Journal of Finsler Geometry and its Applications
Vol. 6, No. 1 (2025), pp 103-113
https://doi.org/10.22098/jfga.2025.16478.1144

Duals, approximate duals and pseudo-duals of generalized frames in Hilbert C^*-modules

Morteza Mirzaee Azandaryani*

Department of Mathematics, University of Qom, Qom, Iran

E-mail: morteza_ma62@yahoo.com; m.mirzaee@qom.ac.ir

Abstract. The present paper considers duals, approximate duals and pseudo-duals of generalized frames in Hilbert C^*-modules. In particular, the ones constructed by bounded operators inserted between the synthesis and analysis operators of Bessel sequences are focused and characterized. Moreover, the mentioned notions for modular g-Riesz bases are studied and some of their properties are obtained.

Keywords: Hilbert C^*-module, Generalized frame, Modular g-Riesz basis, Dual, Approximate dual, Pseudo-dual.

1. **Introduction**

A Hilbert C^*-module is considered as a generalization of a Hilbert space by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers. Here, we briefly recall the definition and some basic properties of Hilbert C^*-modules and the adjointable operators defined on them.

Suppose that $\mathfrak{A}$ is a unital C^*-algebra and $\mathcal{H}$ is a left $\mathfrak{A}$-module such that the linear structures of $\mathfrak{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathfrak{A}$-module

*Corresponding Author
AMS 2020 Mathematics Subject Classification: 42C15

if $\mathcal{H}$ is equipped with an $\mathfrak{A}$-valued inner product $\langle\cdot,\cdot\rangle\colon \mathcal{H}\times\mathcal{H}\longrightarrow\mathfrak{A}$, that is sesquilinear, positive definite and respects the module action. In other words

(i) $\langle \alpha x+\beta y, z\rangle = \alpha\langle x,z\rangle+\beta\langle y,z\rangle$, for each $\alpha,\beta\in\mathbb{C}$ and $x,y,z\in\mathcal{H}$;
(ii) $\langle ax,y\rangle = a\langle x,y\rangle$, for each $a\in\mathfrak{A}$ and $x,y\in\mathcal{H}$;
(iii) $\langle x,y\rangle=\langle y,x\rangle^*$, for each $x,y\in\mathcal{H}$;
(iv) $\langle x,x\rangle\geq 0$, for each $x\in\mathcal{H}$ and if $\langle x,x\rangle=0$, then $x=0$.

For each $x\in\mathcal{H}$, we define $\|x\|:=\|\langle x,x\rangle\|^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with the norm $\|\cdot\|$, it is called a *Hilbert $\mathfrak{A}$-module* or a *Hilbert C^*-module* over $\mathfrak{A}$.

A Hilbert $\mathfrak{A}$-module $\mathcal{H}$ is called *finitely generated* if there exists a finite set $\{x_1,\ldots,x_n\}\subseteq\mathcal{H}$ such that every element $x\in\mathcal{H}$ can be expressed as an $\mathfrak{A}$-linear combination $x=\sum_{i=1}^n a_i x_i, a_i\in\mathfrak{A}$. A Hilbert $\mathfrak{A}$-module $\mathcal{H}$ is *countably generated* if there exists a countable set $\{x_i\}_{i\in\mathbb{I}}\subseteq\mathcal{H}$ such that $\mathcal{H}$ equals the norm-closure of $\mathfrak{A}$-linear hull of $\{x_i\}_{i\in\mathbb{I}}$.

For each a in a C^*-algebra $\mathfrak{A}$, we have $|a|=(a^*a)^{\frac{1}{2}}$ and we define $|x|:=\langle x,x\rangle^{\frac{1}{2}}$, for each $x\in\mathcal{H}$. The *center* of $\mathfrak{A}$ is denoted by $\mathcal{Z}(\mathfrak{A})$ and is defined by

$$\mathcal{Z}(\mathfrak{A})=\{a\in\mathfrak{A}: ab=ba, \forall b\in\mathfrak{A}\}.$$

We note that $\mathcal{Z}(\mathfrak{A})$ is a commutative C^*-subalgebra of $\mathfrak{A}$. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert $\mathfrak{A}$-modules. The operator $T\colon\mathcal{H}\longrightarrow\mathcal{K}$ is called *adjointable* if there exists an operator $T^*\colon\mathcal{K}\longrightarrow\mathcal{H}$ such that $\langle T(x),y\rangle=\langle x,T^*(y)\rangle$, for each $x\in\mathcal{H}$ and $y\in\mathcal{K}$. Every adjointable operator T is automatically bounded and $\mathfrak{A}$-linear (that is, $T(ax)=aT(x)$ for each $x\in\mathcal{H}$ and $a\in\mathfrak{A}$). We denote the set of all adjointable operators from $\mathcal{H}$ into $\mathcal{K}$ by $\mathcal{L}(\mathcal{H},\mathcal{K})$ and the set of all bounded operators from $\mathcal{H}$ into $\mathcal{K}$ is denoted by $\mathcal{B}(\mathcal{H},\mathcal{K})$. Note that $\mathcal{L}(\mathcal{H},\mathcal{H})$ is a C^*-algebra and we denote it by $\mathcal{L}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H},\mathcal{H})$ is denoted by $\mathcal{B}(\mathcal{H})$, for more details see [13].

Frames for Hilbert spaces were introduced in [5]. Then, Frank and Larson in [6] presented a general approach to the frame theory in Hilbert C^*-modules.

Let $\mathcal{H}$ be a Hilbert $\mathfrak{A}$-module. A family $\{f_i\}_{i\in\mathbb{I}}\subseteq\mathcal{H}$ is a *frame* for $\mathcal{H}$, if there exist real constants $0<A\leq B<\infty$, such that for each $x\in\mathcal{H}$,

$$A\langle x,x\rangle\leq\sum_{i\in\mathbb{I}}\langle x,f_i\rangle\langle f_i,x\rangle\leq B\langle x,x\rangle. \tag{1.1}$$

The numbers A and B are called the lower and upper bound of the frame, respectively. In this case, we call it an (A,B) frame. If only the second inequality is required, we call it a *Bessel sequence*. If the sum in (1.1) converges in norm, the frame is called *standard*.

Let $\mathcal{F}=\{f_i\}_{i\in\mathbb{I}}$ and $\mathcal{G}=\{g_i\}_{i\in\mathbb{I}}$ be standard Bessel sequences in $\mathcal{H}$. Then we say that $\mathcal{G}$ (resp. $\mathcal{F}$) is an *alternate dual* or a *dual* of $\mathcal{F}$ (resp. $\mathcal{G}$), if $x=\sum_{i\in\mathbb{I}}\langle x,f_i\rangle g_i$ or equivalently $x=\sum_{i\in\mathbb{I}}\langle x,g_i\rangle f_i$, for each $x\in\mathcal{H}$. For more results about frames in Hilbert C^*-modules, see [6, 1].

Generalized frames or g-frames in Hilbert spaces were introduced in [21] and generalized to Hilbert C^*-modules in [10]:

Let $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$ be a sequence of Hilbert $\mathfrak{A}$-modules. A sequence $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) \colon i \in \mathbb{I}\}$ is called a *g-frame* for $\mathcal{H}$ with respect to $\{\mathcal{H}_i \colon i \in \mathbb{I}\}$ if there exist real constants $A_\Lambda, B_\Lambda > 0$ such that for each $x \in \mathcal{H}$,

$$A_\Lambda\langle x, x\rangle \leq \sum_{i\in\mathbb{I}}\langle \Lambda_i x, \Lambda_i x\rangle \leq B_\Lambda\langle x, x\rangle.$$

A_Λ and B_Λ are g-frame bounds of Λ. In this case, we call it an (A_Λ, B_Λ) g-frame. The g-frame is *standard* if for each $x \in \mathcal{H}$, the sum converges in norm. If only the second-hand inequality is required, Λ is called a *g-Bessel sequence*. If $A_\Lambda = B_\Lambda$, the g-frame is called *tight* and if $A_\Lambda = B_\Lambda = 1$, the g-frame is called *Parseval*.

If $\{\mathcal{H}_i \colon i \in \mathbb{I}\}$ is a sequence of Hilbert $\mathfrak{A}$-modules, then

$$\oplus_{i\in\mathbb{I}}\mathcal{H}_i = \left\{x = \{x_i\}_{i\in\mathbb{I}} : x_i \in \mathcal{H}_i \text{ and } \sum_{i\in\mathbb{I}}\langle x_i, x_i\rangle \text{ is norm convergent in } \mathfrak{A}\right\},$$

is a Hilbert $\mathfrak{A}$-module with pointwise operations and $\mathfrak{A}$-valued inner product

$$\langle x, y\rangle = \sum_{i\in\mathbb{I}}\langle x_i, y_i\rangle,$$

where $x = \{x_i\}_{i\in\mathbb{I}}$ and $y = \{y_i\}_{i\in\mathbb{I}}$.

For a standard g-Bessel sequence Λ, the operator $T_\Lambda \colon \oplus_{i\in\mathbb{I}}\mathcal{H}_i \longrightarrow \mathcal{H}$ which is defined by $T_\Lambda(\{g_i\}_{i\in\mathbb{I}}) = \sum_{i\in\mathbb{I}}\Lambda_i^* g_i$ is called the *synthesis operator* of Λ. T_Λ is adjointable and $T_\Lambda^*(x) = \{\Lambda_i x\}_{i\in\mathbb{I}}$. The operator $S_\Lambda \colon \mathcal{H} \longrightarrow \mathcal{H}$ which is defined by $S_\Lambda x = T_\Lambda T_\Lambda^*(x) = \sum_{i\in\mathbb{I}}\Lambda_i^*\Lambda_i(x)$, is called the *operator* of Λ. If Λ is a standard (A_Λ, B_Λ) g-frame, then $A_\Lambda \cdot \mathrm{I}_\mathcal{H} \leq S_\Lambda \leq B_\Lambda \cdot \mathrm{I}_\mathcal{H}$.

Recall that if $\Lambda = \{\Lambda_i\}_{i\in\mathbb{I}}$ and $\Gamma = \{\Gamma_i\}_{i\in\mathbb{I}}$ are standard g-Bessel sequences such that $\sum_{i\in\mathbb{I}}\Gamma_i^*\Lambda_i x = x$ or equivalently $\sum_{i\in\mathbb{I}}\Lambda_i^*\Gamma_i x = x$, for each $x \in \mathcal{H}$, then Γ (resp. Λ) is called a *g-dual* of Λ (resp. Γ). The *canonical g-dual* for an (A, B) standard g-frame $\Lambda = \{\Lambda_i\}_{i\in\mathbb{I}}$ is defined by $\widetilde{\Lambda} = \{\tilde{\Lambda}_i\}_{i\in\mathbb{I}}$, where $\tilde{\Lambda}_i = \Lambda_i S_\Lambda^{-1}$ which is an $(\frac{1}{B}, \frac{1}{A})$ standard g-frame and for each $x \in \mathcal{H}$, we have

$$x = \sum_{i\in\mathbb{I}}\Lambda_i^*\tilde{\Lambda}_i x = \sum_{i\in\mathbb{I}}\tilde{\Lambda}_i{}^*\Lambda_i x.$$

For more results about g-frames in Hilbert C^*-modules, see [10, 22].

Duals play an important role in frame theory and its applications. Approximate duals and pseudo-duals can also be helpful, in particular, when it is difficult to find a dual. Approximate duality for frames and g-frames in Hilbert spaces has been investigated and studied in [4, 12] and its generalization to Hilbert C^*-modules has been presented in [14]. Some properties of pseudo-duals in Hilbert spaces were obtained in [3], then pseudo-duals for continuous frames and continuous g-frames were considered in [19, 20], respectively.

In [9], the authors, using a bounded operator Q (inserted between the synthesis and analysis operators), defined a new kind of duals for a fusion frame. Then, Q-duals and Q-approximate duals for frames, g-frames and fusion frames in Hilbert spaces were introduced in [15, 16] (also, see [23]). Afterwards, in [17], Q-duals and Q-approximate duals were considered for frames in Hilbert C^*-modules, where Q is assumed to be bounded and not necessarily adjointable. Also, Q-pseudo-duals for frames in Hilbert C^*-modules were introduced and studied in [18].

In the present paper, the concepts of Q-duals, Q-approximate duals and Q-pseudo-duals for generalized frames or g-frames are presented and some of their properties are obtained. Here, all C^*-algebras are unital and all Hilbert C^*-modules are finitely or countably generated.

2. **Duals, approximate duals and pseudo-duals of standard g-frames**

In this section, some properties of Q-duals, Q-approximate duals and Q-pseudo-duals of standard g-frames are obtained. Mainly, their characterizations are considered and some equivalent conditions for duality, approximate duality and pseudo-duality are presented. Indeed, most of the obtained results in [20] are generalized to Hilbert C^*-modules.

Definition 2.1. *Let $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i),\ \ i \in \mathbb{I}\}$ and $\Gamma = \{\Gamma_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i),\ \ i \in \mathbb{I}\}$ be two standard g-Bessel sequences and let $Q \in \mathcal{B}(\oplus_{i\in\mathbb{I}}\mathcal{H}_i)$. Then*

(i) *Λ is called a Q-pseudo-dual for Γ if the operator $S_{\Lambda,Q,\Gamma} := T_\Lambda Q T_\Gamma^*$ is invertible.*

(ii) *Λ is called a Q-approximate dual for Γ if $\|T_\Lambda Q T_\Gamma^* - I_{\mathcal{H}}\| < 1$.*

(iii) *Λ is called a Q-dual for Γ if $T_\Lambda Q T_\Gamma^* = I_{\mathcal{H}}$.*

If $Q = I_{(\oplus_{i\in\mathbb{I}}\mathcal{H}_i)}$, then a Q-pseudo-dual (resp. a Q-approximate dual, a Q-dual) is called a pseudo-dual (resp. an approximate dual, a dual) and $S_{\Lambda,Q,\Gamma}$ is denoted by $S_{\Lambda,\Gamma}$.

Theorem 2.2. *Let Λ and Γ be two standard g-frames and let $T \in \mathcal{L}(\mathcal{H})$. Then $\Lambda T := \{\Lambda_i T \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i),\ \ i \in \mathbb{I}\}$ and $\Gamma T := \{\Gamma_i T \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i),\ \ i \in \mathbb{I}\}$ are two standard g-Bessel sequences. Moreover, the following statements are equivalent:*

(i) *There exists some $Q \in \mathcal{L}(\oplus_{i\in\mathbb{I}}\mathcal{H}_i)$ such that the operator $S_{\Lambda T,Q,\Gamma T}$ is left-invertible in $\mathcal{L}(\mathcal{H})$.*

(ii) *There exists some $Q \in \mathcal{L}(\oplus_{i\in\mathbb{I}}\mathcal{H}_i)$ such that the operator $S_{\Lambda T,Q,\Gamma T}$ is right-invertible in $\mathcal{L}(\mathcal{H})$.*

(iii) *The operator T is left-invertible in $\mathcal{L}(\mathcal{H})$.*

Proof. For every f in $\mathcal{H}$, we get

$$\left\| \sum_{i\in\mathbb{I}} |\Lambda_i(Tf)|^2 \right\| \leq B_\Lambda \|Tf\|^2 \leq B_\Lambda \|T\|^2 \|f\|^2.$$

Now, Theorem 3.1 in [22] implies that ΛT is a standard g-Bessel sequence. Similarly, ΓT is also a standard g-Bessel sequence.

(i) $\Rightarrow$ (iii). It is easy to see that $T_{\Lambda T} = T^* T_\Lambda$ and $T_{\Gamma T} = T^* T_\Gamma$. Hence, for each $f \in \mathcal{H}$, we have

$$S_{\Lambda T,Q,\Gamma T}(f) = T_{\Lambda T} Q T^*_{\Gamma T}(f) = (T^* T_\Lambda) Q (T^*_\Gamma T) f = T^*(T_\Lambda Q T^*_\Gamma) T f.$$

Since $S_{\Lambda T,Q,\Gamma T}$ is left-invertible, we conclude that T is left-invertible.

(ii) $\Rightarrow$ (iii). Since $S_{\Lambda T,Q,\Gamma T}$ is right-invertible, we conclude that T^* is right-invertible which is equivalent to say that T is left-invertible.

(iii) $\Rightarrow$ (i), (ii). Let L_T be a left inverse of T and define

$$Q := T^*_{\tilde{\Lambda}} L^*_T T_{\tilde{\Gamma}}.$$

Then $S_{\Lambda T,Q,\Gamma T} = T$ which is left-invertible and if $Q := T^*_{\tilde{\Lambda}} L_T T_{\tilde{\Gamma}}$, we have $S_{\Lambda T,Q,\Gamma T} = T^*$ which is right-invertible. □

Corollary 2.3. *Let Λ and Γ be two standard g-frames and let $T \in \mathcal{L}(\mathcal{H})$. Then $\Lambda T := \{\Lambda_i T \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i), \quad i \in \mathbb{I}\}$ and $\Gamma T := \{\Gamma_i T \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i), \quad i \in \mathbb{I}\}$ are two standard g-Bessel sequences. Moreover, if ΛT is a Q-pseudo-dual for ΓT, then T is left-invertible.*

The next result is an immediate consequence of the stated proof for the Theorem 2.2.

Corollary 2.4. *Let Λ and Γ be two standard g-frames and let $T \in \mathcal{L}(\mathcal{H})$. Then $\Lambda T := \{\Lambda_i T \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i), \quad i \in \mathbb{I}\}$ and $\Gamma T := \{\Gamma_i T \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i), \quad i \in \mathbb{I}\}$ are two standard g-Bessel sequences. Moreover, if Λ is a Q-pseudo-dual for Γ and T is invertible, then ΛT is a Q-pseudo-dual for ΓT.*

Theorem 2.5. *Let $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i), \quad i \in \mathbb{I}\}$, $\Gamma = \{\Gamma_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i), \quad i \in \mathbb{I}\}$ and $\Theta = \{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i), \quad i \in \mathbb{I}\}$ be three standard g-Bessel sequences. Then*

(i) *$\Theta - \Lambda := \{\Theta_i - \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i), \quad i \in \mathbb{I}\}$ that for every $f \in \mathcal{H}$*

$$(\Theta_i - \Lambda_i) f = \Theta_i f - \Lambda_i f$$

is a standard g-Bessel sequence.

(ii) *If $\|T_{\Theta-\Lambda} Q T^*_\Gamma\| < 1$ and Λ is a Q-dual for Γ, then Θ is a Q-approximate dual for Γ.*

Proof. (i) For each $f \in \mathcal{H}$, we have

$$\begin{aligned}\left\| \sum_{i\in\mathbb{I}} |(\Theta_i - \Lambda_i) f|^2 \right\| &\le B_\Theta \|f\|^2 + B_\Lambda \|f\|^2 + 2 \left\| \sum_{i\in\mathbb{I}} |\Theta_i f|^2 \right\|^{\frac{1}{2}} \left\| \sum_{i\in\mathbb{I}} |\Lambda_i f|^2 \right\|^{\frac{1}{2}} \\ &\le B_\Theta \|f\|^2 + B_\Lambda \|f\|^2 + 2\sqrt{B_\Theta B_\Lambda} \|f\|^2,\end{aligned}$$

So $\Theta - \Lambda$ is a standard g-Bessel sequence.

(ii) It is obvious that $T_{\Theta-\Lambda} = T_\Theta - T_\Lambda$, so

$$T_\Theta Q T_\Gamma^* = T_{\Theta-\Lambda} Q T_\Gamma^* + T_\Lambda Q T_\Gamma^* = T_{\Theta-\Lambda} Q T_\Gamma^* + I_{\mathcal{H}}.$$

Consequently

$$\|T_\Theta Q T_\Gamma^* - I_{\mathcal{H}}\| = \|T_{\Theta-\Lambda} Q T_\Gamma^*\| < 1,$$

which means that Θ is a Q-approximate dual of Γ.

This completes the proof. □

Theorem 2.6. *Let $\Gamma = \{\Gamma_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i), \quad i \in \mathbb{I}\}$ be a standard g-Bessel sequence. Then, the following statements are equivalent:*

(i) Γ *has a Q-dual, for some $Q \in \mathcal{B}(\oplus_{i\in\mathbb{I}}\mathcal{H}_i)$.*
(ii) Γ *has a Q-approximate dual, for some $Q \in \mathcal{B}(\oplus_{i\in\mathbb{I}}\mathcal{H}_i)$.*
(iii) Γ *has a Q-pseudo-dual, for some $Q \in \mathcal{B}(\oplus_{i\in\mathbb{I}}\mathcal{H}_i)$.*
(iv) Γ *is a standard g-frame.*
(v) Γ *is a Q-dual of itself, for some $Q \in \mathcal{B}(\oplus_{i\in\mathbb{I}}\mathcal{H}_i)$.*
(vi) Γ *is a Q-approximate dual of itself, for some $Q \in \mathcal{B}(\oplus_{i\in\mathbb{I}}\mathcal{H}_i)$.*
(vii) Γ *is a Q-pseudo-dual of itself, for some $Q \in \mathcal{B}(\oplus_{i\in\mathbb{I}}\mathcal{H}_i)$.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (iii) and (v) $\Rightarrow$ (i) are obvious.

(iii) $\Rightarrow$ (iv). Assume that there exists some $Q \in \mathcal{B}(\oplus_{i\in\mathbb{I}}\mathcal{H}_i)$ such that Λ is a Q-pseudo-dual of Γ, which is, $S_{\Lambda,Q,\Gamma} := T_\Lambda Q T_\Gamma^*$ is invertible. Then, for each $f \in \mathcal{H}$, we obtain that

$$\|f\| = \|S^{-1}_{\Lambda,Q,\Gamma} S_{\Lambda,Q,\Gamma} f\| \leq \|S^{-1}_{\Lambda,Q,\Gamma}\| \|T_\Lambda\| \|Q\| \|T_\Gamma^* f\|,$$

so

$$\frac{\|f\|^2}{\|S^{-1}_{\Lambda,Q,\Gamma}\|^2 \|T_\Lambda\|^2 \|Q\|^2} \leq \left\| \sum_{i\in\mathbb{I}} |\Gamma_i f|^2 \right\| \leq B_\Gamma \|f\|^2.$$

Now, Theorem 3.1 in [22] yields that Γ is a standard g-frame.

(iv) $\Rightarrow$ (v). Assume that $\Gamma = \{\Gamma_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i), \quad i \in \mathbb{I}\}$ is a standard g-frame for $\mathcal{H}$, so

$$\widetilde{\Gamma} := \{\Gamma_i S_\Gamma^{-1} \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i), \quad i \in \mathbb{I}\}$$

is a standard g-frame for $\mathcal{H}$, and $\widetilde{\Gamma}$ is a dual for Γ, so $T_{\widetilde{\Gamma}}$ and $T^*_{\widetilde{\Gamma}}$ are bounded. Then $Q := T^*_{\widetilde{\Gamma}} T_{\widetilde{\Gamma}}$ is a bounded operator and we have

$$T_\Gamma Q T_\Gamma^* = T_\Gamma (T^*_{\widetilde{\Gamma}} T_{\widetilde{\Gamma}}) T_\Gamma^* = I_{\mathcal{H}},$$

meaning that Γ is a Q-dual of itself. □

Proposition 2.7. *Let Λ and Γ be two standard g-Bessel sequences and let $Q \in \mathcal{B}(\oplus_{i\in\mathbb{I}}\mathcal{H}_i)$. Then, the following statements are equivalent:*

(i) Λ *is a Q-pseudo-dual (Q-approximate dual) of* Γ.

(ii) Γ *is a standard g-frame and there exist an invertible operator* $S \in \mathcal{B}(\mathcal{H})$ *(*$S \in \mathcal{B}(\mathcal{H})$ *with* $\|S - I_{\mathcal{H}}\| < 1$*) and some* $R \in \mathcal{B}(\oplus_{i\in\mathbb{I}}\mathcal{H}_i, \mathcal{H})$ *such that*

$$T_\Lambda Q = S(S_\Gamma^{-1}T_\Gamma + R(I_{\oplus_{i\in\mathbb{I}}\mathcal{H}_i} - T_\Gamma^* S_\Gamma^{-1} T_\Gamma)).$$

Proof. (i) $\Rightarrow$ (ii). Since Λ is a Q-pseudo-dual (Q-approximate dual) of Γ, by Theorem 2.6, Γ is a standard g-frame. Let

$$S := S_{\Lambda,Q,\Gamma}, \quad R := S_{\Lambda,Q,\Gamma}^{-1} T_\Lambda Q \in \mathcal{B}(\oplus_{i\in\mathbb{I}}\mathcal{H}_i, \mathcal{H}).$$

Then, it is easy to see that

$$S(S_\Gamma^{-1}T_\Gamma + R(I_{\oplus_{i\in\mathbb{I}}\mathcal{H}_i} - T_\Gamma^* S_\Gamma^{-1} T_\Gamma)) = T_\Lambda Q.$$

(ii) $\Rightarrow$ (i). If Γ is a standard g-frame and there are operators $S \in \mathcal{B}(\mathcal{H})$ and $R \in \mathcal{B}(\oplus_{i\in\mathbb{I}}\mathcal{H}_i, \mathcal{H})$ such that S is invertible ($\|S - I_{\mathcal{H}}\| < 1$) and

$$T_\Lambda Q = S(S_\Gamma^{-1}T_\Gamma + R(I_{\oplus_{i\in\mathbb{I}}\mathcal{H}_i} - T_\Gamma^* S_\Gamma^{-1} T_\Gamma)),$$

then it is obtained that

$$\begin{aligned} T_\Lambda Q T_\Gamma^* &= S(T_\Gamma T_\Gamma^*)^{-1} T_\Gamma T_\Gamma^* + SRT_\Gamma^* - SRT_\Gamma^*(T_\Gamma T_\Gamma^*)^{-1} T_\Gamma T_\Gamma^* \\ &= S + SRT_\Gamma^* - SRT_\Gamma^* = S. \end{aligned}$$

Since S is invertible ($\|S - I_{\mathcal{H}}\| < 1$), Λ is a Q-pseudo-dual (Q-approximate dual) of Γ. □

Corollary 2.8. *Let* Γ *and* Λ *be two standard g-Bessel sequences, and let* $Q \in \mathcal{B}(\oplus_{i\in\mathbb{I}}\mathcal{H}_i)$*. Then* Λ *is a Q-dual of* Γ *if and only if* Γ *is a standard g-frame and there exists some* $R \in \mathcal{B}(\oplus_{i\in\mathbb{I}}\mathcal{H}_i, \mathcal{H})$ *such that*

$$T_\Lambda Q = (T_\Gamma T_\Gamma^*)^{-1} T_\Gamma + R(I_{\oplus_{i\in\mathbb{I}}\mathcal{H}_i} - T_\Gamma^* S_\Gamma^{-1} T_\Gamma).$$

3. Duals, approximate duals and pseudo-duals of modular g-Riesz bases

In this section, duals, approximate duals and pseudo-duals of modular g-Riesz bases are considered. We mention that Riesz bases in Hilbert C^*-modules were introduced in [6]. It was shown in [7] that a Riesz basis in a Hilbert C^*-module can possess more than one dual and a dual of a Riesz basis is not necessarily a Riesz basis. The authors in [2, 7, 8, 11] studied the Riesz bases with only one dual (the canonical dual). This kind of Riesz bases is called a modular Riesz basis and its generalization, introduced in [11], is called a modular g-Riesz basis.

Definition 3.1. *Let* $\Gamma = \{\Gamma_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i), \quad i \in \mathbb{I}\}$ *be a standard g-frame. Then* Γ *is called a modular g-Riesz basis if it has only one dual, i.e.,* $\tilde{\Gamma} := \{\Gamma_i S_\Gamma^{-1} \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i), \quad i \in \mathbb{I}\}$ *(the canonical dual) is the only dual of* Γ.

Theorem 3.2. *Let* $\Gamma = \{\Gamma_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i), \quad i \in \mathbb{I}\}$ *be a standard g-frame. Then, the following statements are equivalent:*

(i) Γ *is a modular g-Riesz basis.*
(ii) *For every adjointable, invertible operator* T *on* $\mathcal{H}$, $\Gamma \circ T := \{\Gamma_i \circ T \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i), \quad i \in \mathbb{I}\}$ *is a modular g-Riesz basis.*
(iii) *Every pseudo-dual of* Γ *is a modular g-Riesz basis.*
(iv) *Every approximate dual of* Γ *is a modular g-Riesz basis.*
(v) *Every dual of* Γ *is a modular g-Riesz basis.*

Proof. (i) $\Rightarrow$ (ii). Let Γ be a modular g-Riesz basis and let T be an invertible operator on $\mathcal{H}$. For each $f \in \mathcal{H}$, we get

$$\left\| \sum_{i\in\mathbb{I}} |\Gamma_i T f|^2 \right\| \leq B_\Gamma \|Tf\|^2 \leq B_\Gamma \|T\|^2 \|f\|^2. \tag{3.1}$$

Since

$$\|Tf\| = \|S_\Gamma^{-1} S_\Gamma T f\| \leq \|S_\Gamma^{-1}\| \|T_\Gamma\| \|T_\Gamma^* T f\|,$$

we have

$$\|Tf\|^2 \leq \|S_\Gamma^{-1}\|^2 \|T_\Gamma\|^2 \left\| \sum_{i\in\mathbb{I}} |\Gamma_i T f|^2 \right\|.$$

Now, the invertibility of T implies that

$$\frac{\|f\|^2}{\|T^{-1}\|^2 \|S_\Gamma^{-1}\|^2 \|T_\Gamma\|^2} \leq \left\| \sum_{i\in\mathbb{I}} |\Gamma_i T f|^2 \right\| \leq B_\Gamma \|T\|^2 \|f\|^2,$$

so $\Gamma \circ T$ is a standard g-frame. Now, let Λ be a dual of $\Gamma \circ T$. Then

$$I_\mathcal{H} = T_\Lambda T_{\Gamma\circ T}^* = T_\Lambda (T^* T_\Gamma)^* = T_\Lambda T_\Gamma^* T.$$

Thus

$$T^{-1} = T_\Lambda T_\Gamma^*,$$

so

$$T_{\Lambda\circ T^*} T_\Gamma^* = T T_\Lambda T_\Gamma^* = T T^{-1} = I_\mathcal{H}.$$

Hence, $\Lambda \circ T^*$ is a dual of Γ. Since Γ is a modular g-Riesz basis, it possesses just one dual which is $\tilde{\Gamma}$, so $\tilde{\Gamma} = \Lambda \circ T^*$, and since T is invertible, we have

$$\tilde{\Gamma} {T^*}^{-1} = \Lambda$$

which means that $\Gamma \circ T$ has only one dual, consequently $\Gamma \circ T$ is a modular g-Riesz basis.

(ii) $\Rightarrow$ (i). Since for each invertible operator T on $\mathcal{H}$, $\Gamma \circ T$ is a modular g-Riesz basis, the statement holds for $T := I_\mathcal{H}$.

(i) $\Rightarrow$ (iii). Let Λ be a pseudo-dual for Γ. Then, by considering $T := S_{\Gamma,\Lambda}^{-1}$, $\Lambda \circ T$ is a dual for Γ, so $\Lambda \circ T = \tilde{\Gamma}$, consequently

$$\Lambda = \tilde{\Gamma} \circ T^{-1}.$$

Now, the same argument stated for the proof of the implication (i) $\Rightarrow$ (ii) yields that Λ is modular g-Riesz basis.

The implications (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) are obvious.
(v) $\Rightarrow$ (i). Since every dual of Γ is a modular g-Riesz basis, $\tilde{\Gamma} := \Gamma S_\Gamma^{-1}$ is also a modular g-Riesz basis, so it has only one dual. Now, it is easy to see that Γ has also only one dual. $\square$

Proposition 3.3. *Let* $\Gamma = \{\Gamma_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i), \quad i \in \mathbb{I}\}$ *be a standard g-frame. Then, the following statements are equivalent:*

(i) *There exists a dual for* Γ *which is a modular g-Riesz basis.*
(ii) *There exists some approximate dual for* Γ *which is a modular g-Riesz basis.*
(iii) *There exists some pseudo-dual for* Γ *which is a modular g-Riesz basis.*
(iv) *There exists some adjointable, invertible operator* T *on* $\mathcal{H}$ *such that* $\Gamma \circ T$ *is a modular g-Riesz basis.*
(v) Γ *is a modular g-Riesz basis.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (iv). Suppose that there exists a pseudo-dual for Γ like $\Theta = \{\Theta_i\}_{i\in\mathbb{I}}$ which is a modular g-Riesz basis, so $S_{\Theta,\Gamma}$ and $S_{\Gamma,\Theta}$ are invertible. Now, for $R := S_{\Gamma,\Theta}^{-1}$, it is easy to see that $\Theta \circ R := \{\Theta_i \circ R\}_{i\in\mathbb{I}}$ is a standard g-Bessel sequence and

$$\sum_{i\in\mathbb{I}} (\Theta_i \circ R)^* \Gamma_i f = S_{\Theta,\Gamma}^{-1} \sum_{i\in\mathbb{I}} \Theta_i^* \Gamma_i f = f,$$

so $\Theta \circ R$ is a dual of Γ. Therefore

$$I_\mathcal{H} = T_\Gamma T_{\Theta\circ R}^* = T_\Gamma (R^* T_\Theta)^* = T_\Gamma T_\Theta^* R.$$

Now, by the invertibility of R, it is obtained that

$$T_\Gamma T_\Theta^* = R^{-1},$$

so $\Gamma \circ R^*$ is a dual of Θ because

$$T_{\Gamma\circ R^*} T_\Theta^* = R T_\Gamma T_\Theta^* = R R^{-1} = I_\mathcal{H}.$$

On the other hand, since Θ is a modular g-Riesz basis, we get

$$\Gamma \circ R^* = \tilde{\Theta} = \Theta S_\Theta^{-1}.$$

Now by considering $T := R^* S_\Theta$, we have

$$\Gamma \circ T = \Gamma \circ R^* S_\Theta = \Theta \circ S_\Theta^{-1} S_\Theta = \Theta$$

and it is concluded that $\Gamma \circ T$ is a modular g-Riesz basis.

(iv) $\Rightarrow$ (v). Suppose that there exists an invertible operator T on $\mathcal{H}$ such that $\Lambda := \Gamma \circ T$ is a modular g-Riesz basis. If Θ_1, Θ_2 are two duals for Γ, then we have

$$T_{\Theta_1} T_\Gamma^* = I_\mathcal{H} = T_\Gamma T_{\Theta_1}^*.$$

Thus

$$T_{\Theta_1\circ T^{-1*}}T^*_{\Gamma\circ T} = T^{-1}T_{\Theta_1}(T^*T_\Gamma)^* = T^{-1}T_{\Theta_1}T^*_\Gamma T = I_{\mathcal{H}}.$$

Hence $\Theta_1 \circ T^{-1^*}$ is a dual for Λ. Similarly, we can obtain that $\Theta_2 \circ T^{-1^*}$ is also a dual for Λ. Since Λ is a modular g-Riesz basis, $\Theta_1 \circ T^{-1^*} = \Theta_2 \circ T^{-1^*}$, so $\Theta_1 = \Theta_2$ which implies that Γ is a modular g-Riesz basis.

The implication (v) $\Rightarrow$ (i) can be obtained using $\tilde{\Gamma}$ (the canonical dual of Γ) as a dual of Γ which is a modular g-Riesz basis. $\square$

References

1. L. Arambasic, *On frames for countably generated Hilbert C^*-modules*, Proc. Amer. Math. Soc. **135** (2007), 469-478.
2. L. Arambasic and D. Bakic, *Frames and outer frames for Hilbert C^*-modules*, Linear and multilinear algebra. **65** (2017), 381-431.
3. D. Bakic and T. Beric, *On excesses of frames*, Glasnik Matematicki. **50** (2015), 415-427.
4. O. Christensen and R.S. Laugesen, *Approximate dual frames in Hilbert spaces and applications to Gabor frames*, Sampl Theory Signal Image Process. **9** (2011), 77-90.
5. R.J. Duffin and A.C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), 341-366.
6. M. Frank and D.R. Larson, *Frames in Hilbert C^*-modules and C^*-algebras*, J. Operator Theory. **48** (2002), 273-314.
7. D. Han, W. Jing, D. Larson and R. Mohapatra, *Riesz bases and their dual modular frames in Hilbert C^*-modules*, J. Math. Anal. Appl. **343** (2008), 246-256.
8. M. Hasannasab, *Modular Riesz bases versus Riesz bases in Hilbert C^*-modules*, Asian-European Journal of Mathematics. **14** (2021), 2050149.
9. S.B. Heineken, P.M. Morillas, A.M. Benavente and M.I. Zakowicz, *Dual fusion frames*, Arch. Math. **103** (2014), 355-365.
10. A. Khosravi and B. Khosravi, *Fusion frames and g-frames in Hilbert C^*-modules,* Int. J. Wavelets Multiresolut. Inf. Process. **6** (2008), 433-446.
11. A. Khosravi and B. Khosravi, *G-frames and modular Riesz bases*, Int. J. Wavelets Multiresolut. Inf. Process. **10** (2012), 1250013-1-1250013-12.
12. A. Khosravi and M. Mirzaee Azandaryani, Approximate duality of g-frames in Hilbert spaces, Acta. Math. Sci. **34** (2014), 639-652.
13. E.C. Lance, *Hilbert C^*-modules- A Toolkit for operator algebraists*. London Math. Soc. Lecture Note Ser., Vol. 210, Cambridge University Press, Cambridge, England (1995).
14. M. Mirzaee Azandaryani, *Approximate duals and nearly Parseval frames*, Turk. J. Math. **39** (2015), 515-526.
15. M. Mirzaee Azandaryani, On the approximate duality of g-frames and fusion frames, U. P. B. Sci. Bull. Ser A. **79** (2017), 83-93.
16. M. Mirzaee Azandaryani, An operator theory approach to the approximate duality of Hilbert space frames, J. Math. Anal. Appl. **489** (2020), 1-13 (124177).
17. M. Mirzaee Azandaryani, *Approximate duals, Q-approximate duals and morphisms of Hilbert C^*-modules*, Iran. J. Sci. Technol. Trans. Sci. **44** (2020), 1685-1694.
18. M. Mirzaee Azandaryani, *Pseudo-duals of frames and modular Riesz bases in Hilbert C^*-modules*, Sahand Communications in Mathematical Analysis. **21(4)** (2024), 173-189.
19. M. Mirzaee Azandaryani and Z. Javadi, *Pseudo-duals of continuous frames in Hilbert spaces*, J. Pseudo-Differ. Oper. Appl. **13** (2022), 1-16.

20. M. Mirzaee Azandaryani and Z. Javadi, *Pseudo-duals and closeness of continuous g-frames in Hilbert spaces*, AUT Journal of Mathematics and Computing. To appear.
21. W. Sun, *G-frames and g-Riesz bases*, J. Math. Anal. Appl. **332** (2006), 437-452.
22. X. Xiao and X. Zeng, *Some properties of g-frames in Hilbert C^*-modules*, J. Math. Anal. Appl. **363** (2010), 399-408.
23. X. Xiao, G. Zhao and G. Zhou, *Q-duals and Q-approximate duals of g-frames in Hilbert spaces*, Numerical Functional Analysis and Optimization. **44** (2023), 510-528.

Received: 02.01.2025
Accepted: 15.04.2025