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# A family of graphs generated by Hilbert spaces

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**Abstract.** In this paper, we introduce a novel class of graphs based on Hilbert spaces, termed Hilbert graphs. Constructed using the inner product defined on a Hilbert space, Hilbert graphs leverage the concept of orthogonality, where orthogonal elements correspond to adjacent vertices. We demonstrate that Hilbert graphs are regular and vertex transitive, with the clique number equal to the dimension of the corresponding Hilbert space. Our investigation encompasses various properties of Hilbert graphs, including connectivity, girth, diameter, and chromatic number, and we draw comparisons with Cayley graphs and zero divisor graphs.

**Keywords:** Hilbert Graph, orthogonality, regular graph, vertex transitive graph.

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## 1. **Introduction**

Assigning a graph to other mathematical objects is an investigation subject to mathematicians in both fields. One of the earliest assignments is Cayley graph which is stablished by Cayley to illustrate and translate the concepts of group theory into graph theory [14]. Since every ring is an abelian group with the summation, Akhtar et.al. extended the concept of Cayley graphs to rings [2] and shortly after, lots of authors have found lots of properties of such graphs. Similar generalizations of Cayley graphs have been investigated by researchers [1]. Some other ideas for assignment of a graph to a ring are stablished by I. Beck [10]. Beck's idea stablished on colouring of a ring in the sence that no two zero divisor elements of a ring in which their multiplication is zero have the same colour. He conjectured if the chromatic number and clique number of such graphs coincide. But Anderson et.al [3] found some counter example and brought a new definition for zero divisor graphs and this concept was investigated by several authors. After that some authors assigned a graph to other mathematical objects which were similar to the zero divisor graphs (For instance see [3, 6, 7, 8, 9, 11, 12, 13]). This motivated us to define a new kind of graph which is based upon a Hilbert space.

The possible structure for the full automorphism group of a graph is investigated by lots of authors, specially when we assign a graph to some mathematical object, because the symmetric properties of a graph is completely related to its automorphism group [19]. Some of symmetric properties such as vertex transitivity and edge transitivity of a graph arise from the automorphism group of the graph. For instance every Cayley graph of a group and consequently every Cayley graph of a ring is vertex transitive, since the group itself acts transitively on vertices[20] and some of them are edge transitive as well [18]. Even more, in a very special case of Cayley graph $\Gamma$ of a group $G$, the normalizer of $G$ in $\Gamma$ acts transitively on the set of edges of $\Gamma$ which is called normal edge transitive [5, 16, 24]. But zero divisor graph of a ring is not vertex transitive. We will prove thought the Hilbert space does not act on the vertices of Hilbert graph, but the Hilbert graph is still vertex transitive. We also find the girth of a Hilbert graph, which is similar to the zero divisor graph's in some cases. Therefore, we introduced a graph which has some properties of Cayley graphs as well as some properties of the zero divisor graph of a ring, i.e., it generalizes both concepts.

We will also prove that the chromatic number and the clique number of a Hilbert graph coincide and are equal to the dimension of the Hilbert space. Thus, as in [21, 23], we have introduced a family of graphs in which the Beck's conjecture holds.

## 2. **Hilbert graphs and their graph properties**

In this section, we consider Hilbert spaces over the field $\mathbb{F} = \mathbb{R}$. Through this paper, by a basis of a Hilbert space, we mean a Hilbert basis [15].

For the real Hilbert space $H$, define the following equivalence relation on $H \setminus \{0\}$:

$$\forall x, y \in H \quad x \sim y \Longleftrightarrow \exists \lambda > 0 \quad x = \lambda y.$$

The collection of all equivalence classes of the previous relation is denoted by $H_0$. For simplicity, we denote each equivalence class $[x]$ by $x$. If we pick an element $x$ of $H_0$ such that $||x|| = 1$, we may easily assume that $H_0 = S$ where $S$ is the unit ball

$$\{x \in H : ||x|| = 1\}.$$

Now, we are ready to present the following definition. The Hilbert graph of a Hilbert space $H$ is a graph $\Gamma(H) = (V, E)$ where $V = H_0$ and $E = \{(x, y) \in H_0 \times H_0 :< x, y >= 0\}$. In other words, two vertices $x$ and $y$ are adjecent if and only if $x$ and $y$ are orthogonal. Note that the previous definition is well-defined since $< \lambda x, y >=< x, \lambda y >= \lambda < x, y >$ for all $\lambda \in \mathbb{R}$ and $x, y \in H_0$. Also, it is clear that $\Gamma(H)$ is a simple graph.

**Example 2.1.** $\Gamma(H) = \overline{K_2}$ *iff* $H = \mathbb{R}$.

**Example 2.2.** *If* $H = \mathbb{R}^2$*, then the set of vertices of* $\Gamma(H)$ *is* $V = S^1$*. Also* $\Gamma(H)$ *is a regular graph of degree two.* $N(i, -i) = \{j, -j\}$ *which has two elements, but* $N(i, j) = \emptyset$*.*

We recall that a regular graph is a graph where each vertex has the same number of neighbors; i.e., all vertices have the same degree.

we have the following general fact about the Hilbert graphs.

**Theorem 2.3.** *For any Hilbert space* $H$*, the Hilbert graph* $\Gamma(H)$ *is regular.*

*Proof.* Let $x, y \in V$. Let $S = \{h \in H : ||h|| = 1\}$ and $N(x)$ be the set of vertices of $\Gamma(H)$ adjacent to $x$. Then

$$N(x) = \mathrm{span}(\{x\})^{\perp} \cap S$$

and

$$N(y) = \mathrm{span}(\{y\})^{\perp} \cap S.$$

Since $\mathrm{span}(\{x\})^{\perp}$ and $\mathrm{span}(\{y\})^{\perp}$ are hyperspaces in $H$, there are maximal orthonormal sets $\mathcal{B}_1 = \{v_i\}_{i \in I}$ and $\mathcal{B}_2 = \{w_i\}_{i \in I}$ such that

$$N(x) = \{v = \sum_{i \in I} \lambda_i v_i : \sum_{i \in I} \lambda_i^2 = 1\}$$

and

$$N(y) = \{w = \sum_{i \in I} \lambda_i w_i : \sum_{i \in I} \lambda_i^2 = 1\}.$$

Clearly, the map $\psi : N(x) \to N(y)$ defined by

$$\psi(\sum_{i\in I} \lambda_i v_i) := \sum_{i\in I} \lambda_i w_i$$

is a bijection, therefore $\mathrm{card}(N(x)) = \mathrm{card}(N(y))$. It completes the proof. $\square$

Next we will discuss the connectivity, girth and diameter of a Hilbert graph. Recall that a graph is connected when there is a path between every pair of vertices. In a connected graph, there are no unreachable vertices. A graph that is not connected is disconnected. The girth of a graph is the length of a shortest cycle contained in the graph. If the graph does not contain any cycles (i.e., it's an acyclic graph), its girth is defined to be infinity. The distance between two vertices in a graph is the number of edges in a shortest path connecting them. Diameter of a graph is the maximum distance between pairs of vertices of the graph.

**Theorem 2.4.** *For any Hilbert space $H$, we have*

(1) *If* $\dim H = 2$, *then* $\Gamma(H)$ *is disconnected,* $gr\Gamma(H) = 4$ *and* $\mathrm{diam}\Gamma(H) = \infty$,
(2) *If* $\dim H > 2$, *then* $\Gamma(H)$ *is connected,* $gr\Gamma(H) = 3$ *and* $\mathrm{diam}\Gamma(H) = 2$.

*Proof.* If $\dim H = 2$, then it is easily seen that the cardinal number of each connected component is 4 but $V(H)$ is infinite, implying $\Gamma(H)$ is disconnected, hence $\mathrm{diam}\Gamma(H) = \infty$ and $\Gamma(H)$ is disjoint union of 4 circles, i.e., its girth number is 4.

Now suppose $\dim H > 2$. For any two vertices $x$ and $y$, $\mathrm{span}(\{x, y\})$ is a pure subspace of $H$, Thus $\mathrm{span}(\{x, y\})^\perp$ is nonempty. Suppose $z \in \mathrm{span}(\{x, y\})^\perp$. Hence $z$ is adjacent to both $x$ and $y$, and $\mathrm{diam}\Gamma(H) = 2$. Every 3 elements of an orthogonal basis is a circle, i.e., girth of $\Gamma(H)$ is 3. $\square$

2.1. **Clique number and Chromatic number of $\Gamma(H)$.** A clique is subset of vertices of an undirected graph, such that its induced subgraph is complete; that is, every two distinct vertices in a clique are adjacent. Cliques are one of the basic concepts of graph theory and are used in many other mathematical problems and constructions on graphs. Cliques have also been studied in computer science: the task of finding whether there is a clique of a given size in a graph is NP-complete. Clique number of a graph $G$ is the number of vertices of maximum clique in the graph and denoted by $\omega(G)$. The chromatic polynomial counts the number of ways a graph can be coloured using no more than a given number of colours and denoted by $\chi(G)$.

**Theorem 2.5.** *Let $H$ be a Hilbert space. Then the clique number of $\Gamma(H)$ and the dimension of $H$ coincide.*

*Proof.* Let $\mathcal{B}$ be an orthogonal basis for the Hilbert space $H$. By definition, $\mathcal{B}$ will be a clique of $\Gamma(H)$, and hence $\omega(\Gamma(H)) \geqslant \dim H$.

Suppose that we have a clique $\mathcal{C}$ such that $\mathrm{card}(\mathcal{C}) \geqslant \mathrm{card}(\mathcal{B})$. Thus $\mathcal{C}$ contains pairwise orthogonal elements of $H$. Therefore, we can extend $\mathcal{C}$ to a basis for $H$ with cardinality greater than $\mathcal{B}$. A contradiction, i.e., $\omega(\Gamma(H)) \leqslant \dim H$. $\square$

It is well known that $\chi(G) \geqslant \omega(G)$ for a graph $G$. Thus Theorem 2.5 bring us a lower bound for the chromatic number. In the next theorem we will prove the equality.

**Theorem 2.6.** *For any given Hilbert space $H$, the chromatic number of $\Gamma(H)$ is equal to the dimension of $H$.*

*Proof.* Suppose that $\mathcal{B}$ is an orthonormal basis for $H$ and $\mathrm{card}(\mathcal{B}) = \alpha$. If we assign $\alpha$ disjoint colours to the disjoint elements of $\mathcal{B}$ and denote them by $\omega_b$ for each $b \in \mathcal{B}$, we then can colour all vertices of $\Gamma(H)$ in the following way.

Assume that $a$ is a common neighbourhood of all but one element of $\mathcal{B}$. Suppose $a$ is adjacent to $\mathcal{B} \setminus \{b\}$, thus $a$ belongs to $(\mathcal{B} \setminus \{b\})^{\perp}$ which is $< b >$. But $< b > \cap S^{n-1} = \{b, -b\}$, i.e., $a = b$ or $a = -b$. In this case we can assign the colour $\omega_b$ to $a$, since if $c$ is adjacent to $a$, one of the following cases may happen.

Case I: $c$ is also adjacent to $\mathcal{B} \setminus \{b\}$, thus

$$c \in (\mathcal{B} \setminus \{b\})^{\perp} =< b >$$

which implies $c = b$ or $c = -b$. But $< \pm b, b >= \pm 1 \neq 0$, a contradiction with the asumption.

Case II: $c$ is adjacent to $a$, but is not adjacent to at least one element of $\mathcal{B}$ different from $b$ such as $d$. We then can assign $\omega_d$ to $c$.

Therefore, every vertex of $\Gamma(H)$ which is adjacent to all but one elements of $\mathcal{B}$ can be coloured by these $\alpha$ colours.

Now if $\mathcal{D} = \mathcal{B} \setminus N(a)$ has more than one element, by axiom of choice we can choose a $d \in \mathcal{D}$. We now can assign $\omega_d$ to $a$, because if $c$ is a neighborhood of $a$ not in $\mathcal{B}$, one of the following cases may happen.

Case III: $N(c) \cap \mathcal{B} = \mathcal{B} \setminus \{d\}$. Similar arguments in case I, shows $c = \pm d$, which is not adjacent to $a$, by the assumption. A contradiction.

Case IV: $N(c) \cap \mathcal{B}$ has at least one more element except for $d$, such as $e$. We then can assign $\omega_e$ to $c$.

Thus we can continue the procedure to colour all vertices of $\Gamma(H)$. $\square$

Theorem 2.6 also bring us a graph which is not finitely colourable.

**Corollary 2.7.** *If $H$ is an infinite dimensional Hilbert space then $\Gamma(H)$ can not be finitely colourable.*

We also can find some graphs with countable colourable.

**Corollary 2.8.** *For a Hilbert space $H$, $\chi(\Gamma(H)) = \aleph_0$ if and only if $H$ is separable.*

*Proof.* A Hilbert space is separable if and only if it admits a countable orthonormal basis. This fact combined with Theorem 2.6 completes the proof. $\square$

2.2. **Dominating number.** A set $D \subseteq V$ of vertices in a graph $\Gamma = (V, E)$ is a dominating set if every vertex $v \in V$ is an element of $D$ or adjacent to an element of $D$. The domination number $\gamma(\Gamma)$ of a graph $\Gamma$ is the minimum cardinality of a dominating set of $\Gamma$ (see [25, 17, 22]).

**Theorem 2.9.** *For any given Hilbert space with $\dim H \geqslant 2$, the domination number of $\Gamma(H)$ is infinite.*

*Proof.* Suppose that a finite subset of $V(G)$ such as $D$ is a dominating set of $G$. Let $D = \{a_1, \ldots, a_n\}$ for some integer $n$. For every element of $D$ such as $a$, the neighborhood of $a$ is the hyperspace $< a >^{\perp}$. $D$ is a dominating set, therefore

$$V(\Gamma) = H_0 = (D \cup < a_1 >^{\perp} \cup \cdots \cup < a_n >^{\perp}) \cap H_0.$$

Thus $H$ is the union of a finite set $D$ and finite union of it's pure subspaces, which is a contradiction. $\square$

2.3. **Antipodal of $\Gamma(H)$.** In 1971 Smith [26] initiated the concept of *antipodal graph of a graph* $G$ as the graph $A(G)$ having the same vertex set as that of $G$ and two vertices are adjacent if they are at the distance of $\mathrm{diam}(G)$ in $G$. A graph is *antipodal* if it is the antipodal graph $A(H)$ of some graph $H$.

The following theorem is proved by Aravamudhan et. al [4].

**Theorem 2.10.** *A graph $G$ is an antipodal graph if and only if it is the antipodal graph of its complement.*

We have the following theorem for $\Gamma(H)$.

**Theorem 2.11.** *Let $H$ be a Hilbert space with $\dim H \geqslant 3$, then*

(1) *The antipodal of $\Gamma(H)$ is the complement of $\Gamma(H)$;*
(2) *$\Gamma(H)$ is an antipodal graph.*

*Proof.* (1) By Theorem 2.4, $\Gamma(H)$ is connected and every pair of vertices are either adjacent or have a distance equal to the diameter of the graph $\Gamma(H)$. Therefore, two vertices $x$ and $y$ are not adjacent if and only if $d(x, y) = \mathrm{diam}\Gamma(H)$, i.e.,

$$A(\Gamma(H)) = \Gamma(H)^c.$$

(2) Apply Theorem 2.10 and part 1.

This completes the proof. $\square$

## 3. **Group properties of** $\Gamma(H)$

A bijection mapping $\sigma : V \to V'$ is an isomorphism between two graphs $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ if and only if it preserves the edges. An isomorphism from a graph to itself is an automorphism of a graph. The set of automorphisms of a given graph $\Gamma$ is a group with the composition operation and denoted by $Aut(\Gamma)$. The more bigger is the automorphism group of a graph, the more symmetric is the graph. Finding the automorphism group of a graph may be too hard to determine in general, but lots of symmetric properties of such a graph may be discovered. Since $Aut(\Gamma)$ is a permutaion group of the vertex set of a given graph $\Gamma$ , it acts on the set of vertices of $\Gamma$ as well as the set of edges of $\Gamma$. If $Aut(\Gamma)$ acts transitive on the set of vertices or edges of $\Gamma$, then the graph $\Gamma$ is called vertex transitive or edge transitive respectively.

**Theorem 3.1.** *$H_1$ and $H_2$ are isomorphic Hilbert spaces if and only if $\Gamma(H_1)$ and $\Gamma(H_2)$ are isomorphic graphs.*

*Proof.* Let $H_1$ and $H_2$ be isomorphic Hilbert spaces. There exists an inner-product preserving isomorphism $\phi : H_1 \to H_2$. Let $V_i$ be the set of the vertices of $\Gamma(H_i)$ $(i = 1, 2)$. Define

$$\psi : V_1 \to V_2$$

by

$$\psi([x]) := [\phi(x)].$$

$\psi$ is well-defined, since if $[x] = [y]$ then $x = \lambda y$ for some $\lambda > 0$, therefore $\phi(x) = \lambda\phi(y)$ which implies

$$\psi([x]) = [\phi(x)] = [\lambda\phi(y)] = [\phi(y)] = \psi([y]).$$

On the other hand, if $[x]$ and $[y]$ are adjacent then $< x, y >= 0$. Since $\phi$ is inner-product preserving then $< \phi(x), \phi(y) >= 0$, therefore $\psi([x]) = [\phi(x)]$ is adjacent to $\psi([y]) = [\phi(y)]$. It shows that $\psi$ preserves the adjancy. Finally, $\psi$ is bijective since $\phi$ is bijective.

Conversely, let $\Gamma(H_1)$ and $\Gamma(H_2)$ be isomorpohic graphs. Then the clique number of $\Gamma(H_1)$ coincides to the clique number of $\Gamma(H_2)$. Therefore, by Theorem 2.5, we have $\dim H_1 = \dim H_2$, so $H_1$ and $H_2$ are isomorphic Hilbert spaces. It completes the proof. □

**Theorem 3.2.** *For any Hilbert space $H$, the Hilbert graph $\Gamma(H)$ is vertex transitive.*

*Proof.* We consider the following cases:

Case I: Let $\dim H = n < +\infty$. By Theorem 3.1, we may assume that $H = \mathbb{R}^n$. Let $\mathcal{B} = \{e_1, e_2, ..., e_n\}$ be the standard basis of $\mathbb{R}^n$. First let $x \in V = S^{n-1}$. Extend the set $\{x\}$ to a basis $\mathcal{B}' = \{x, x_2, x_3, ..., x_n\}$ for $\mathbb{R}^n$.

Apply the Gram-Schmidt process to find an orthonormal basis $\mathcal{B}'' = \{x, v_2, v_3, ..., v_n\}$ for $\mathbb{R}^n$. Now consider the matrix $A_x$ with columns $x, v_2, v_3, ..., v_n$. Since $\mathcal{B}''$ is an orthonormal basis then $A_x^t A_x = I$. We also have $A_x e_1 = x$. Since $\det A_x = \pm 1$, we may assume that $\det A_x = 1$ by replacing $v_n$ by $-v_n$ if necessary. Note that we still have $A_x e_1 = x$.

For $y \neq x$, applying the previous procedure, we may find an orthogonal matrix $A_y$ with $\det A_y = 1$ and $A_y e_1 = y$. Therefore,

$$e_1 = A_x^{-1} x = A_y^{-1} y$$

and so, $x = A_x A_y^{-1} y$. It suffices to set $\psi = (A_x A_y^{-1})|_V$.

Case II: Let $\dim H = \aleph_0$. Let $\mathcal{B} = \{e_1, e_2, e_3, ...\}$ be an orthonormal basis for $H$. First let $x \in V = \{h \in H : ||h|| = 1\}$. Extend the set $\{x\}$ to a basis $\mathcal{B}' = \{x, x_2, x_3, ...\}$ for $H$. Apply the Gram-Schmidt process to find an orthonormal basis $\mathcal{B}'' = \{x, v_2, v_3, ...\}$ for $H$. Define the linear operator $T_x : H \to H$ by $T_x(e_1) = x$ and $T_x(e_j) = v_j$ $(j \geqslant 2)$. $T_x$ is an inner-product preserving linear operator, since if $v = \sum_{j=1}^n \lambda_j e_j$ and $w = \sum_{k=1}^n \mu_k e_k$ then

$$\begin{aligned}
< T_x(v), T_x(w) > &= < T_x(\sum_{j=1}^n \lambda_j e_j), T_x(\sum_{k=1}^n \mu_k e_k) > \\
&= < \sum_{j=1}^n \lambda_j T_x(e_j), \sum_{k=1}^n \mu_k T_x(e_k) > \\
&= < \lambda_1 x + \sum_{j=2}^n \lambda_j v_j, \mu_1 x + \sum_{k=2}^n \mu_k v_k > \\
&= \lambda_1 \mu_1 \underbrace{||x||^2}_{1} + \sum_{k=2}^n \lambda_1 \mu_k \underbrace{< x, v_k >}_{0} + \sum_{j=2}^n \mu_1 \lambda_j \underbrace{< v_j, x >}_{0} \\
&+ \sum_{j=2}^n \sum_{k=2}^n \lambda_j \mu_k \underbrace{< v_j, v_k >}_{\delta_{jk}} \\
&= \sum_{j=1}^n \lambda_j \mu_j \\
&= < v, w > .
\end{aligned}$$

For $y \neq x$, let $T_y : H \to H$ be as in the previous discussion, i.e., $T_y$ is an inner-product operator such that $T_y e_1 = y$. Therefore,

$$e_1 = T_x^{-1} x = T_y^{-1} y$$

and so, $x = T_x T_y^{-1} y$. It suffices to set $\psi = (T_x T_y^{-1})|_V$.

Case III: Let $\dim H = \alpha \geqslant 2^{\aleph_0}$. Let $\mathcal{B} = \{e_i\}_{i\in I}$ be an orthonormal basis for $H$ where $\mathrm{card}(I) = \alpha$. Fix $j_0 \in I$. Let $x \in V$. Then $\mathrm{span}(\{x\})^\perp$ is a hyperspace in $H$. Let $\mathcal{B}' = \{v_i\}_{i\in I}$ be an orthonormal basis for $\mathrm{span}(\{x\})^\perp$. (Note that the elements of $\mathcal{B}'$ may be indexed by $I$). Therefore $\mathcal{B}'' = \{x\} \cup \mathcal{B}'$ is an orthonormal basis for $H$. Define the linear operator $T_x : H \to H$ by

$$T_x(e_{j_0}) = x, \quad \text{and} \quad T_x(e_i) = v_i \ (i \neq j_0).$$

For $y \neq x$, let also $T_y : H \to H$ be an inner-product operator such that $T_y e_{j_0} = y$. Again, as in case II, we may set $\psi = (T_x T_y^{-1})|_V$.

Then, we get the proof. □

**Corollary 3.3.** *Given any cardinal number $\alpha$, there exists a regular vertex transitive graph $\Gamma$ such that $\chi(\Gamma) = \omega(\Gamma) = \alpha$. Moreover, if $\alpha \geq 3$, the graph is also antipodal.*

*Proof.* Let $\alpha$ be a cardinal number. Let $I$ be any set such that $\mathrm{card}(I) = \alpha$. Set

$$l^2(I) := \{(x_i)_{i\in I} : x_i \in \mathbb{R}, \sum_{i\in I} |x_i|^2 < +\infty\}.$$

It is known that $H = l^2(I)$, equipped by the inner-product

$$< (x_i)_{i\in I}, (y_i)_{i\in I} >:= \sum_{i\in I} x_i y_i$$

is a Hilbert space with $\dim H = \mathrm{card}(I) = \alpha$ [15]. Now, set $\Gamma = \Gamma(H)$. By Theorems 2.3, 2.6 and 3.2, $\Gamma$ is a regular vertex transitive graph such that

$$\chi(\Gamma) = \omega(\Gamma) = \dim H = \alpha.$$

Finally, If $\alpha \geq 3$, by Theorem 2.11, the graph is also antipodal. □

## Conclusion

In this paper, we introduced the concept of Hilbert graphs, a novel class of graphs constructed using the inner product defined on Hilbert spaces. By leveraging the concept of orthogonality, we demonstrated that Hilbert graphs are regular and vertex transitive, with the clique number equal to the dimension of the corresponding Hilbert space. Our exploration of various properties, including connectivity, girth, diameter, and chromatic number, revealed intriguing parallels and distinctions between Hilbert graphs and other well-known graph structures such as Cayley graphs and zero divisor graphs. The significance of Hilbert graphs lies in their potential to unify concepts from functional analysis, graph theory, and algebra. By providing a new framework for understanding the relationships between these fields, Hilbert graphs not only generalize existing concepts but also open new avenues for research and applications. Additionally, we suggested potential applications in quantum mechanics and the generalization of Hilbert graphs to other inner product spaces.

Our findings contribute to a deeper understanding of graph theory and its intersections with other mathematical disciplines, offering new insights and opportunities for future research. The study of Hilbert graphs provides a rich and promising area for further investigation, with the potential to uncover new connections and applications across various fields of mathematics.

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