : \mathbb{R} - \{a\} \to \mathbb{R}$ by $f(t) = \frac{b-y}{a-t}$, where $f(t)$ equals to the slope of the line segment from (a, b) to (t, y). We know that f is strictly decreasing on its domain. Since $\mathbb{R}^2_1$ is globally hyperbolic, $(t, y) \in J^+(a, b)$ if and only if the line segment from (a, b) to (t, y) is a causal curve and by definition of f this is equivalent to say that $|f(t)| \geqslant 1$. Since f is strictly decreasing, $f(a + b - y) = -1$, and $f(a - b + y) = 1$ then

$$-\infty = \lim_{t \to a^-} f(t) < f(t) \leqslant -1 \Leftrightarrow a - (y - b) \leqslant t < a \Leftrightarrow -(y - b) \leqslant t - a < 0$$

and

$$1 \leqslant f(t) < \infty = \lim_{t \to a^+} f(t) \Leftrightarrow a < t \leqslant a + (y - b) \Leftrightarrow 0 < t - a < y - b \text{ (see Figure 1).}$$

Therefore, if $x \neq a$ and $b < y$ we have $(x, y) \in J^+(a, b)$ if and only if $0 < |x - a| \leqslant y - b$
If $x = a$ and $b < y$, then $\gamma : (b, y) \to \mathbb{R}^2_1$ given by $\gamma(t) = (a, t)$ is a future directed timelike curve from (a, b) to (x, y), then $(x, y) \in J^+(a, b)$.
Hence, we prove that if $b < y$, then $(x, y) \in J^+(a, b)$ if and only if $|x - a| \leqslant y - b$. This complete the proof. □

Proposition 4.2 has a time dual as the following proposition and we can prove it by a similar approach as in proposition 4.2 (see Figure 2).

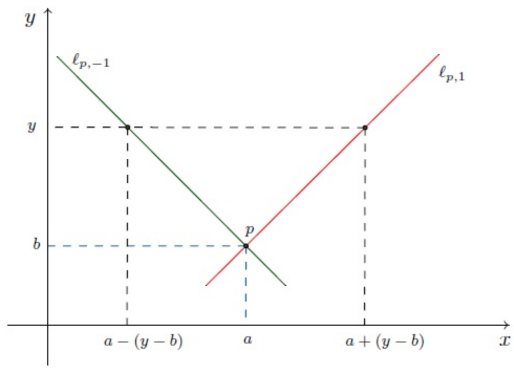

FIGURE 1. $(x, y) \in J^+(a, b) \Leftrightarrow (x, y) = (a, b)$ or $b > y$ and $|x - a| \leqslant y - b$

Proposition 4.3. *Let $(a, b), (x, y) \in \mathbb{R}^2_1$. Then, $(x, y) \in J^-(a, b)$ if and only if (x, y) satisfies in one of the following conditions;*
(i) $(x, y) = (a, b)$.
(ii) $b > y$ and $|x - a| \leqslant b - y$.

Remark 4.4. *Applying proposition 4.1, we can find the continuous function $f : \mathbb{R} \to \mathbb{R}$ such that the map $F : \mathbb{R} \to \Sigma$ defined by $F(t) = (t, f(t))$ is a homeomorphism. Therefore,*

$$\Sigma = F(\mathbb{R}) = \{(t, f(t)) : t \in \mathbb{R}\}.$$

Let $A \in \mathcal{A}$ (recall that $\mathcal{A}$ is the set of all compact and connected subsets of Σ). Since F is a homeomorphism, $F^{-1}(A)$ is a compact and connected subsets of $\mathbb{R}$ and there exist $a, b \in \mathbb{R}$ such that $F^{-1}(A) = [a, b]$. Therefore, $A = \{(t, f(t)) : a \leqslant t \leqslant b\}$.
Now, set $p = (b, f(b))$ and $q = (a, f(a))$. Let us consider the lines $\ell_{p,1}$, $\ell_{p,-1}$ and $\ell_{q,1}$, $\ell_{q,-1}$ as follows,

$$\ell_{p,1} : y = x - b + f(b),$$

$$\ell_{p,-1} : y = -x + b + f(b),$$

$$\ell_{q,1} : y = x - a + f(a),$$

$$\ell_{q,-1} : y = -x + a + f(a).$$

We know that $\ell_{p,1}$ is perpendicular to $\ell_{q,-1}$ and $\ell_{p,-1}$ is perpendicular to $\ell_{q,1}$ are perpendicular. We can find the points of their intersections by solving the following systems of linear equations.

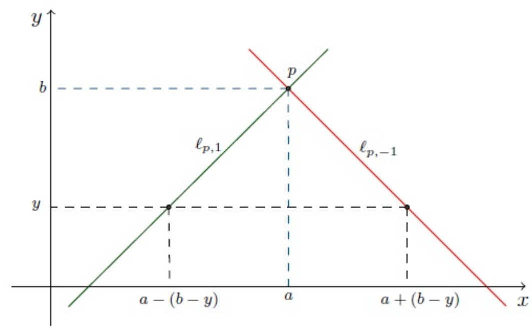

FIGURE 2. $(x,y) \in J^-(a,b) \Leftrightarrow (x,y) = (a,b)$ or $b > y$ and $|x-a| \leqslant b-y$

$$\begin{cases} y = x - b + f(b) \\ y = -x + a + f(a) \end{cases} \tag{4.1}$$

$$\begin{cases} y = -x + b + f(b) \\ y = x - a + f(a) \end{cases} \tag{4.2}$$

We set R and L as the solution of the system of linear equations (4.1) and (4.2), respectively (see Figure 3). It is easy to see that

$$R = \Big(\frac{a+b}{2} - \frac{f(b)-f(a)}{2}, -\frac{b-a}{2} + \frac{f(a)+f(b)}{2}\Big)$$

and

$$L = \Big(\frac{a+b}{2} + \frac{f(b)-f(a)}{2}, \frac{b-a}{2} + \frac{f(a)+f(b)}{2}\Big).$$

There are some interesting properties between the points L, R, p and q, where we state them as follows.

Proposition 4.5. *$L \in J^+(p) \cap J^+(q)$ and $R \in J^-(p) \cap J^-(q)$.*

Proof. We know that the points $p = (b, f(b))$ and $q = (a, f(a))$ are on the spacelike Cauchy surface Σ (see Figure 3).

Step 1: In this step we want to show that $L \in J^+(p)$. Using Theorem 3.5, we have

$$\frac{f(b)-f(a)}{b-a} < 1 \Rightarrow \frac{f(a)-f(b)}{b-a} > -1 \Rightarrow 1 + \frac{f(a)-f(b)}{b-a} > 0$$

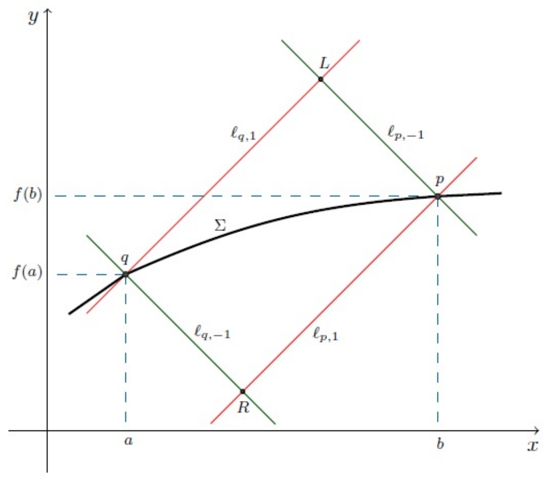

FIGURE 3. Spacelike Cauchy surface Σ

and this yields that

$$b-a+f(a)-f(b)>0 \Rightarrow b-a+f(b)+f(a)>2f(b).$$

Thus,

$$\frac{b-a}{2}+\frac{f(b)+f(a)}{2}>f(b). \tag{4.3}$$

Since the line $\ell_{p,-1}$ passes through the point L, we deduce that the coordinate of the point L satisfies in the equation of the line $\ell_{p,-1}$ and we have

$$f(b)-\big(\frac{a+b}{2}+\frac{f(b)-f(a)}{2}\big)=\big(\frac{b-a}{2}+\frac{f(b)+f(a)}{2}\big)-b.$$

Then,

$$\Big|\big(\frac{a+b}{2}+\frac{f(b)-f(a)}{2}\big)-f(b)\Big|=\big(\frac{b-a}{2}+\frac{f(b)+f(a)}{2}\big)-b.$$

This yeilds that,

$$\Big|\big(\frac{a+b}{2}+\frac{f(b)-f(a)}{2}\big)-f(b)\Big|\leqslant\big(\frac{b-a}{2}+\frac{f(b)+f(a)}{2}\big)-b. \tag{4.4}$$

In view of Proposition 4.2 and inequalities 4.3 and 4.4, we see that $L\in J^+(p)$.

Step 2: In this step we will prove that $L\in J^+(q)$. Using Theorem 3.5, we have

$$\frac{f(b)-f(a)}{b-a}>-1 \Rightarrow 1+\frac{f(b)-f(a)}{b-a}>0$$

and this yields that

$$b-a+f(b)-f(a)>0 \Rightarrow b-a+f(b)+f(a)>2f(a).$$

Thus,

$$\frac{b-a}{2} + \frac{f(b)+f(a)}{2} > f(a). \tag{4.5}$$

Since the line $\ell_{q,1}$ passes through the point L, we deduce that the coordinate of the point L satisfies in the equation of the line $\ell_{q,1}$ and we have

$$\big(\frac{a+b}{2} + \frac{f(b)-f(a)}{2}\big) - a = \big(\frac{b-a}{2} + \frac{f(b)+f(a)}{2}\big) - f(a).$$

Then,

$$\Big|\big(\frac{a+b}{2} + \frac{f(b)-f(a)}{2}\big) - a\Big| = \big(\frac{b-a}{2} + \frac{f(b)+f(a)}{2}\big) - f(a).$$

This yields that,

$$\Big|\big(\frac{a+b}{2} + \frac{f(b)-f(a)}{2}\big) - a\Big| \leqslant \big(\frac{b-a}{2} + \frac{f(b)+f(a)}{2}\big) - f(a). \tag{4.6}$$

In view of Proposition 4.2 and inequalities (4.5) and (4.6), we see that $L \in J^+(q)$. Therefore, by claims of Step 1 and Step 2 we can conclude that $L \in J^+(p) \cap J^+(q)$.

Step 3: In this step we will show that $R \in J^-(p)$. Using Theorem 3.5, we have

$$\frac{f(b)-f(a)}{b-a} > -1 \Rightarrow f(b)-f(a) > -(b-a)$$

and this yields that

$$a-b+f(a)-f(b) < 0 \Rightarrow a-b+f(a)+f(b) < 2f(b).$$

Thus,

$$\big(-\frac{b-a}{2} + \frac{f(b)+f(a)}{2} < f(b)\big). \tag{4.7}$$

Since the line $\ell_{p,1}$ passes through the point R, we deduce that the coordinate of the point R satisfies in the equation of the line $\ell_{p,1}$ and we have

$$b - \big(\frac{a+b}{2} - \frac{f(b)-f(a)}{2}\big) = f(b) - \big(-\frac{b-a}{2} + \frac{f(a)+f(b)}{2}\big).$$

Then,

$$\Big|b - \big(\frac{a+b}{2} - \frac{f(b)-f(a)}{2}\big)\Big| = f(b) - \big(-\frac{b-a}{2} + \frac{f(a)+f(b)}{2}\big).$$

This yields that,

$$\Big|\big(\frac{a+b}{2}-\frac{f(b)-f(a)}{2}\big)-b\Big| \leqslant f(b)-\big(-\frac{b-a}{2}+\frac{f(a)+f(b)}{2}\big). \tag{4.8}$$

In view of Proposition 4.3 and inequalities (4.7) and (4.8), we see that $R \in J^-(p)$.

Step 4: In this step we want to prove that $R \in J^-(q)$. Using Theorem 3.5, we have

$$\frac{f(b)-f(a)}{b-a} < 1 \Rightarrow -1+\frac{f(b)-f(a)}{b-a} < 0$$

and this yields that

$$a-b+f(b)-f(a) < 0 \Rightarrow a-b+f(a)+f(b) < 2f(a).$$

Thus,

$$\big(-\frac{b-a}{2}+\frac{f(a)+f(b)}{2}\big) < f(a) \tag{4.9}$$

Since the line $\ell_{q,-1}$ passes through the point R, we deduce that the coordinate of the point R satisfies in the equation of the line $\ell_{q,-1}$ and we have

$$\big(\frac{a+b}{2}-\frac{f(b)-f(a)}{2}\big)-a = f(a)-\big(-\frac{b-a}{2}+\frac{f(a)+f(b)}{2}\big).$$

Then,

$$\Big|\big(\frac{a+b}{2}-\frac{f(b)-f(a)}{2}\big)-a\Big| = f(a)-\big(-\frac{b-a}{2}+\frac{f(a)+f(b)}{2}\big).$$

This yeilds that,

$$\Big|\big(\frac{a+b}{2}-\frac{f(b)-f(a)}{2}\big)-a\Big| \leqslant f(a)-\big(-\frac{b-a}{2}+\frac{f(a)+f(b)}{2}\big). \tag{4.10}$$

In view of Proposition 4.3 and inequalities (4.9) and (4.10), we see that $R \in J^-(q)$. By using the results of Step 3 and Step 4, we have $R \in J^-(p) \cap J^-(q)$.

These complete the proof. □

Corollary 4.6. *$L \in I^+(\Sigma)$ and $R \in I^-(\Sigma)$.*

Proof. Applying Proposition 4.5, we see that $L \in J^+(p)$. Since $p \in \Sigma$, we deduce that $L \in J^+(\Sigma) = \Sigma \cup I^+(\Sigma)$. The inextendible future directed causal curve $\gamma : \mathbb{R} \to \mathbb{R}^2_1$ defined by $\gamma(t) = (t, -t+b+f(b))$, passes through the points p and L. In view of Proposition 2.3, we infer that L is not a member of Σ and we conclude $L \in I^+(\Sigma)$.

Employing a similar approach as above, we see that $R \in I^-(\Sigma)$. □

Corollary 4.7. $L \in I^+(R)$.

Proof. We know that

$$R = \Big(\frac{a+b}{2} - \frac{f(b)-f(a)}{2}, -\frac{b-a}{2} + \frac{f(a)+f(b)}{2}\Big),$$

and

$$L = \Big(\frac{a+b}{2} + \frac{f(b)-f(a)}{2}, \frac{b-a}{2} + \frac{f(a)+f(b)}{2}\Big).$$

The slope of the line passes through the points R and L is

$$m = \frac{(\frac{b-a}{2} + \frac{f(a)+f(b)}{2}) - (-\frac{b-a}{2} + \frac{f(a)+f(b)}{2})}{(\frac{a+b}{2} + \frac{f(b)-f(a)}{2}) - (\frac{a+b}{2} - \frac{f(b)-f(a)}{2})} = \frac{b-a}{f(b)-f(a)}$$

Moreover, we know that the points $p = (b, f(b))$ and $q = (a, f(a))$ are on the spacelike Cauchy surface Σ. Using Theorem 3.5, we see that

$$\Big|\frac{f(b)-f(a)}{b-a}\Big| < 1 \Rightarrow \Big|\frac{b-a}{f(b)-f(a)}\Big| > 1,$$

This means that the line segment passes through the points R and L in the globally hyperbolic spacetime $\mathbb{R}^2_1$ is timelike, and then we conclude that $L \in I^+(R)$. □

Proposition 4.8. *Let f, a, L and R be those notions which have been stated in Remark 4.4. If $t < a$ then $(t, f(t)) \notin J^-(L)$ and $(t, f(t)) \notin J^+(R)$.*

Proof. At first we show that $(t, f(t)) \notin J^-(L)$. If $(t, f(t)) = L$, then the future directed causal curve $\gamma : \mathbb{R} \to \mathbb{R}^2_1$ defined by $\gamma(s) = -s + a + f(a)$ intersects spacelike Cauchy surface Σ in two distinct points $q = (a, f(a))$ and $(t, f(t))$. In view of Proposition 2.3, it is a contradiction. It yields that $(t, f(t)) \neq L$.
We know that one and only one of the following statements is true,

(i) $f(t) = \dfrac{b-a}{2} + \dfrac{f(b)+f(a)}{2}$,

(ii) $f(t) > \dfrac{b-a}{2} + \dfrac{f(b)+f(a)}{2}$,

(iii) $f(t) < \dfrac{b-a}{2} + \dfrac{f(b)+f(a)}{2}$.

Let (i) be true. Since $(t, f(t)) \neq L$, we must have $t \neq \dfrac{a+b}{2} + \dfrac{f(b)+f(a)}{2}$. Using Proposition 4.3, one can observe $(t, f(t)) \notin J^{-}(L)$.
Let (ii) be true. In view of Proposition 4.3, we see $(t, f(t)) \notin J^{-}(L)$.
Ultimately, let (iii) be true. Applying Theorem 3.5, we have

$$\frac{f(b)-f(t)}{b-t} > -1 \Rightarrow \frac{f(b)-f(t)}{t-b} < 1 \Rightarrow t-b > f(b)-f(t).$$

It yields that

$$t - \frac{a}{2} - \frac{b}{2} - \frac{f(b)}{2} + \frac{f(a)}{2} > \frac{b}{2} - \frac{a}{2} + \frac{f(b)}{2} + \frac{f(a)}{2} - f(t).$$

Then,

$$t - \left(\frac{a+b}{2} + \frac{f(b)-f(a)}{2}\right) > \left(\frac{b-a}{2} + \frac{f(b)+f(a)}{2}\right) - f(t).$$

By Proposition 4.3, we have $(t, f(t)) \notin J^{-}(L)$ and the proof in this case is complete. Now it is enough to show that $(t, f(t)) \notin J^{+}(R)$. If $(t, f(t)) = R$ then the future directed causal curve $\alpha : \mathbb{R} \to \mathbb{R}^2_1$ defined by $\alpha(s) = s - a + f(a)$ intersects spacelike Cauchy surface Σ in two distinct points $q = (a, f(a))$ and $(t, f(t))$. In view of Proposition 2.3, it is a contradiction. It yields that $(t, f(t)) \neq R$.
We know that one and only one of the following statements is true,

(iv) $f(t) = -\dfrac{b-a}{2} + \dfrac{f(b)+f(a)}{2}$,

(v) $f(t) > -\dfrac{b-a}{2} + \dfrac{f(b)+f(a)}{2}$,

(vi) $f(t) < -\dfrac{b-a}{2} + \dfrac{f(b)+f(a)}{2}$.

Let (iv) be true. Since $(t, f(t)) \neq R$, we must have $t \neq \dfrac{a+b}{2} - \dfrac{f(b)-f(a)}{2}$. Using Proposition 4.2, one can observe $(t, f(t)) \notin J^{+}(R)$.
Let (v) be true. In view of Proposition 4.2, we see $(t, f(t)) \notin J^{+}(R)$.
Ultimately, let (vi) be true. Applying Theorem 3.5, we have

$$\frac{f(a)-f(t)}{a-t} > -1 \Rightarrow \frac{f(a)-f(t)}{t-a} < 1 \Rightarrow t-a < f(a)-f(t).$$

It yields that

$$t - \frac{a}{2} - \frac{b}{2} + \frac{f(b)}{2} - \frac{f(a)}{2} < -f(t) - \frac{b}{2} + \frac{a}{2} + \frac{f(a)}{2} + \frac{f(b)}{2}.$$

Then,

$$t - \Big(\frac{a+b}{2} - \frac{f(b)-f(a)}{2}\Big) < -\Big(f(t) - \big(-\frac{b-a}{2} + \frac{f(b)+f(a)}{2}\big)\Big).$$

By proposition 4.2, we have $(t, f(t)) \notin J^-(L)$ and this complete the proof. □

Applying a similar approach as the proof of proposition 4.8, we can prove the following proposition.

Proposition 4.9. *Let f, b, L and R be those notions which have been stated in Remark 4.4. If $b < t$ then $(t, f(t)) \notin J^-(L)$ and $(t, f(t)) \notin J^+(R)$.*

Proposition 4.10. *Let f, a, b, L and R be those notions which have been stated in Remark 4.4. If $a \leqslant t \leqslant b$ then $(t, f(t)) \in J^-(L)$ and $(t, f(t)) \in J^+(R)$.*

Proof. For showing $(t, f(t)) \in J^-(L)$ by Proposition 4.3 we will prove the following inequqlities

$$f(t) < \frac{b-a}{2} + \frac{f(b)+f(a)}{2},$$

$$\Big|t - \Big(\frac{a+b}{2} + \frac{f(b)-f(a)}{2}\Big)\Big| < \Big(\frac{b-a}{2} + \frac{f(b)+f(a)}{2}\Big) - f(t).$$

Applying Theorem 3.5, we have

$$\frac{f(b)-f(t)}{b-t} > -1 \Rightarrow \frac{f(t)-f(b)}{b-t} < 1.$$

It yields that
$$\frac{f(t)-f(b)}{(b-t)(t-a)} < \frac{1}{t-a} = \frac{b-t}{(b-t)(t-a)}. \tag{4.11}$$

On the other hand in view of Theorem 3.5, we have

$$\frac{f(t)-f(a)}{t-a} < 1$$

Therefore,
$$\frac{f(t)-f(a)}{(t-a)(b-t)} < \frac{1}{b-t} = \frac{t-a}{(b-t)(t-a)}. \tag{4.12}$$

Using (4.11) and (4.12), we obtain that

$$\frac{f(t)-f(b)}{(b-t)(t-a)}+\frac{f(t)-f(a)}{(t-a)(b-t)}<\frac{b-t}{(b-t)(t-a)}+\frac{t-a}{(b-t)(t-a)}.$$

It yields that

$$f(t)-f(b)+f(t)-f(a)<b-t+t+a\Rightarrow 2f(t)<b-a+f(b)+f(a).$$

Then, we conclude

$$f(t)<\frac{b-a}{2}+\frac{f(b)+f(a)}{2}.$$

We apply Theorem 3.5 again and we see that

$$\frac{f(b)-f(t)}{b-t}>-1\Rightarrow\frac{f(b)-f(t)}{t-b}<1\Rightarrow t-b<f(b)-f(t).$$

It implies that

$$2t-2b<2f(b)-2f(t)\Rightarrow 2t-a-b-f(b)+f(a)<b-a+f(b)+f(a)-2f(t).$$

Then, we observe that

$$t\big(-\big(\frac{a+b}{2}+\frac{f(b)-f(a)}{2}\big)<\big(\frac{b-a}{2}+\frac{f(b)+f(a)}{2}\big)-f(t)\big). \tag{4.13}$$

Using Theorem 3.5 one more time, we have

$$\frac{f(t)-f(a)}{t-a}<1\Rightarrow f(t)-f(a)<t-a.$$

Thus, we infer that

$$2f(t)-2f(a)<2t-2a\Rightarrow 2f(t)-b+a-f(b)-f(a)<2t-a-b-f(b)+f(a).$$

Therefore,

$$\Big(-\Big(\big(\frac{b-a}{2}+\frac{f(b)+f(a)}{2}\big)-f(t)\Big)<t\big(-\big(\frac{a+b}{2}+\frac{f(b)-f(a)}{2}\big)\big). \quad (4.14)$$

Using (4.13) and (4.14), one can observe that

$$\Big|t-\big(\frac{a+b}{2}+\frac{f(b)-f(a)}{2}\big)\Big|<\big(\frac{b-a}{2}+\frac{f(b)+f(a)}{2}\big)-f(t).$$

Hence, we prove that $(t, f(t)) \in J^-(L)$.

In view of Proposition 4.2 and a similar approach as above, one can prove $(t, f(t)) \in J^+(R)$. □

Theorem 4.11. *Let A, L and R be those notions which have been stated in Remark 4.4. Then $S_L^+ = A$ and $S_R^- = A$.*

Proof. Let $r \in S_L^+ = J^-(L) \cap \Sigma$. Applying Proposition 4.1, there exists $t \in \mathbb{R}$ such that $r = (t, f(t))$. Since $r = (t, f(t)) \in J^-(L)$, we must have $a \leqslant t \leqslant b$ because if $t < a$ or $t > b$ then in view of proposition 4.8 and Propostion 4.9, we have $(t, f(t)) \notin J^-(L)$. Then, we see that $r = (t, f(t)) \in A$. It means that $S_L^+ \subset A$.

Now, let $(t, f(t)) \in A$. Using Proposition 4.10, we obtain $(t, f(t)) \in J^-(L)$. Since $A \subset \Sigma$, we have $(t, f(t)) \in J^-(L) \cap \Sigma = S_L^+$. Then, we infer that $A \subset S_L^+$. Therefore, $S_L^+ = A$.
Applying a similar approach as above, we see that $S_R^- = A$. □

We know that every future or past causally admissible subset of Σ is compact and connected. By the above theorem we can show that every compact and connected subset A of Σ is a future or past causally admissible subset. Therefore, we have the following corollary.

Corollary 4.12. *Let Σ be anon-compact spacelike Cuachy surface on $\mathbb{R}_1^2$ and let C^+, C^- and C be respectively the future admissible, past admissible and admissible system on Σ. Then $C^+ = C^- = C$.*

Theorem 4.13. *Let C be causally admissible system on Σ and let $\mathcal{A}$ be the set of all compact and connected subsets of Σ. Then, $C = \mathcal{A}$.*

Proof. For each $p \in J^+(\Sigma)$ and $q \in J^-(\Sigma)$, we know that the sets S_p^+ and S_q^- are compact and connected subsets of Σ. Therefore, $C \subset \mathcal{A}$.
Let A be a compact and connected subset of Σ. In view of Proposition 4.11, there are the points L and R such that $S_L^+ = A$ and $S_R^- = A$. It means that A is a future causal set and a past causal set of two-dimensional Minkowski spacetime $\mathbb{R}_1^2$, respectively. Therefore, we have $A \in C$. It yields that $\mathcal{A} \subset C$. Hence, we prove that $C = \mathcal{A}$. □

Theorem 4.14. *Let Σ and $\Sigma^{'}$ be two non-compact spacelike Cauchy surfaces of two-dimensional Minkowski spacetime $\mathbb{R}_1^2$ and let $f : \Sigma \to \Sigma^{'}$ be a bijaction. Then the following statements are equivalent,*
(i) f is a future causally admissible function,
(ii) f is a past causally admissible function,
(iii) f is a causally admissible function,
(iv) f is a homeomorphism.

Proof. By 4.12 we can see that (i), (ii) and (iii) are equivalent. The proof of (iii)⇒(iv) has been obtained by Theorem 3.4, which is say that every causally admissible function $f : \Sigma \to \Sigma^{'}$ can be extend to a causally isomorphism between their manifolds such as $\tilde{f} : \mathbb{R}_1^2 \to \mathbb{R}_1^2$. Therefore, $f = \tilde{f}|_{\Sigma} : \Sigma \to \Sigma^{'}$ is a homeomorphism. Now suppose that $f : \Sigma \to \Sigma^{'}$ is a homeomorphism then, for every $S \in C$, $f(S)$ is a compact connected subset of $\Sigma^{'}$. So by Theorem 4.13 we have $f(A) \in C^{'}$ and it is show that (iv)⇒(iii). □

Remark 4.15. *By Theorem 4.14 and Theorem 3.4, every homeomorphism between two non-compact Cuachy surfaces Σ, $\Sigma^{'}$ of $\mathbb{R}_1^2$, determines a causal isomorphism of $\mathbb{R}_1^2$ to itself.*

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