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On a family of Einstein like Walker metrics

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Abstract. A four dimensional pseudo-Riemannian manifold of signature $(2, 2)$ is called a Walker manifold if it admits a parallel degenerate plane field. Einstein like metrics are generalizations of Einstein metrics. In this paper, we study the curvature properties of a family of four dimensional Walker manifolds. We give conditions on the coefficients of the metric so that the Ricci tensor of the metric is parallel, cyclic parallel and Codazzi respectively.

Keywords: Einstein manifold, Einstein like manifold, Walker metric.

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1. **Introduction**

A pseudo-Riemannian metric g on a four dimensional manifold M is said to be a Walker metric if there exists a two dimensional null distribution on M, which is parallel with respect to the Levi-Civita connection of g. This type of metrics has been introduced by Walker [12] who has shown that they have a local canonical form depending on three smooth functions. Various curvature properties of some special classes of Walker metrics have been studied in [3] where several examples of neutral metrics with interesting geometric properties have been given. Conditions for a restricted four dimensional Walker manifold to be Einstein, locally symmetric, Einstein and locally conformally flat are given in [5]. Examples of Walker Osserman metrics of signature $(3,3)$ which admits a field of parallel null 3-planes are given in [7, 8]. A lot of examples of Walker structures have appeared, which proved to be important in differential geometry and general relativity as well [6, 11].

Two of the most extensively studied objects in Riemannian geometry and physics are Einstein manifolds and the Riemannian manifolds with constant scalar curvature. We denote by $\mathcal{E}$ the class of Einstein manifolds, by $\mathcal{C}$ the class of Riemannian manifolds with constant scalar curvature and by $\mathcal{P}$ the class of manifolds with parallel Ricci tensor. We have the following inclusion:

$$\mathcal{E} \subset \mathcal{P} \subset \mathcal{C}.$$

Gray [10] considered the space of covariant derivatives of Ricci tensors to introduce two interesting classes of Riemannian manifolds which generalizes the concept of Einstein manifolds. The class of Riemannian manifolds admitting a cyclic parallel Ricci tensor denoted by $\mathcal{A}$ and the class of Riemannian manifolds with Codazzi Ricci tensor denote by $\mathcal{B}$. These two classes of manifolds, consisting on those with cyclic parallel Ricci tensor and Codazzi Ricci tensor have been extensively studied in the Riemannian and affine setting. Note that, the classes of Riemannian manifolds $\mathcal{A}$ and $\mathcal{B}$ lie between the class of Riemannian manifolds with parallel Ricci tensor $\mathcal{P}$ and the class of Riemannian manifolds with constant scalar curvature $\mathcal{C}$.

Calvariso [4] classify three-dimensional homogeneous Lorentzian manifolds, equipped with Einstein-like metrics. Batat et al. [1] study and characterize a class of four-dimensional Walker metrics to being Einstein-like, conformally flat, locally symmetric. Motivated, by the following papers [1, 4], we consider the family of Walker metrics g_a on $\mathcal{O} \subset \mathbb{R}^4$ given by

$$\begin{aligned} g_a = 2(dx_1 \circ dx_3 + dx_2 \circ dx_4) + a(x_1, x_2, x_3, x_4)dx_3 \circ dx_3 \\ +a(x_1, x_2, x_3, x_4)dx_4 \circ dx_4, \end{aligned} \tag{1.1}$$

where a is a fonction depending on (x_1, x_2, x_3, x_4). We shall characterize Walker metrics (1.1) which are parallel Ricci tensor, cyclic parallel Ricci tensor and the Codazzi Ricci tensor. We organized the paper as follow : in section 2, we shall describe the curvature of the metric considered. In section 3, we will give conditions on the coefficients of the metric (1.1) so that the Ricci tensor to being parallel, cyclic parallel and Codazzi respectively.

2. Description of the Metric

For next, we denote by

$$\partial_i := \frac{\partial}{\partial x_i}, \quad a_i := \frac{\partial a(x_1, x_2, x_3, x_3)}{\partial x_i}.$$

A straightforward calculation shows that the non-zero components of the Levi-Civita connection of the metric (1.1) are given by :

$$\begin{aligned} \nabla_{\partial_1}\partial_3 &= \frac{1}{2}a_1\partial_1,\ \nabla_{\partial_1}\partial_4 = \frac{1}{2}a_1\partial_2,\ \nabla_{\partial_2}\partial_3 = \frac{1}{2}a_2\partial_1,\ \nabla_{\partial_2}\partial_4 = \frac{1}{2}a_2\partial_2, \\ \nabla_{\partial_3}\partial_3 &= \frac{1}{2}(aa_1 + a_3)\partial_1 + \frac{1}{2}(aa_2 - a_4)\partial_2 - \frac{1}{2}a_1\partial_3 - \frac{1}{2}a_2\partial_4, \\ \nabla_{\partial_3}\partial_4 &= \frac{1}{2}a_4\partial_1 + \frac{1}{2}a_3\partial_2, \\ \nabla_{\partial_4}\partial_4 &= \frac{1}{2}(aa_1 - a_3)\partial_1 + \frac{1}{2}(aa_2 + a_4)\partial_2 - \frac{1}{2}a_1\partial_3 - \frac{1}{2}a_2\partial_4. \end{aligned} \tag{2.1}$$

A curve $\gamma(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ in (1.1) is a geodesic if and only if the following equations are satisfied :

$$\begin{aligned} 0 &= \ddot{x}_1 + a_1\dot{x}_1\dot{x}_3 + a_2\dot{x}_2\dot{x}_3 + \frac{1}{2}(aa_1 + a_3)\dot{x}_3\dot{x}_3 + a_4\dot{x}_3\dot{x}_4 \\ &\quad + \frac{1}{2}(aa_1 - a_3)\dot{x}_4\dot{x}_4, \\ 0 &= \ddot{x}_2 + a_1\dot{x}_1\dot{x}_4 + a_2\dot{x}_2\dot{x}_4 + \frac{1}{2}(aa_2 - a_4)\dot{x}_3\dot{x}_3 + a_3\dot{x}_3\dot{x}_4 \\ &\quad + \frac{1}{2}(aa_2 + a_4)\dot{x}_4\dot{x}_4, \\ 0 &= \ddot{x}_3 - \frac{a_1}{2}\dot{x}_3\dot{x}_3 - \frac{a_1}{2}\dot{x}_4\dot{x}_4 = 0, \\ 0 &= \ddot{x}_4 - \frac{a_2}{2}\dot{x}_3\dot{x}_3 - \frac{a_2}{2}\dot{x}_4\dot{x}_4. \end{aligned}$$

Recall that, a pseudo-Riemannian manifold (M, g) is geodesically complete if all geodesics exist for all time. The above partial differential equations system is hard to solve. So the geodesically completeness of the Walker metric (1.1) is not easy to prove.

Using (2.1), we can completely determine the curvature tensor of the metric (1.1) by the following formula:

$$\mathcal{R}(\partial_i, \partial_j)\partial_k = \Big([\nabla_{\partial_i}, \nabla_{\partial_j}] - \nabla_{[\partial_i, \partial_i]}\Big)\partial_k.$$

Then, taking into account (1.1), we can determine all components of the $(0, 4)$-curvature tensor

$$R_{ijkl} = g_a(\mathcal{R}\big(\partial_i, \partial_j)\partial_k, \partial_l\big).$$

We obtain that, the non-zero component of the $(0,4)$-curvature tensor of the metric (1.1) are given by

$$\begin{aligned}
R_{1313} &= \frac{1}{2}a_{11}, \quad R_{1323} = \frac{1}{2}a_{12},\\
R_{1424} &= \frac{1}{2}a_{12}, \quad R_{1334} = \frac{1}{4}(a_1a_2 - 2a_{14}),\\
R_{1414} &= \frac{1}{2}a_{11}, \quad R_{1434} = \frac{1}{4}(2a_{13} - a_1^2),\\
R_{2323} &= \frac{1}{2}a_{22}, \quad R_{2334} = \frac{1}{4}(a_2^2 - 2a_{24}),\\
R_{2424} &= \frac{1}{2}a_{22}, \quad R_{2434} = \frac{1}{4}(2a_{23} - a_1a_2),\\
R_{3434} &= \frac{1}{4}(2a_{33} + 2a_{44} - aa_1^2 - aa_2^2).
\end{aligned} \tag{2.2}$$

By (2.2), we can calculate the compoments ρ_{ij} with respect to ∂_i of the Ricci tensor of the metric (1.1). We find that, the non-zero component of the Ricci tensor are given by :

$$\begin{aligned}
\rho_{13} &= \frac{1}{2}a_{11},\ \rho_{14} = \frac{1}{2}a_{12},\ \rho_{23} = \frac{1}{2}a_{12},\ \rho_{24} = \frac{1}{2}a_{22},\\
\rho_{33} &= \frac{1}{2}(a_2^2 + aa_{11} + aa_{22} - 2a_{24}),\\
\rho_{34} &= \frac{1}{2}(-a_1a_2 + a_{14} + a_{23}),\\
\rho_{44} &= \frac{1}{2}(a_1^2 + aa_{11} - 2a_{13} + aa_{22}).
\end{aligned} \tag{2.3}$$

The scalar curvature defined by $\tau = \mathrm{trace}\rho$ of the metric (1.1) is

$$\tau = a_{11} + a_{22}.$$

Let $\mathcal{F}$ be the Einstein tensor defined by

$$\mathcal{F}(X,Y) := \rho(X,Y) - \frac{\tau}{4}\cdot g(X,Y).$$

We have

$$\begin{aligned}
\mathcal{F}_{13} &= -\mathcal{F}_{24} = \frac{1}{4}(a_{11} - a_{22}),\\
\mathcal{F}_{14} &= \mathcal{F}_{23} = \frac{1}{2}a_{12},\\
\mathcal{F}_{33} &= \frac{1}{4}(2a_2^2 + aa_{11} + aa_{22} - 4a_{24}),\\
\mathcal{F}_{34} &= \frac{1}{2}(-a_1a_2 + a_{14} + a_{23}),\\
\mathcal{F}_{44} &= \frac{1}{4}(2a_1^2 + aa_{11} - 4a_{13} + aa_{22}).
\end{aligned}$$

The Walker metric (1.1) is Einstein if $\mathcal{F}(X,Y)=0$ for any vector fields X,Y. This is equivalent to

$$a_{11}-a_{22}=0, \quad a_{12}=0, \quad 2a_2^2+aa_{11}+aa_{22}-4a_{24}=0,$$
$$a_1a_2-a_{14}-a_{23}=0, \quad 2a_1^2+aa_{11}+aa_{22}-4a_{13}=0.$$

The non-zero components of the Ricci operator Q given by

$$g(Q(X),Y)=\rho(X,Y)$$

with respect to ∂_i are

$$\begin{aligned} Q_{11} &= \rho_{13}, \quad Q_{12}=\rho_{14}, \quad Q_{13}=\rho_{33}-a\rho_{13}, \\ Q_{14} &= \rho_{34}-a\rho_{14}, \quad Q_{21}=\rho_{14}, \quad Q_{22}=\rho_{24}, \\ Q_{23} &= \rho_{34}-a\rho_{14}, \quad Q_{24}=\rho_{44}-a\rho_{24}, \quad Q_{33}=\rho_{13}, \\ Q_{34} &= \rho_{14}, \quad Q_{43}=\rho_{14}, \quad Q_{44}=\rho_{24}. \end{aligned} \tag{2.4}$$

Then, according to the above, it is easy to see that the eigenvalues of the Ricci operator are solutions of

$$\left[(\rho_{13}-\lambda)(\rho_{24}-\lambda)-\rho_{14}^2\right]^2=0.$$

If $\rho_{13}=\rho_{24}$ and $\rho_{14}=0$, then $\lambda=\rho_{13}=\frac{1}{2}a_{11}$ is the only Ricci eigenvalue. In this case, it is easy to see that the corresponding eigenspace is not four-dimensional (and so, Q is not diagonalizable), unless

$$\rho_{13}-\rho_{24}=\rho_{14}=\rho_{33}-a\rho_{13}=\rho_{34}=\rho_{44}-a\rho_{13}=0$$

that is, a satisfies

$$\begin{aligned} a_{11}-a_{22} &= a_{12} \\ &= a_2^2+aa_{11}-2a_{24} \\ &= a_1a_2-a_{14}-a_{23} \\ &= a_1^2+aa_{11}-2a_{13} \\ &= 0. \end{aligned} \tag{2.5}$$

If $\rho_{13}\neq\rho_{24}$ or $\rho_{14}\neq 0$, then Q admits the eigenvalues

$$\lambda_\varepsilon=\frac{\rho_{13}+\rho_{24}+\varepsilon\sqrt{(\rho_{13}-\rho_{24})^2+\rho_{14}^2}}{2},$$

where $\varepsilon=\pm1$, each of multiplicity 2. In this case, it is easily seen by (1.1) that Q is not diagonalizable, unless

$$\begin{aligned} 2\rho_{14}\rho_{34}-2a\rho_{14}^2-\rho_{13}\rho_{44}+2a\rho_{13}\rho_{24}-\rho_{24}\rho_{33} &= 0 \\ \rho_{44}-a\rho_{24}+\rho_{33}-a\rho_{13} &= 0, \end{aligned} \tag{2.6}$$

equivalently, equations above (2.6) are equivalent to requiring that the defining function a satisfies :

$$\begin{aligned} 0 &= a_1^2 - 2a_{13} + aa_{22} + a_2^2 + aa_{11} - 2a_{24} \\ &= a_{12}(-a_1a_2 + a_{14} + a_{23} - aa_{12}) \\ &\quad -\frac{1}{2}a_{11}(a_1^2 + aa_{11} - 2a_{13}) - \frac{1}{2}a_{22}(a_2^2 + aa_{22} - 2a_{24}). \end{aligned} \tag{2.7}$$

Note that (2.5) implies (2.6). Hence, we can state the following result :

Proposition 2.1. *The Walker metric (1.1) has a diagonalizable Ricci operator only if its defining function a satisfies (2.7).*

Note that, Ricci-parallel Riemannian manifolds have a diagonalizable Ricci curvature and are therefore isometric to a product of Einstein manifolds, at least locally. This is no longer true for pseudo-Riemannian manifolds [2].

We can now calculate the covariant derivative of the Ricci tensor $\nabla\rho$ of the metric (1.1). By definition :

$$(\nabla_{\partial_i}\rho)_{jk} = \nabla_{\partial_i}\rho(\partial_j, \partial_k) - \rho(\nabla_{\partial_i}\partial_j, \partial_k) - \rho(\partial_j, \nabla_{\partial_i}\partial_k),$$

By using (2.1) and (2.3), we prove the following.

Proposition 2.2. *The non-zero components of the covariant derivative $\nabla\rho$ of the Walker metric (1.1) are given by*

$$\begin{aligned} (\nabla_{\partial_1}\rho)_{13} &= \frac{1}{2}a_{111}, \\ (\nabla_{\partial_3}\rho)_{13} &= \frac{1}{4}(2a_{113} + a_2a_{12}), \\ (\nabla_{\partial_1}\rho)_{14} &= \frac{1}{2}a_{121}, \\ (\nabla_{\partial_4}\rho)_{13} &= \frac{1}{4}(2a_{114} - a_1a_{12}), \\ (\nabla_{\partial_3}\rho)_{14} &= \frac{1}{4}(2a_{123} - a_1a_{12}), \\ (\nabla_{\partial_4}\rho)_{14} &= \frac{1}{4}(2a_{124} - a_1a_{22} + a_1a_{11} + a_2a_{12}), \\ (\nabla_{\partial_3}\rho)_{23} &= \frac{1}{4}(2a_{123} - a_2a_{11} + a_1a_{12} + a_2a_{22}), \end{aligned}$$

$$
\begin{aligned}
(\nabla_{\partial_1}\rho)_{24} &= \frac{1}{2}a_{122},\\
(\nabla_{\partial_2}\rho)_{24} &= \frac{1}{2}a_{222},\\
(\nabla_{\partial_3}\rho)_{24} &= \frac{1}{4}(2a_{223} - a_2a_{12}),\\
(\nabla_{\partial_4}\rho)_{24} &= \frac{1}{4}(2a_{224} + a_1a_{12}),\\
(\nabla_{\partial_4}\rho)_{23} &= \frac{1}{4}(2a_{124} - a_2a_{12}),\\
(\nabla_{\partial_1}\rho)_{33} &= \frac{1}{2}(2a_2a_{12} + aa_{111} + a_1a_{22} + aa_{221} - 2a_{241}),\\
(\nabla_{\partial_2}\rho)_{33} &= \frac{1}{2}(2a_2a_{22} + aa_{112} + a_2a_{22} + aa_{222} - 2a_{242}),\\
(\nabla_{\partial_3}\rho)_{33} &= \frac{1}{2}(3a_2a_{23} + aa_{113} + a_3a_{22} + aa_{223} - 2a_{243} - aa_2a_{12}\\
&\quad +a_4a_{12} + aa_1a_{22} - 2a_1a_{24} + a_2a_{14}),\\
(\nabla_{\partial_4}\rho)_{33} &= \frac{1}{2}(2a_2a_{24} + aa_{114} + a_4a_{22} + aa_{224} - 2a_{244} - a_3a_{12}),\\
(\nabla_{\partial_1}\rho)_{34} &= \frac{1}{2}(a_{141} + a_{231} - 2a_1a_{12} - a_2a_{11}),\\
(\nabla_{\partial_2}\rho)_{34} &= \frac{1}{2}(-2a_2a_{12} - a_1a_{22} + a_{142} + a_{232}),\\
(\nabla_{\partial_3}\rho)_{34} &= \frac{1}{4}(-4a_2a_{13} - a_1a_{23} + 2a_{143} + 2a_{233} - 2a_3a_{12} - aa_1a_{12}\\
&\quad +a_4a_{22} + a_1a_{14} + aa_2a_{11} - a_4a_{11}),\\
(\nabla_{\partial_4}\rho)_{34} &= \frac{1}{4}(-a_2a_{14} - 4a_1a_{24} + 2a_{144} + 2a_{234} - 2a_4a_{12} - a_3a_{22}\\
&\quad +a_3a_{11} - aa_2a_{12} + aa_1a_{22} + a_2a_{23}),\\
(\nabla_{\partial_1}\rho)_{44} &= \frac{1}{2}(3a_1a_{11} + aa_{111} - 2a_{131} + aa_{122}),\\
(\nabla_{\partial_2}\rho)_{44} &= \frac{1}{2}(2a_1a_{12} + a_2a_{11} + aa_{112} - 2a_{132} + aa_{222}),\\
(\nabla_{\partial_3}\rho)_{44} &= \frac{1}{2}(2a_1a_{13} + a_3a_{11} + aa_{113} - 2a_{133} + aa_{223} - a_4a_{12}),\\
(\nabla_{\partial_4}\rho)_{44} &= \frac{1}{2}(3a_1a_{14} + a_4a_{11} + aa_{114} - 2a_{134} + aa_{224} - aa_1a_{12} + a_3a_{12}\\
&\quad +a_1a_{23} + aa_2a_{11} - 2a_2a_{13}).
\end{aligned}
$$

3. **Einstein Like Walker Metrics**

Einstein-like metrics were introduced and first studied by Gray [10] in the Riemannian framework as natural generalizations of Einstein metrics. Since they are defined through conditions on the Ricci tensor, their definition extends at once to the affine and pseudo-Riemannian setting [9].

Next, we assume that, the defining function a on the Walker metric described by (1.1) is on the following form :

$$a(x_1, x_2, x_3, x_4) = x_1 b(x_3, x_4) + x_2 c(x_3, x_4) + d(x_3, x_4), \tag{3.1}$$

where b, c, d are $\mathcal{C}^\infty(\mathcal{O})$ real valued functions. Note that $\mathcal{O} \subset \mathbb{R}^4$. From scalar curvature formula, the Walker metric (3.1) has vanishing scalar curvature.

We start by given conditions such that the Walker metric (3.1) should be Einstein. Recall that, a pseudo-Riemannian manifold (M, g) is said to be Einstein if its metric tensor g satisfies $\rho = \lambda \cdot g$, where ρ denotes the Ricci tensor and $\lambda \in \mathbb{R}$ is a constant.

Proposition 3.1. *The Walker metric described by (1.1) and (3.1) is Einstein if and only if the following eqations are satisfied :*

$$b_3 = \frac{1}{2}b^2, \quad c_4 = \frac{1}{2}c^2, \quad b_4 + c_3 = bc.$$

That means the functions b and c have the following forms :

$$b(x_3, x_4) = -\frac{2\gamma}{\alpha + \beta x_3 + \gamma x_4}, \quad et \quad c(x_3, x_4) = -\frac{2\beta}{\alpha + \beta x_3 + \gamma x_4}.$$

Example 3.2. *Let consider*

$$\mathcal{O} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4, 4 - x_3 - x_4 \neq 0\}.$$

We set: $b = c = \frac{2}{4 - x_3 - x_4}$. Then for

$$a(x_1, x_2, x_3, x_4) = \frac{2(x_1 + x_2)}{4 - x_3 - x_4} + d(x_3, x_4),$$

the Walker metric (1.1) and (3.1) is Einstein.

A pseudo-Riemannian manifold (M, g) has parallel Ricci tensor or belongs to class $\mathcal{P}$ if and only if its Ricci tensor satisfy :

$$(\nabla_X \rho)(Y, Z) = 0, \tag{3.2}$$

for all vector fields X, Y, Z tangent to M.

Proposition 3.3. *The Walker metric g_a described by (1.1) and (3.1) has parallel Ricci tensor if and only if the functions b and c satisfies*

$$\begin{aligned}
&b^2 = 2b_3 + k(x_4), && c^2 = 2c_4 + l(x_3),\\
&3cc_3 - 2c_{34} - 2bc_4 + b_4 c = 0, && 4cb_3 + bc_3 - 2b_{34} - 2c_{33} - bb_4 = 0,\\
&cb_4 + 4bc_4 - 2b_{44} - 2c_{34} - cc_3 = 0, && 3bb_4 - 2b_{34} + bc_3 - 2cb_3 = 0.
\end{aligned}$$

Proof. From (3.1), applying (3.2), we have : $g_a \in \mathcal{P}$ if and only if :

$$\begin{aligned}(\nabla_{\partial_3}\rho)_{33} &= 0,\\ (\nabla_{\partial_3}\rho)_{34} &= 0,\\ (\nabla_{\partial_3}\rho)_{44} &= 0,\\ (\nabla_{\partial_4}\rho)_{33} &= 0,\\ (\nabla_{\partial_4}\rho)_{34} &= 0,\\ (\nabla_{\partial_4}\rho)_{44} &= 0.\end{aligned}$$

The above system is equivalent to :

$$\begin{aligned}&3cc_3 - 2c_{34} - 2bc_4 + b_4c = 0,\\ &4cb_3 + bc_3 - 2b_{34} - 2c_{33} - bb_4 = 0,\\ &bb_3 - b_{33} = 0,\ cc_4 - c_{44} = 0,\\ &cb_4 + 4bc_4 - 2b_{44} - 2c_{34} - cc_3 = 0,\\ &3bb_4 - 2b_{34} + bc_3 - 2cb_3 = 0.\end{aligned}$$

This completes the proof. □

A comparison between Proposition 3.1 and Proposition 3.3 shows at once that Ricci-parallel Walker metrics (3.1) which are not Einstein form a quite large class, depending on two arbitrary non-vanishing one-variable functions k and l. Note that an irreducible Ricci-parallel Riemannian manifold is necessarily Einstein [10].

Recall that, a pseudo-Riemannian manifold (M, g) has cyclic parallel Ricci tensor or belongs to class $\mathcal{A}$ if and only if its Ricci tensor ρ satisfy

$$(\nabla_X\rho)(Y, Z) + (\nabla_Y\rho)(X, Z) + (\nabla_Z\rho)(Y, X) = 0, \qquad (3.3)$$

for all vector fields X, Y, Z tangent to M. The relation (3.3) is equivalent to requiring that ρ is a Killing tensor, that is,

$$(\nabla_X\rho)(X, X) = 0. \qquad (3.4)$$

Proposition 3.4. *The Walker metric g_a described by (1.1) and (3.1) has cyclic parallel Ricci tensor or belongs to class $\mathcal{A}$ if and only if the functions b and c satisfies :*

$$3cc_3 - 2c_{34} - 2bc_4 + cb_4 = 0, \qquad 3bb_4 - 2b_{34} + bc_3 - 2cb_3 = 0. \qquad (3.5)$$

Proof. From (3.1), applying (3.4), we have: $g_a \in \mathcal{A}$ if and only if

$$(\nabla_{\partial_3}\rho)_{33} = 0,\ (\nabla_{\partial_4}\rho)_{44} = 0.$$

This is equivalent to (3.5). □

Recall that, a pseudo-Riemannian manifold (M, g) is called Codazzi Ricci tensor or belongs to class $\mathcal{B}$ if and only if its Ricci tensor satisfy

$$(\nabla_X \rho)(Y, Z) = (\nabla_Y \rho)(X, Z), \tag{3.6}$$

for all vector fields X, Y, Z tangent to M.

Proposition 3.5. *The Walker metric g_a described by (1.1) and (3.1) has Codazzi Ricci tensor or belongs to class $\mathcal{B}$ if and only if the functions b and c satisfies*

$$\begin{aligned} 4cb_3 + bc_3 - 2b_{34} - 2c_{33} - bb_4 + 2cc_4 - 2c_{44} = 0, \\ cb_4 + 4bc_4 - 2b_{44} - 2c_{34} - cc_3 + 2bb_3 - 2b_{33} = 0. \end{aligned}$$

Proof. From (3.1), applying (3.6), we have: $g_a \in \mathcal{B}$ if and only if:

$$(\nabla_{\partial_3} \rho)_{34} = (\nabla_{\partial_4} \rho)_{33}, \ (\nabla_{\partial_3} \rho)_{44} = (\nabla_{\partial_4} \rho)_{34},$$

that is equivalent to

$$\begin{aligned} 4cb_3 + bc_3 - 2b_{34} - 2c_{33} - bb_4 + 2cc_4 - 2c_{44} = 0, \\ cb_4 + 4bc_4 - 2b_{44} - 2c_{34} - cc_3 + 2bb_3 - 2b_{33} = 0. \end{aligned}$$

Then, we get the proof. □

With the family of Walker metric (1.1) and (3.1), we have the following inclusion :

$$\mathcal{E} \subset \mathcal{P} = \mathcal{A} \cap \mathcal{B} \subset \mathcal{A} \cup \mathcal{B} \subset \mathcal{C}.$$

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