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On tangent sphere bundles with contact pseudo-metric structures

Narges Ghaffarzadeh[a] and Morteza Faghfouri[a*]

[a]Department of Pure Mathematics,
Faculty of Mathematical Sciences,
University of Tabriz, Tabriz, Iran.

E-mail: n.ghaffarzadeh@tabrizu.ac.ir
E-mail: faghfouri@tabrizu.ac.ir

Abstract. In this paper, we introduce a contact pseudo-metric structure on a tangent sphere bundle $T_\varepsilon M$. We prove that the tangent sphere bundle $T_\varepsilon M$ is (κ, μ)-contact pseudo-metric manifold if and only if the manifold M is of constant sectional curvature. Also, we show that this structure on the tangent sphere bundle is K-contact if and only if the base manifold has constant curvature ε.

Keywords: Contact pseudo-metric structure, tangent sphere bundle, unit tangent sphere bundle, Sasaki pseudo-metric.

1. **Introduction**

In 1956, S. Sasaki [7] introduced a Riemannian metric on tangent bundle TM and tangent sphere bundle T_1M over a Riemannian manifold M. Thereafter, that metric was called the Sasaki metric. In 1962, Dombrowski [3] also showed at each $Z \in TM$, $TM_Z = HTM_Z \oplus VTM_Z$, where HTM_Z and VTM_Z orthogonal subspaces of dimension n, called horizontal and vertical distributions,

*Corresponding Author
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respectively. He defined an almost Kählerian structure on TM and proved that it is Kählerian manifold if M is flat. In the same year, Tachibana and Okumura [9] showed that the tangent bundle space TM of any non-flat Riemannian space M always admits an almost Kählerian structure, which is not Kählerian. Tashiro [11] introduced a contact metric structure on the unit tangent sphere bundle T_1M and prove that contact metric structure on T_1M is K-contact if and only if M has constant curvature 1, in which case the structure is Sasakian.

Kowalski [5] computed the curvature tensor of Sasaki metric. Thus, on T_1M, $R(X,Y)\xi$ can be computed by the formulas for the curvature of TM.

In [1], Blair et al. introduced (κ,μ)-contact Riemannian manifolds and proved that, the tangent sphere bundle T_1M is a (κ,μ)-contact Riemannian manifold if and only if the base manifold M is of constant sectional curvature c.

In [10], Takahashi introduced contact pseudo-metric structures (η, g), where η is a contact one-form and g a pseudo-Riemannian metric associated to it. These structures are a natural generalization of contact metric structures. Recently, contact pseudo-metric manifolds have been studied by Calvaruso and Perrone [2, 6] and authors of this paper [4] introduced and studied (κ,μ)-contact pseudo-metric manifolds.

In this paper, we suppose that (M,g) is pseudo-metric manifold and define pseudo-metric on TM. Also, we introduce contact pseudo-metric structures (φ,ξ,η,G) on $T_\varepsilon M$ and prove that

$$\bar{R}(X,Y)\xi = c(2\varepsilon - c)\Big\{\eta(Y)X - \eta(X)Y\Big\} - 2c\Big\{\eta(Y)\,\mathbf{h}X - \eta(X)\,\mathbf{h}Y\Big\}$$

if and only if the base manifold M is of constant sectional curvature. That is, the tangent sphere bundle $T_\varepsilon M$ is a (κ,μ)-contact pseudo-metric manifold iff the base manifold M is of constant sectional curvature c. Also, the contact pseudo-metric structure (φ,ξ,η,G) on $T_\varepsilon M$ is K-contact if and only if the base manifold (M,g) has constant curvature ε.

2. **Preliminaries**

Let (M,g) be a pseudo-metric manifold, ∇ the associated Levi-Civita connection and $R = [\nabla,\nabla] - \nabla_{[,]}$ the curvature tensor. The tangent bundle of M, denoted by TM, consists of pairs (x,u), where $x \in M$ and $u \in T_xM$,(i.e., $TM = \cup_{x\in M} T_xM$). The mapping $\pi : TM \to M, \pi(x,u) = x$ is the natural projection and for all $(x,u) \in TM$, the connection map $\mathcal{K} : TTM \to TM$ is given by $\mathcal{K}(X_*u) = \nabla_u X$, where $X : M \to TM$ is a vector field on M [3].

The tangent space $T_{(x,u)}TM$ splits into the vertical subspace $VTM_{(x,u)}$ and the horizontal subspace $HTM_{(x,u)}$ are given by $VTM_{(x,u)} := \ker \pi_*|_{(x,u)}$ and $HTM_{(x,u)} := \ker \mathcal{K}|_{(x,u)}$:

$$T_{(x,u)}TM = VTM_{(x,u)} \oplus HTM_{(x,u)}.$$

For every $X \in T_xM$, there is a unique vector $X^h \in HTM_{(x,u)}$, such that

$$\pi_*(X^h) = X.$$

It is called the horizontal lift of X to (x, u). Also, there is a unique vector $X^v \in VTM_{(x,u)}$, such that

$$X^v(df) = Xf, \forall f \in C^\infty(M).$$

X^v is called the vertical lift of X to (x, u). The maps $X \mapsto X^h$ between T_xM and $HTM_{(x,u)}$, and $X \mapsto X^v$ between T_xM and $VTM_{(x,u)}$ are isomorphisms. Hence, every tangent vector $\bar{Z} \in T_{(x,u)}TM$ can be decomposed $\bar{Z} = X^h + Y^v$ for uniquely determined vectors $X, Y \in T_xM$. The horizontal (respectively, vertical) lift of a vector field X on M to TM is the vector field X^h (respectively, X^v) on M, whose value at the point (x, u) is the horizontal (respectively, vertical) lift of X_x to (x, u).

A system of local coordinate $(x^1, \dots, x^n)$ on an open subset U of M induces on $\pi^{-1}(U)$ of TM a system of local coordinate $(\bar{x}^1, \dots, \bar{x}^n; u^1, \dots, u^n)$ as follows:

$$\bar{x}^i(x, u) = (x^i \circ \pi)(x, u) = x^i(x),$$
$$u^i(x, u) = dx^i(u) = ux^i$$

for $i = 1, \dots, n$ and $(x, u) \in \pi^{-1}(U)$. With respect to the induced local coordinate system, the horizontal and vertical lifts of a vector field $X = X^i \frac{\partial}{\partial x^i}$ on U are given by

$$X^h = (X^i \circ \pi)\frac{\partial}{\partial \bar{x}^i} - u^b((X^a\Gamma^i_{ab}) \circ \pi)\frac{\partial}{\partial u^i}, \tag{2.1}$$

$$X^v = (X^i \circ \pi)\frac{\partial}{\partial u^i}, \tag{2.2}$$

where Γ^i_{jk} are the local components of ∇, i.e.,

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = \Gamma^i_{jk}\frac{\partial}{\partial x^i}.$$

From (2.1) and (2.2), one can easily calculate the brackets of vertical and horizontal lifts:

$$[X^h, Y^h] = [X, Y]^h - v\{R(X, Y)u\}, \tag{2.3}$$

$$[X^h, Y^v] = (\nabla_X Y)^v, \tag{2.4}$$

$$[X^v, Y^v] = 0, \tag{2.5}$$

for all $X, Y \in \Gamma(TM)$. We use some notation, due to M. Sekizawa ([8]). Let T be a tensor field of type $(1, s)$ on M and $X_1, \dots, X_{s-1} \in \Gamma(TM)$, the vertical vector field $v\big\{T(X_1, \dots, u, \dots, X_{s-1})\big\}$ on $\pi^{-1}(U)$ is given by

$$v\{T(X_1, \dots, u, \dots, X_{s-1})\} := u^a(T(X_1, \dots, \frac{\partial}{\partial x^a}, \dots, X_{s-1}))^v.$$

Analogously, one defines the horizontal vector field $h\{T(X_1,\ldots,u,\ldots,X_{s-1})\}$ and $h\{T(X_1,\ldots,u,\ldots,u,\ldots,X_{s-2})\}$ and the vertical vector field $v\{T(X_1,\ldots,u,\ldots,u,\ldots,X_{s-2})\}$. Note that these vector fields do not depend on the choice of coordinates on U.

If f is a smooth function on M and X is a vector field on M, then

$$\begin{aligned} X^h(f\circ\pi) &= (Xf)\circ\pi, \\ X^v(f\circ\pi) &= 0. \end{aligned} \tag{2.6}$$

In particular, we write $X = X^i\frac{\partial}{\partial x^i}$ on U, and then we have

$$\begin{aligned} X^h(\bar{x}^i) &= X^i\circ\pi, \\ X^v(\bar{x}^i) &= 0. \end{aligned} \tag{2.7}$$

Further, from (2.1) and (2.2), we have

$$\begin{aligned} X^h(u^i) &= -u^b(X^a\Gamma^i_{ab})\circ\pi, \\ X^v(u^i) &= X^i\circ\pi. \end{aligned} \tag{2.8}$$

Let (M,g) be a pseudo-metric manifold. On the tangent bundle TM, we can define a pseudo-metric $\tilde{g}$ to be

$$\begin{aligned} \tilde{g}(X^h,Y^h) &= \tilde{g}(X^v,Y^v) = g(X,Y)\circ\pi, \\ \tilde{g}(X^h,Y^v) &= 0 \end{aligned} \tag{2.9}$$

for all $X,Y\in\Gamma(TM)$. We call it Sasaki pseudo-metric. According (2.9), If $\{E_1,\ldots,E_n\}$ is an orthonormal frame field on U then $\{E_1^v,\ldots,E_n^v,E_1^h,\ldots,E_n^h\}$ is an orthonormal frame field on $\pi^{-1}(U)$. So, we have the following:

Proposition 2.1. *If the index of g is ν then the index of the Sasaki pseudo-metric $\tilde{g}$ is 2ν.*

Let $\tilde{\nabla}$ be the Levi-Civita connection of $\tilde{g}$. It is easy to check that for $X,Y\in\Gamma(TM)$ and $(x,u)\in TM$(see [5] for more details):

$$\begin{aligned} \tilde{\nabla}_{X^v}Y^v &= 0, \\ \tilde{\nabla}_{X^v}Y^h &= \frac{1}{2}h\Big\{R(u,X)Y\Big\}, \\ \tilde{\nabla}_{X^h}Y^v &= (\nabla_XY)^v + \frac{1}{2}h\Big\{R(u,Y)X\Big\}, \\ \tilde{\nabla}_{X^h}Y^h &= (\nabla_XY)^h - \frac{1}{2}v\Big\{R(X,Y)u\Big\}. \end{aligned} \tag{2.10}$$

3. The curvature of the unit tangent sphere bundle with pseudo-metric

Let $(TM,\tilde{g})$ be the tangent bundle of (M,g) endowed with its Sasaki pseudo-metric. We consider the hypersurface $T_\varepsilon M = \Big\{(x,u)\in TM | g_x(u,u)=\varepsilon\Big\}$, which we call the unit tangent sphere bundle. A unit normal vector field N on $T_\varepsilon M$ is the (vertical) vector field

$$N = u^i \frac{\partial}{\partial u^i} = u^i (\frac{\partial}{\partial x^i})^v.$$

N is independent of the choice of local coordinates and it is defined globally on TM. We introduce some more notation. If $X \in T_x M$, we define the tangential lift of X to $(x, u) \in T_\varepsilon M$ by

$$X^t_{(x,u)} = X^v_{(x,u)} - \varepsilon g(X, u) N_{(x,u)}. \tag{3.1}$$

Clearly, the tangent space to $T_\varepsilon M$ at (x, u) is spanned by vectors of the form X^h and X^t, where $X \in T_x M$. Note that $u^t_{(x,u)} = 0$. The tangential lift of a vector field X on M to $T_\varepsilon M$ is the vertical vector field X^t on $T_\varepsilon M$, whose value at the point $(x, u) \in T_\varepsilon M$ is the tangential lift of X_x to (x, u). For a tensor field T of type $(1, s)$ on M and $X_1, \dots, X_{s-1} \in \Gamma(TM)$, we define the vertical vector fields $t\{T(X_1, \dots, u, \dots, X_{s-1})\}$ and $t\{T(X_1, \dots, u, \dots, u, \dots, X_{s-2})\}$ on $T_\varepsilon M$ in the obvious way.
In what follows, we will give explicit expressions for the brackets of vector fields on $T_\varepsilon M$ involving tangential lifts, the Levi-Civita connection and the associated curvature tensor of the induced metric $\bar{g}$ on $T_\varepsilon M$.

First, for the brackets of vector fields on $T_\varepsilon M$ involving tangential lifts, we obtain

$$[X^h, Y^t] = (\nabla_X Y)^t, \tag{3.2}$$

$$[X^t, Y^t] = \varepsilon g(X, u) Y^t - \varepsilon g(Y, u) X^t. \tag{3.3}$$

Next, we denote by $\bar{g}$ the pseudo-metric induced on $T_\varepsilon M$ from $\tilde{g}$ on TM as follows:

$$\begin{aligned} &\bar{g}(X^h, Y^h) = g(X, Y), \\ &\bar{g}(X^t, Y^t) = g(X, Y) - \varepsilon g(X, u) g(Y, u), \\ &\bar{g}(X^h, Y^t) = 0 \end{aligned} \tag{3.4}$$

Proposition 3.1. *The Levi-Civita connection $\bar{\nabla}$ of $(T_\varepsilon M, \bar{g})$ is described completely by*

$$\begin{aligned} \bar{\nabla}_{X^t} Y^t &= -\varepsilon g(Y, u) X^t, \\ \bar{\nabla}_{X^t} Y^h &= \frac{1}{2} h\{R(u, X)Y\}, \\ \bar{\nabla}_{X^h} Y^t &= (\nabla_X Y)^t + \frac{1}{2} h\{R(u, Y)X\}, \\ \bar{\nabla}_{X^h} Y^h &= (\nabla_X Y)^h - \frac{1}{2} t\{R(X, Y)u\} \end{aligned} \tag{3.5}$$

for all vector fields X and Y on M.

Proof. This is obtained by an easy calculation using (2.10) and the following equation

$$\bar{\nabla}_{\bar{A}}\bar{B} = \tilde{\nabla}_{\bar{A}}\bar{B} - \varepsilon\tilde{g}(\bar{\nabla}_{\bar{A}}\bar{B}, N)N,$$

for vector fields $\bar{A}, \bar{B}$ tangent to $T_\varepsilon M$. □

Proposition 3.2. *The curvature tensor $\bar{R}$ of $(T_\varepsilon M, \bar{g})$ is described completely by*

$$\bar{R}(X^t, Y^t)Z^t = \varepsilon\Big(- \bar{g}(X^t, Z^t)Y^t + \bar{g}(Z^t, Y^t)X^t\Big), \tag{3.6}$$

$$\begin{aligned}\bar{R}(X^t, Y^t)Z^h = (R(X,Y)Z)^h - \varepsilon\Big(g(Y,u)h(R(X,u)Z) + g(X,u)h(R(u,Y)Z)\Big)\\ +\frac{1}{4}h\Big\{[R(u,X), R(u,Y)]Z\Big\},\end{aligned} \tag{3.7}$$

$$\begin{aligned}\bar{R}(X^h, Y^t)Z^t = -\frac{1}{2}(R(Y,Z)X)^h + \frac{\varepsilon}{2}\Big(g(Y,u)h(R(u,Z)X) + g(Z,u)h(R(Y,u)X)\Big)\\ -\frac{1}{4}h\Big\{R(u,Y)R(u,Z)X\Big\},\end{aligned} \tag{3.8}$$

$$\begin{aligned}\bar{R}(X^h, Y^t)Z^h = \frac{1}{2}\Big(R(X,Z)Y\Big)^t - \frac{\varepsilon}{2}g(Y,u)t\Big\{R(X,Z)u\Big\}\\ -\frac{1}{4}t\Big\{R(X, R(u,Y)Z)u\Big\} + \frac{1}{2}h\Big\{(\nabla_X R)(u,Y)Z\Big\},\end{aligned} \tag{3.9}$$

$$\begin{aligned}\bar{R}(X^h, Y^h)Z^t = \Big(R(X,Y)Z\Big)^t - \varepsilon g(Z,u)t\Big\{R(X,Y)u\Big\}\\ +\frac{1}{4}t\Big\{R(Y, R(u,Z)X)u - R(X, R(u,Z)Y)u\Big\}\\ +\frac{1}{2}h\Big\{(\nabla_X R)(u,Z)Y - (\nabla_Y R)(u,Z)X\Big\},\end{aligned} \tag{3.10}$$

$$\begin{aligned}\bar{R}(X^h, Y^h)Z^h = \Big(R(X,Y)Z\Big)^h + \frac{1}{2}h\Big\{R(u, R(X,Y)u)Z\Big\}\\ -\frac{1}{4}h\Big\{R(u, R(Y,Z)u)X - R(u, R(X,Z)u)Y\Big\}\\ +\frac{1}{2}t\Big\{(\nabla_Z R)(X,Y)u\Big\}\end{aligned} \tag{3.11}$$

for all vector fields X, Y and Z on M.

Proof. The proof is made by using the following equation and equation (3.5) for the covariant derivative, (2.3), (3.2) and (3.3) for the brackets are explicitly calculated.

$$\bar{R}(\bar{A}, \bar{B})\bar{C} = \bar{\nabla}_{\bar{A}}\bar{\nabla}_{\bar{B}}\bar{C} - \bar{\nabla}_{\bar{B}}\bar{\nabla}_{\bar{A}}\bar{C} - \bar{\nabla}_{[\bar{A},\bar{B}]}\bar{C}.$$

□

4. The contact pseudo-metric structure of the unit tangent sphere bundle

First, we give some basic facts on contact pseudo-metric structures. A pseudo-Riemannian manifold (M^{2n+1}, g) has a contact pseudo-metric structure if it admits a vector field ξ, a one-form η and a $(1,1)$-tensor field φ satisfying

$$\begin{aligned} &\eta(\xi) = 1, \\ &\varphi^2(X) = -X + \eta(X)\xi, \\ &g(\varphi X, \varphi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y), \\ &d\eta(X,Y) = g(X, \varphi Y), \end{aligned} \tag{4.1}$$

where $\varepsilon = \pm 1$ and $X, Y \in \Gamma(TM)$. In this case, $(M, \varphi, \xi, \eta, g)$ is called a contact pseudo-metric manifold. In particular, the above conditions imply that the characteristic curves, i.e., the integral curves of the characteristic vector field ξ, are geodesics.

If ξ is in addition a Killing vector field with respect to g, then the manifold is said to be a K-contact (pseudo-metric) manifold. Another characterizing property of such contact pseudo-metric manifolds is the following:
geodesics which are orthogonal to ξ at one point, always remain orthogonal to ξ. This yields a second special class of geodesics, the φ-geodesics.

Next, if $(M^{2n+1}, \varphi, \xi, \eta, g)$ is a contact pseudo-metric manifold satisfying the additional condition $N_\varphi(X,Y) + 2d\eta(X,Y)\xi = 0$ is said to be Sasakian, where

$$N_\varphi(X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$$

is the Nijenhuis torsion tensor of φ.

A contact pseudo-metric structure is a Sasakian structure if and only if R satisfies

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{4.2}$$

In particular, one can show that the characteristic vector field ξ is a Killing vector field. Hence, a Sasakian manifold is also a K-contact manifold. In a contact pseudo-metric manifold $M^{2n+1}(\varphi, \xi, \eta, g)$, defined the $(1,1)$-tensor field $\mathbf{h}$ by

$$\mathbf{h}X = \frac{1}{2}(L_\xi \varphi)(X),$$

where L denotes the Lie derivative. The tensors $\mathbf{h}$ is self-adjoint operator satisfying([2, 6])

$$\mathbf{h}\varphi = -\varphi\,\mathbf{h}, \tag{4.3}$$

$$\mathbf{h}\xi = 0, \tag{4.4}$$

$$\nabla_X \xi = -\varepsilon\varphi X - \varphi\,\mathbf{h}X. \tag{4.5}$$

(see [2, 6] for more details). If a contact pseudo-metric manifold satisfying

$$R(X,Y)\xi = \varepsilon\kappa\Big(\eta(Y)X - \eta(X)Y\Big) + \varepsilon\mu\Big(\eta(Y)\,\mathbf{h}X - \eta(X)\,\mathbf{h}Y\Big),$$

we call (κ, μ)-contact pseudo-metric manifold, where $(\kappa, \mu) \in \mathbb{R}^2$. the (κ, μ)-contact pseudo-metric manifold is Sasakian iff $\kappa = \varepsilon$ and hence $\mathbf{h} = 0$, by (4.2). (see [4] for more details).

Take now an arbitrary pseudo-metric manifold (M, g). One can easily define an almost complex structure J on TM in the following way:

$$JX^h = X^v, \quad JX^v = -X^h \tag{4.6}$$

for all vector fields X on M. From (2.3), (2.4) and (2.5), we have the almost complex structure J is integrable if and only if (M, g) is flat. From the definition (2.9) of the pseudo-metric $\tilde{g}$ on TM, it follows immediately that $(TM, \tilde{g}, J)$ is almost Hermitian. Moreover, J defines an almost Kählerian structure. It is a Kähler manifold only when (M, g) is flat[3].
Next, we consider the unit tangent sphere bundle $(T_\varepsilon M, \bar{g})$, which is isometrically embedded as a hypersurface in $(TM, \tilde{g})$ with unit normal field N. Using the almost complex structure J on TM, we define a unit vector field ξ', a one-form η' and a $(1,1)$-tensor field φ' on $T_\varepsilon M$ by

$$\xi' = -JN, \quad JX = \varphi' X + \eta'(X)N. \tag{4.7}$$

In local coordinates, ξ', η' and φ' are described by

$$\begin{aligned} &\xi' = u^i \Big(\frac{\partial}{\partial x^i}\Big)^h, \\ &\eta'(X^t) = 0, \quad \eta'(X^h) = \varepsilon g(X, u), \\ &\varphi'(X^t) = -X^h + \varepsilon g(X, u)\xi', \\ &\varphi'(X^h) = X^t, \end{aligned} \tag{4.8}$$

where $X, Y \in \Gamma(TM)$. It is easily checked that these tensors satisfy the conditions (4.1) excepts or the last one, where we find $\varepsilon\bar{g}(X, \varphi' Y) = 2d\eta'(X, Y)$. So strictly speaking, $(\varphi', \xi', \eta', \bar{g})$ is not a contact pseudo-metric structure. Of course, the difficulty is easily rectified and

$$\eta = \frac{1}{2}\eta', \quad \xi = 2\xi', \quad \varphi = \varepsilon\varphi', \quad G = \frac{1}{4}\bar{g}$$

is taken as the standard contact pseudo-metric structure on $T_\varepsilon M$. In local coordinates, with respect to induce the local coordinates (x^i, u^i) on TM, the characteristic vector field is given by

$$\xi_{(x,u)} = 2u^i \Big(\frac{\partial}{\partial x^i}\Big)^h = 2u^h.$$

By using (3.2) and (3.3), we have

$$\begin{aligned} &L_\xi X^h = 2\Big(u^i [\frac{\partial}{\partial x^i}, X]^h - v\{R(u, X)u\} + u^b X^a \Gamma^i_{ab} (\frac{\partial}{\partial x^i})^h\Big), \\ &L_\xi X^t = 2\Big((\nabla_u X)^t - X^h + \varepsilon g(X, u) u^h\Big). \end{aligned} \tag{4.9}$$

Before beginning our theorems, we explicitly obtain the covariant derivatives of ξ. For a horizontal tangent vector field, we may use a horizontal lift again. Then

$$\bar{\nabla}_{X^h}\xi = \tilde{\nabla}_{X^h}\xi = -v\Big\{R(X, u)u\Big\}$$

and hence for any horizontal vector X^h at $(x,u)\in T_\varepsilon M$, we have

$$\bar{\nabla}_{X^h}\xi = -v\{R(X,u)u\} = -t\{R(X,u)u\}.$$

For a vertical vector field X^v tangent to $T_\varepsilon M$, we have

$$\bar{\nabla}_{X^v}\xi = \tilde{\nabla}_{X^v}\xi = -2\varepsilon\varphi X^v - h\{R(X,u)u\}.$$

Since $J(\frac{\partial}{\partial x^i})^h = (\frac{\partial}{\partial x^i})^v$, or in terms of tangential lifts of a vector X on M,

$$\bar{\nabla}_{X^t}\xi = -2\varepsilon\varphi X^t - h\{R(X,u)u\}.$$

Comparing with $\bar{\nabla}_X\xi = -\varepsilon\varphi X - \varphi\,\mathbf{h}X$ on $T_\varepsilon M$ for a vertical vector V and a horizontal vector X orthogonal to ξ, $\mathbf{h}V$ and $\mathbf{h}X$ are given by

$$\mathbf{h}V = \varepsilon V - \varepsilon v\{R(\mathcal{K}V,u)u\} \quad \text{and} \quad \mathbf{h}X = -\varepsilon X + \varepsilon h\{R(\pi_* X,u)u\}. \tag{4.10}$$

Theorem 4.1. *The tangent sphere bundle $T_\varepsilon M$ is (κ,μ)-contact pseudo-metric manifold if and only if the base manifold M is of constant sectional curvature c and $\kappa = \varepsilon c(2\varepsilon - c), \mu = -2\varepsilon c$.*

Proof. Assume that the manifold M is a pseudo-metric manifold of constant curvature c. Then from equations (3.6-3.11), for X,Y orthogonal to ξ, we have $\bar{R}(X,Y)\xi = 0$ and for a vertical vector V, we get $\bar{R}(V,\xi)\xi = c^2V$ and also, for a horizontal vector X orthogonal to ξ, we obtain $\bar{R}(X,\xi)\xi = (4\varepsilon c - 3c^2)X$. Moreover, from equations (4.10),

$$\mathbf{h}V = (\varepsilon - c)V \text{ and } \mathbf{h}X = (c-\varepsilon)X. \tag{4.11}$$

Thus for all X,Y on $T_\varepsilon M$, the curvature tensor on $T_\varepsilon M$ satisfies

$$\bar{R}(X,Y)\xi = c(2\varepsilon - c)\Big(\eta(Y)X - \eta(X)Y\Big) - 2c\Big(\eta(Y)\,\mathbf{h}X - \eta(X)\,\mathbf{h}Y\Big). \tag{4.12}$$

Conversely, if the contact pseudo-metric structure on $T_\varepsilon M$ satisfies the condition

$$\bar{R}(X,Y)\xi = \varepsilon\kappa\Big(\eta(Y)X - \eta(X)Y\Big) + \varepsilon\mu\Big(\eta(Y)\,\mathbf{h}X - \eta(X)\,\mathbf{h}Y\Big),$$

then

$$\bar{R}(X,\xi)\xi = \varepsilon\kappa X + \varepsilon\mu\,\mathbf{h}X, \tag{4.13}$$

for any X orthogonal to ξ. Now, for a vector u on M, that $g(u,u) = \varepsilon$ define a symmetric the Jacobi operator with respect to u, that is, $\psi_u : \langle u\rangle^\perp \to \langle u\rangle^\perp$ by

$$\psi_u X = R(X,u)u.$$

By placing the equation (4.10) in (4.13), we get

$$\bar{R}(V,\xi)\xi = (\varepsilon\kappa + \mu)V - \mu\, v\{\psi_u \mathcal{K}V\}. \tag{4.14}$$

Also using equations (3.6-3.11), we have

$$\bar{R}(V,\xi)\xi = -v\Big\{R(R(u,\mathcal{K}V)u,u)u\Big\} = v\Big\{\psi_u^2\mathcal{K}V\Big\}. \tag{4.15}$$

From a comparison of equations (4.14) and (4.15), the operator ψ_u satisfies the equation

$$\psi_u^2 + \mu\psi_u - (\varepsilon\kappa + \mu)I = 0. \tag{4.16}$$

In a similar way, for a horizontal X orthogonal to ξ,

$$\bar{R}(X,\xi)\xi = (\varepsilon\kappa - \mu)X + \mu\, h\{\psi_u\pi_* X\}, \tag{4.17}$$

and, from equations (3.6-3.11), we obtain

$$\bar{R}(X,\xi)\xi = h\Big\{4\psi_u\pi_* X - 3\psi_u^2\pi_* X\Big\}, \tag{4.18}$$

From a comparison of equations (4.17) and (4.18), we have

$$3\psi_u^2 + (\mu - 4)\psi_u + (\varepsilon\kappa - \mu)I = 0. \tag{4.19}$$

Since ψ_u is symmetric operator, then the eigenvalues a of ψ_u are real numbers and satisfy the quadratic equations

$$a^2 + \mu a - (\varepsilon\kappa + \mu) = 0, \tag{4.20}$$

$$a^2 + \frac{\mu - 4}{3}a + \frac{\varepsilon\kappa - \mu}{3} = 0. \tag{4.21}$$

According to the equations (4.16) and (4.19), the minimal polynomial of ψ_u divides the quadratic equations (4.20) and (4.21). Hence, the minimal polynomial of ψ_u has degree at most 2. If the minimal polynomial of ψ_u is of degree two, then ψ_u has two eigenvalues, therefore, $\mu = -2$ and $\kappa = \varepsilon$. Thus $a^2 + \mu a - (\varepsilon\kappa + \mu) = (a-1)^2 = 0$ that is, $a = 1$. If the minimal polynomial of ψ_u is of degree one, then a is the only eigenvalue of ψ_u. Anyway, $a = -\frac{\mu}{2}$. Hence, we have

$$\psi_u X = R(X,u)u = -\frac{\mu}{2}X.$$

We suppose that $g(X,X) = 1$ and X orthogonal to u. Then

$$K(X,u) = \frac{g(R(X,u)u, X)}{g(u,u)g(X,X) - g(X,u)^2} = -\varepsilon\frac{\mu}{2}, \tag{4.22}$$

where $K(X,u)$ is the sectional curvatures of the nondegenerate plane $\{X,u\}$. Therefore, (M,g) is a space of constant curvature $c = -\dfrac{\varepsilon\mu}{2}$ and $\kappa = \varepsilon c(2\varepsilon - c)$. □

We now have the following theorem about the K-contact structure.

Theorem 4.2. *The contact pseudo-metric structure (φ,ξ,η,G) on $T_\varepsilon M$ is K-contact if and only if the base manifold (M,g) has constant curvature ε, in which case the structure on $T_\varepsilon M$ is Sasakian.*

Proof. We assume that the contact pseudo-metric structure (φ,ξ,η,G) on $T_\varepsilon M$ is K-contact. In this case, ξ is Killing vector field and equivalently $\mathbf{h} = 0$. By using (4.10), for all $(x,u) \in T_\varepsilon M$ and for horizontal lift X^h of X orthogonal to u, we have

$$R(X,u)u = X. \tag{4.23}$$

We suppose that $g(X,X) = 1$. Then

$$K(X,u) = \frac{g(R(X,u)u,X)}{g(u,u)g(X,X) - g(X,u)^2} = \varepsilon. \tag{4.24}$$

Therefore, (M,g) is a space of constant curvature ε. Conversely, if M has constant curvature $c = \varepsilon$, by using (4.12), we have $\kappa = \varepsilon$, then $T_\varepsilon M$ is a Sasakian manifold. Hence M is K-contact. □

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